BIMO 2

Combinatorics Handout

12 March 2022

In combinatorics, we sometimes would like to study a network of things. For example, if we plot out every single Facebook user as points, and connect two users if they are friends, we will obtain a diagram that is made up of points and lines. These kind of diagrams will be called graphs.

1 Basic Definitions

- A *graph* is a pair $G = (V_G, E_G)$, where V_G (or V, if the context is clear) is a finite nonempty set and E_G (or E, if the context is clear) is a multiset of two-element submultisets of V. The elements in V and E are called *vertices* and *edges* respectively.

- If $e = \{u, v\} \in E$, then u and v are said to be the *endpoints* of e; and e is said to be *incident* to u and v; and u and v are said to be adjacent.

- If $u = v$, then e is a *loop* (Figure 1). If $e_1 = e_2 = \{u, v\}$ both appear in E, then e_1 and e_2 are *parallel edges* (Figure 2).

- A graph with no loops nor parallel edges is called a *simple graph* (Figure 3). Therefore, a simple graph is a pair $G = (V, E)$ where V is a finite nonempty set and E is a set of two-element subsets of V. Most graphs will be assumed simple unless stated otherwise.

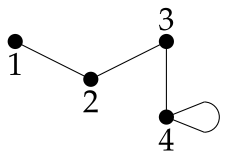

Figure 1: Loops

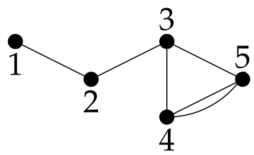

Figure 2: Parallel edges

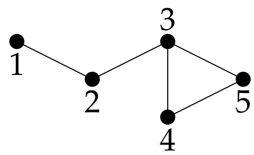

Figure 3: Simple graph

- The degree $\deg_G(v)$ of a vertex v is the number of edges incident to it.

- A *walk* is a sequence of vertices $(v_1, v_2, \cdots, v_n)$ where v_i and v_{i+1} are adjacent for all $i = 1, \cdots, n-1$. A *trail* is a walk with no repeated edges. A *path* is a trail with no repeated vertices.

- (Closed trails) A *circuit* $(v_1, \cdots, v_n)$ is trail where $v_1 = v_n$. A *cycle* is a circuit with no repeated vertices, apart from the first and last one.

- The complete graph K_n on n vertices is where every pair of vertices are adjacent.

- A subgraph $H = (V_H, E_H)$ of $G = (V_G, E_G)$ is a graph where $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

- A subgraph H of G is *spanning* if $V_H = V_G$.
- Given a subset $S \subseteq V_G$, the subgraph *induced* by S is

$$(S, \{e \in E_G : \text{the endpoints of } e \text{ are both in } S\}).$$

- The complement $\overline{G}$ of a graph G is obtained by switching the state of every pair of distinct vertices (adjacent $\leftrightarrow$ not adjacent).

1. (Handshake Lemma) $\sum_{v \in V} \deg(v) = 2|E|$.
2. There is a walk from u to v if and only if there is a path from u to v.

2 Connected Graphs

- A graph is said to be *connected* if there is walk between any two vertices.
- Any graph can be decomposed into a disjoint union of connected subgraphs. These are called the *connected components* (or *components*) of the graph. The number of components of G is denoted by $k(G)$.
- An edge of G is a *cut-edge* (or *bridge / isthmus*) if its removal increases $k(G)$.
- For any two vertices u, v, the distance $d_G(u, v)$ of u, v is the length of the shortest path between u and v. We say $d(u, v) = \infty$ if u, v lie in different components.

1. An edge is cut-edge if and only if it does not belong to any cycle.
2. The following conditions are equivalent for a connected graph G:

   a) G contains no cycles.

   b) All edges of G are cut-edges.

   c) There is a unique path between any two distinct vertices in G.

   d) $|E| = |V| - 1$.

   A graph satisfying such conditions is called a *tree*. A graph all of whose components are trees is called a *forest*.
3. For any forest G, $|E| = |V| - k(G)$.
4. Any connected graph G contains a spanning tree – a spanning subgraph that is a tree.

3 Eulerian Trails and Eulerian Circuits

- A trail or circuit is *Eulerian* if it traverses every edge.
- A vertex is *odd* if its degree is odd.

1. G has an Eulerian circuit if and only if there are no odd vertices.
2. G has an Eulerian trail if and only if there are at most two odd vertices.

4 Colourings

- Given a set of colours, a *vertex-colouring* is a colouring of the vertices such that no two monochromatic vertices are adjacent. The minimal number of colours required to vertex-colour G is denoted by $\chi(G)$, the chromatic number of G.
- Given a set of colours, an *edge-colouring* is a colouring of the edges such that no two monochromatic edges are incident to a common vertex. The minimal number of colours required to edge-colour G is denoted by $\chi_e(G)$, the edge chromatic number of G.
- A set of vertices in G is *independent* its induced subgraph has no edges. The size of the maximum independent set of G is denoted by $\beta(G)$.
- A *k-partite graph* is a graph that can be vertex-coloured with k colours.
- A *complete k-partite graph* $K_{n_1,n_2,\cdots,n_k}$ is the graph $G = (V, E)$ where $V = V_1 \sqcup V_2 \sqcup V_3 \sqcup \cdots \sqcup V_k$, $|V_i| = n_i$, V_i are independent sets, and every vertex of V_i is adjacent to every vertex of V_j for all $i \neq j$.

1. $\chi(G)\beta(G) \geq n$.
2.

5 Subgraphs

- A subgraph H of G that is K_n is called an *n-clique* of G.
- The Türan graph $T(n, k)$ is the complete k-partite graph $K_{n_1,\cdots,n_k}$ on n vertices such that $\max(n_i) - \min(n_i) \leq 1$.

Let $n = |V(G)|$.

1. (Mantel) If G has no 3-cycles (triangles), then $|E| \leq \left\lfloor \dfrac{n^2}{4} \right\rfloor$.

2. (Türan) If G has no $(k+1)$-cliques, then $|E_G| \leq |E_{T(n,k)}|$. Equality can hold.

3. If G has no 4-cycles, then $|E| \leq \frac{n}{4}\left(1+\sqrt{4n+3}\right)$.

- A *simple digraph*, or *simple directed graph*, is a pair $G = (V, E)$ where E is instead a set of ordered pairs of V. It is basically a graph in which every edge is assigned a direction. Here we will just call them *digraphs*.

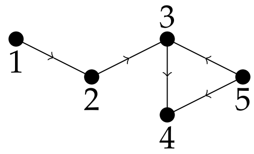

Figure 4: Digraph