Complex Numbers

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1 Introduction

Before this we have been working with a field of numbers called the reals. However, if numbers are only real numbers, equations like

$$x^2 + 1 = 0 \tag{1}$$

are not solvable since $\sqrt{-1}$ is not defined on the real number line. Therefore, humans have invented the new number i, symbolizing the fundamental unit of complex numbers.
Complex numbers may sound intriguing at first, but it is really the same logic as the time you had first learned negative numbers - they look extraneous but they are actually really useful.

2 What is a complex number?

All complex numbers come in the form

$$a + bi$$

where a, b are reals. a is called the real part; b is the imaginary part. Here $i = \sqrt{-1}$. Two complex numbers are equal if and only if their real parts AND imaginary parts are equal.

3 The Four Basic Operations

1. $(4 + i) + (3 - 10i) =$
2. $(4 + i)(3 - 10i) =$
3. $\frac{4+i}{3-10i} =$

4 The Complex Plane (Argand Diagram)

Just like the way every real number can be associated with a point on the real number line, every complex number can be associated with a point on the complex number plane. Similar to the Cartesian Plane, the horizontal axis is called the real axis (Re) while the vertical axis is called the imaginary axis (Im).
The complex plane:

Exercise: Plot $3 + 4i$, $-2 + i$, $\pi - 2i$ on your plane.

4.1 Absolute Value of Complex Numbers

Some of you learned absolute value as "switch if negative, remain if positive". But the actual definition of absolute values (or modulus) is how far the number is from the origin. For example, -3 is three units away from the origin, thus $|-3| = 3$. Now it is easy to figure out why the absolute value of a complex number is

$$|a + bi| = \sqrt{a^2 + b^2}$$

The common letter for it is r.

4.2 Argument of Complex Numbers

This is a new thing in complex numbers. Since a complex number associates with a point on the Argand Plane, this point creates some angle from the horizontal axis at the origin. This angle is called the argument of the number and the symbol for it is

$$arg(a + bi) = tan^{-1}\frac{b}{a}$$

The common letter for it is θ.

You have learnt the basics of the complex plane. Now let's study new methods of writing complex numbers.

5 The Trigonometric Form

Make sure that you are familiar with the definition of trigonometric functions. Now figure out why can we write complex numbers in this form:

$$a + bi = r(cos\theta + isin\theta)$$

where r is the absolute value and θ is the argument.

6 The Exponential Form

The proof of this writing form involves the MacLaurin series or integrals, so beginners won't really care about them. The form is:

$$a + bi = re^{i\theta}$$

where r is the absolute value and θ is the argument.

7 Complex Conjugates

The complex conjugate of $a + bi$ is equal to $\overline{a + bi} = a - bi$.

8 Solving Equations

Back to (1), we get

$$\begin{aligned} x^2 &= -1 \\ x &= \pm\sqrt{-1} \\ x &= \pm i \end{aligned}$$

Note that there are no more "positive roots" in complex numbers since $(1 - i)^2 = (-1 + i)^2 = -2i$ but $1 - i$ and $-1 + i$ cannot be determined whether which is positive. Some more examples:

1. Solve for reals k and j in $5 - ki = j + 4i$

2. $(1 + i)(2 + i)(3 + i) =$

3. Solve for $x \in \mathbb{C}$ in $(x + 1)(x + 2)(x + 3)(x + 4) = 15$.

We will discover more examples after we learn the following important theorem.

9 De Moivre's Theorem

De Moivre's Theorem states that when two complex numbers are multiplied to each other, their modulus multiply and their arguments add up. In maths terms, if $A, B \in \mathbb{C}$,

$$|AB| = |A||B|$$
$$argAB = argA + argB$$

Proof:
Let $A = r_1 e^{i\theta_1}, B = r_2 e^{i\theta_2}$. Then,

$$AB = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

which literally proves the statement.

10 More Problems

Before the examples, let's study a 2008 problem in the Chinese National Team Training Test.

