Finite Differences

Chaang Tze Shen Tristan

January 22, 2022

1 Operators

We define $\Delta f(x) = f(x+1) - f(x)$ as the first order finite difference. The symbol Δ is said to be the finite difference operator. Generally we define

$$\Delta^r f(x) = \Delta(\Delta^{r-1} f(x)) \tag{1}$$

as the r-order finite difference. Also we define the following operators:

1. $Ef(x) = f(x+1)$

2. $If(x) = f(x)$

Hence

$$\begin{aligned} E^n f(x) &= f(x+n) \\ I^n f(x) &= f(x) \\ \Delta &= E - I \end{aligned}$$

These operations will be very useful in proving some of the propositions below.

2 Finite Differences

Let's start by finding a pattern in finite differences of various orders:

$$\begin{aligned} \Delta^2 f(x) &= f(x+2) - 2f(x+1) + f(x) \\ \Delta^3 f(x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x) \\ \Delta^4 f(x) &= f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) \end{aligned}$$

As such, we propose the following:

Proposition 1. Let r be a positive integer, then

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+n-k) \tag{2}$$

Proof. Since $\Delta = E - I$,

$$\Delta^n = (E-I)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E^{n-k} \quad \square$$

Let's look from a different perspective from Proposition 1, where we write a function in terms of finite differences instead.

$$\begin{aligned} f(x+1) &= \Delta f(x) + f(x) \\ f(x+2) &= \Delta^2 f(x) + 2\Delta f(x) + f(x) \\ f(x+3) &= \Delta^3 f(x) + 3\Delta^2 f(x) + 3\Delta f(x) + f(x) \end{aligned}$$

Hence we propose the following:

Proposition 2. For any non-negative integer n,

$$f(x+n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(x) \tag{3}$$

Proof. Since $\Delta = E - I$,

$$E^n = (\Delta + I)^n = \sum_{k=0}^{n} \binom{n}{k} \Delta^k \quad \square$$

Again, this directly brings us to

Proposition 3. For any non-negative integer n,

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0)$$

Proof. Proposition 2. $x = 0$. $\square$

We take one step further: Writing $f(1)+\cdots+f(n)$ as a sum of finite differences.

Proposition 4. The sum of the first n terms of $f(n)$ is

$$f(1)+\cdots+f(n)=\sum_{k=1}^{n}\binom{n}{k}\Delta^{k-1}f(1) \tag{4}$$

Proof. Since $\Delta = E - I$,

$$E^0+\cdots+E^{n-1}=\frac{E^n-I}{E-I}=\frac{(\Delta+I)^n-I}{\Delta}=\sum_{k=0}^{n}\binom{n}{k+1}\Delta^k \quad \square$$

3 Polynomials

Next let's see what happens if we apply finite differences to polynomials:

$$\begin{aligned} f(x) &= a_n x^n + \dots \\ \Delta f(x) &= n a_n x^{n-1} + \dots \\ \Delta^2 f(x) &= n(n-1) a_n x^{n-2} + \dots \end{aligned}$$

Proposition 5. If $f(x)$ is an n-degree polynomial with leading coefficient a_n, then

$$\Delta^n f(x) = n! \cdot a_n \tag{5}$$

Proof. Induct in terms of the leading coefficient.

The following proposition may be very useful and it is very straightforward. We can use it to deal with the degree of a certain polynomial.

Proposition 6. If $f(x)$ is an n-degree polynomial , then

$$\Delta^{n+1} f(x) = 0 \tag{6}$$

Proof. From proposition 5 we have $\Delta^{n+1}f(x) = n!\cdot a_n - n!\cdot a_n = 0$. $\square$

From proposition 3, we can also generalise n to x, writing $f(x)$ as a sum of what we call finite difference polynomials (which is deep down just a binomial choose function!)

Proposition 7. For any n-degree polynomial,

$$f(x) = \sum_{k=0}^{n} \Delta^k f(0) \binom{x}{k} \text{ or simply } \sum_{k=0}^{n} c_k \binom{x}{k}$$

Proof. $f(x) - \sum_{k=0}^{n} \Delta^k f(0) \binom{x}{k}$ has $n+1$ roots, hence 0. $\square$

In fact, the expansion in proposition 7 is unique, and a polynomial is integer valued if and only if $\forall c_i \in \mathbb{Z}$.

