# 2D Projective Geometry

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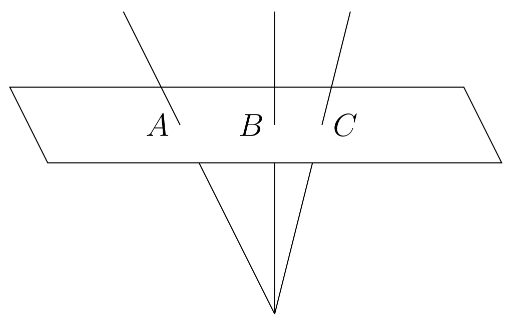
## 1 The Real Projective Plane

In three-dimensional vector space, for every nonzero vector $v$ define a *ray* as the set of vectors $kv$ ($k \in \mathbb{R} \setminus \{0\}$). The real projective plane $\mathbb{RP}^2$ is defined to be the set of all rays in three-dimensional space. Each element of $\mathbb{RP}^2$ is called a *point*, while in reality it is just a ray in three-dimensional space. Hence each point in $\mathbb{RP}^2$ has coordinates with three-entries not all zero.




Even though $\mathbb{RP}^2$ is a three-dimensional concept, it is enough to imagine it as a two-dimensional plane, including an extra line called the *line at infinity* $\ell_\infty$. Every point on $\ell_\infty$ is a *point at infinity* at which every set of parallel lines concur. In this case, every pair of lines must intersect at a point.

## 2 Projective Transformations

A transformation on the projective plane that sends each line to a line is called *projective*. In other words, a projective transformation (abbrv. PT) is a function $T : L \to L$ where $L$ is the set of lines in $\mathbb{RP}^2$. As such, PTs form a group: a composition of PTs is still a PT; each PT has its inverse. The well-known affine transformations (rotations, translations, shear transforms) are obviously also PTs. Under a PT, intersections and cross ratios are preserved. However, note that under a PT sending $\ell_1$ to $\ell_2$, the other points in the plane may not preserve their positions as well. If so, why are PTs so useful then? Let's look at some examples of actions they can do.

**Lemma 2.1.** *There is a PT that sends four points $A, B, C, D$ (no three collinear) to any other four points $A^*, B^*, C^*, D^*$ (no three collinear).*

*Proof.* Apply a PT sending the line connecting $AB \cap CD$ and $AD \cap BC$ to $\ell_\infty$. That means $T(A)B \parallel CD, AD \parallel BC$ are forced to hold, even though we do not know where these points are. Nonetheless, a parallelogram is formed. Take an affine transformation to transform it into a square. Therefore, a PT can transform any quadrilateral into a square $S$. If $ABCD \xrightarrow{T_1} S$ and $A^*B^*C^*D^* \xrightarrow{T_2} S$, then

$$ABCD \xrightarrow{T_1} S \xrightarrow{T_2^{-1}} A^*B^*C^*D^*$$

is indeed a projective transformation. $\square$

**Lemma 2.2.** *Under any PT, conics are sent to conics.*

*Proof.* By definition, a conic is a projection of a circle. □

**Lemma 2.3.** *If we have a circle $\Gamma$ and a line $\ell$ not intersecting $\Gamma$, then there is a PT that fixes $\Gamma$ in place and sends $\ell$ to any other line not intersecting $\Gamma$.*

*Proof.* Apply a PT sending the $\ell$ to $\ell_\infty$. Then $\Gamma$ is sent to another conic. It cannot be a parabola or a hyperbola since they intersect $\ell_infty$. Hence it is a circle or an ellipse. Take an affine transformation to ensure it is a circle. Hence $\Gamma \to \Gamma$ and $\ell \to \ell_\infty$. Then send $\ell_\infty$ back. □

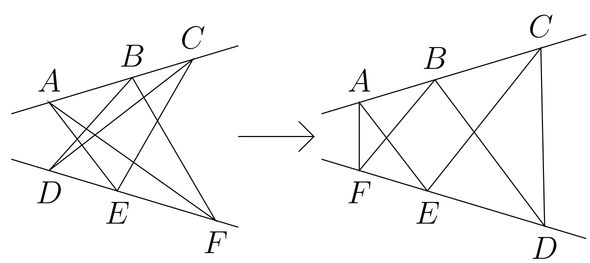
**Lemma 2.4.** *If we have a circle $\Gamma$ and a point $P$ inside $\Gamma$, then there is a PT that fixes $\Gamma$ in place and sends $P$ to any other point inside $\Gamma$.*

*Proof.* This is the dual of the previous lemma using poles and polars. □

We first try to prove some famous theorems:

**Theorem 2.5.** *(Pappus) Let $ABC$ and $DEF$ be two lines. Then the points $AE \cap BD, AF \cap CD, BF \cap CE$ are collinear.*

*Proof.* Take a PT sending the line connecting $AE \cap BD$ and $AF \cap CD$ to $\ell_\infty$. Then chase ratios. □




**Theorem 2.6.** *(Desargues) Let $ABC$ and $DEF$ be two triangles. Then $AD, BE, CF$ are concurrent if and only if $AB \cap DE, AC \cap DF, BC \cap EF$ are collinear.*

*Proof.* Take a PT sending the line connecting $AB \cap DE$ and $AC \cap DF$ to $\ell_\infty$. Chase ratios again. □

**Theorem 2.7.** *(Butterfly) Let $\Gamma$ be a circle and $P$ a point inside. Let $AC, BD, EF$ be two chords passing through $P$. If $P$ is the midpoint of $EF$, then $P$ is the midpoint of $AB \cap EF, CD \cap EF$.*

*Proof.* Translate the midpoint condition into cross ratios. Initially, $(E, F; P, P_{EF}^\infty) = -1$. Take a PT sending $P$ to the center of the circle. Since $EF$ is now the diameter, we know $P_{EF}^\infty$ got sent to another point at infinity. Then the midpoint result is obvious by symmetry. □