Consider an infinite triangular tiling made up with equilateral triangles with side length 1. Let the intersection points be vertices; and vertices with distance 1 be adjacent.
Two frogs A and B are playing a jumping game. "One jump" means a jump to an adjacent vertex. Starting with A, both frogs jump by the following rules:

- Rule (1): A jumps to any adjacent vertex, then B jumps either one time the direction how A jumps; or jumps two time the direction opposite how A jumps.
- Rule (2): If A and B are adjacent to each other, they can undergo rule (1), or A jumps two times keeping adjacent to B while B stays put.

If A and B are adjacent at the beginning, can they undergo a limited number of jumps such that at last, both land on the opponent's starting point?

Solution 1
Let $\omega = \frac{1}{2} + \frac{\sqrt{3}}{2}$.
Denote $\mathbf{A}$ and $\mathbf{B}$ as the variable of the positions of A and B respectively. At the beginning, let $\mathbf{A} = 0$, $\mathbf{B} = 1$. Note that the possible movements can be written by

1. $\Delta(\mathbf{A} - \mathbf{B}) = 0$ Let there be a of them.
2. $\Delta(\mathbf{A} - \mathbf{B}) = 3$ Let there be b of them.
3. $\Delta(\mathbf{A} - \mathbf{B}) = 3\omega$ Let there be c of them.
4. $\Delta(\mathbf{A} - \mathbf{B}) = 3\omega^2$ Let there be d of them.
5. $\Delta(\mathbf{A} - \mathbf{B}) = \sqrt{3}i$ Let there be e of them.
6. $\Delta(\mathbf{A} - \mathbf{B}) = \sqrt{3}i\omega$ Let there be f of them.
7. $\Delta(\mathbf{A} - \mathbf{B}) = \sqrt{3}i\omega^2$ Let there be g of them.

In the end we want to have $\mathbf{A} = 0$, $\mathbf{B} = 1$, which means $\Delta(\mathbf{A} - \mathbf{B}) = -2$ Therefore adding up we have

$$0a + 3b + 3\omega c + 3\omega^2 d + \sqrt{3}ie + \sqrt{3}i\omega f + \sqrt{3}i\omega^2 g = -2$$

Comparing real parts,

$$3b + \frac{3}{2}c + \frac{3}{2}d + \frac{3}{2}f + \frac{3}{2}g = -2$$

Multiplying by 2 on both sides and we get that: the LHS is a multiple of 3, but the RHS is not, a contradiction! Therefore it cannot happen. (Q.E.D)

Solution 2 WLOG let B be directly right of A. Assign each vertex a number, either 1, ω, or ω^2 ($\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}$). Let $A = 1$, $B = \omega$, and every equilateral triangle in the grid will have vertices 1, ω, ω^2. Every time the frogs move, coincidentally the ratio $\frac{\mathbf{A}}{\mathbf{B}}$ does not change ($\mathbf{A}$ and $\mathbf{B}$ denotes the value of positions of frogs A and B). At the start, the ratio is $\frac{1}{\omega}$; if they switched places in the end, the ratio would be $\frac{\omega}{1}$. They are not equal, thus proving it is not possible. (Q.E.D)

Here are some easier problems:

1. Three squares are adjacently placed with each other as in the diagram. What is the value of $\alpha + \beta + \gamma$?

2. $x + \frac{1}{x} = -1$. What is the value of $x^{10} + x^2$? (2017/HuaLuoGeng)

3. Consider a right-angled triangle $\triangle ABC$ with $AC = 3$ and $BC = 4$ and $C = 90°$. Erect a square from its hypotenuse. Let the corner of the square outside the triangle but closer to A be D. Find the length of CD.

4. Let n be a natural number. Prove that the equation
$$z^{n+1} - z^n - 1 = 0$$
has complex roots with modulo 1 if and only if $n + 2$ is divisible by 6. (1987/CMO)

5. Prove Napoleon's Theorem: For any triangle S, erect three equilateral triangles on the three sides of S, and let the centres of these equilateral triangles be P, Q, R. Then $\triangle PQR$ is an equilateral triangle.