Example 1. Let $m \geq n$ be positive integers. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{m-k}{n} \binom{n}{k} = 1.$$

Solution. Note that $\Delta^n f(x)$ is constant if we let

$$\begin{aligned} f(x) &= \binom{m-x}{n} \\ \therefore \text{LHS} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \\ &= (-1)^n \Delta^n f(0) \\ &= (-1)^n \Delta^n f(m-n) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(m-n+k) \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \quad \square \end{aligned}$$

Example 2. Let $P(x)$ be a polynomial of degree n such that

$$P(k) = \binom{n+1}{k}^{-1}$$

for $k = 0, 1, \ldots, n$. Find the value of $P(n+1)$.

Solution. We start with

$$\begin{aligned}\Delta^{n+1}P(x) &= 0\\ \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}P(n+1-i) &= 0\\ P(n+1)+\sum_{i=1}^{n+1}(-1)^i &= 0\\ \therefore P(n+1) &= \begin{cases}0 & \text{if } n \text{ is even;}\\ 1 & \text{if } n \text{ is odd.}\end{cases}\end{aligned}$$

4 Falling Factorials

Let's introduce and define falling factorials.

$$k^{(n)} = k(k-1)\dots(k-n+1).$$

Yes, it is precisely another definition of P_n^k. Take the finite difference

$$\begin{aligned}\Delta k^{(n)} &= (k+1)^{(n)} - k^{(n)}\\ &= (k+1)k^{(n-1)} - k^{(n-1)}(k-n+1)\\ &= n\cdot k^{(n-1)}\end{aligned}$$

and we see it resembles the definition of the derivative! Hence the backward process is analogous, it is called finite integration.

Example 3. Find the value of

$$1^4 + \dots + n^4.$$

Solution.

$$\begin{aligned}\sum_{k=0}^{n} k^4 &= \sum_{k=0}^{n}(k^{(4)} + 6k^{(3)} + 7k^{(2)} + k^{(1)})\\ &= \sum_{k=0}^{n}\Delta\left(\frac{1}{5}k^{(5)} + \frac{3}{2}k^{(4)} + \frac{7}{3}k^{(3)} + \frac{1}{2}k^{(2)}\right)\\ &= \left(\frac{1}{5}(n+1)^{(5)} + \frac{3}{2}(n+1)^{(4)} + \frac{7}{3}(n+1)^{(3)} + \frac{1}{2}(n+1)^{(2)}\right) - 0\\ &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)\end{aligned}$$

Example 4. Find the value of

$$2^3 + 5^3 + 8^3 + \cdots + (3n+2)^3$$

Solution. We have $f(n) = (3n+2)^3$. Then

$$\begin{aligned}
\sum_{k=0}^{n} (3k+2)^3 &= \sum_{k=1}^{n} (27k^{(3)} + 135k^{(2)} + 117k^{(1)} + 8) \\
&= \sum_{k=1}^{n} \Delta \left(\frac{27}{4} k^{(4)} + 45k^{(3)} + \frac{117}{2} k^{(2)} + 8k^{(1)} \right) \\
&= \frac{27}{4}(n+1)^{(4)} + 45(n+1)^{(3)} + \frac{117}{2}(n+1)^{(2)} + 8(n+1)^{(1)} \\
&= \frac{1}{4}(n+1)(3n+4)(9n^2 + 21n + 8)
\end{aligned}$$

Example 5. Find the value of

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+2}$$

Solution. We basically want $(I-E)^n x^{n+2} = (-1)^n \Delta^n x^{n+2}$ evaluated at $x = 0$.

$$\Delta^n (x^{n+2}) = \Delta^n \left(x^{(n+2)} + \left\{ {n+2 \atop n+1} \right\} x^{(n+1)} + \left\{ {n+2 \atop n} \right\} x^{(n)} + \cdots \right)$$

so the answer is

$$(-1)^n n! \left\{ {n+2 \atop n} \right\} = (-1)^n n! \cdot \frac{(n+2)(n+1)n(3n+1)}{24} = (-1)^n \cdot \frac{(n+2)!(3n+1)n}{24}.$$