Stirling Numbers

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In this article we will look at Stirling Numbers.

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1 First Kind

Consider the group S_n of permutations of $[n] = \{1, \cdots, n\}$. Any permutation $\sigma \in S_n$ can be uniquely decomposed into disjoint cycles, for example the permutation $(1, 2, 3, 4, 5) \mapsto (2, 5, 4, 3, 1)$ can be written as $(125)(34)$. This permutation has 2 cycles.

Definition. The number of permutations in S_n with k cycles is denoted by $c(n, k)$.

For any positive n,

- $c(n, n) = 1$ because the only permutation with n cycles is $(1)(2)\cdots(n)$.
- $c(n, 1) = (n-1)!$ because there are $n!/n$ cyclic permutations of $[n]$.
- $c(n, k) = 0$ for $k \leq 0$ or $k > n$.

Proposition 1.1. For any integers n, k, $c(n, k) = c(n-1, k-1) + (n-1) \cdot c(n-1, k)$.

Proof. For any permutation of $[n]$ with k cycles, the cycle that contains n can contain one element (only n), or more than one element. In the former case, those permutations have a one-to-one correspondence with permutations of $[n-1]$ with $k-1$ cycles. In the latter case, those permutations have a one-to-$(n-1)$ correspondence with permutations σ' of $[n-1]$ with k cycles because n can be slotted anywhere in the cyclic decomposition of σ'. □

Proposition 1.2. $x(x+1)\cdots(x+n-1) = \sum_k c(n, k) x^k$.

Proof. Let $x(x+1)\cdots(x+n-1) = \sum_k c_0(n,k)x^k$. Obviously $c_0(1,k) = c(1,k)$. By induction, it remains to verify the recursive property:

$$\begin{aligned}\sum_k c_0(n,k)x^k &= x(x+1)\cdots(x+n-1) \\ &= (x+n-1)\sum_k c_0(n-1,k)x^k \\ &= \sum_k (c(n-1,k-1) + (n-1)\cdot c(n-1,k))x^k. \end{aligned}$$

□

Proposition 1.3. $x(x-1)\cdots(x-n+1) = \sum_k (-1)^{n+k} c(n,k)x^k$.

Proof. Using proposition 2,

$$\begin{aligned}x(x-1)\cdots(x-n+1) &= (-1)^n(-x)(-x+1)\cdots(-x+n-1) \\ &= (-1)^n \sum_k c(n,k)(-x)^k \\ &= \sum_k (-1)^{n+k} c(n,k)x^k.\end{aligned}$$

Definition. The numbers $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n+k}c(n,k)$ are called *Stirling Numbers of the First Kind.*

Define $x^{(n)} = x(x-1)\cdots(x-n+1)$. From proposition 3,

$$x^{(n)} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

2 Second Kind

A *set partition* of $[n]$ is a collection of a disjoint subsets of $[n]$ whose union is $[n]$. For example,

$$\{\{1,2,3\},\{4,6\},\{5\}\}$$

is a set partition of $[6] = \{1,2,3,4,5,6\}$, with 3 parts.

Definition. The number of set partitions of $[n]$ with k parts is denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. These numbers are called *Stirling Numbers of the Second Kind.*

For any positive n,

- $\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1$ because the only set partition with 1 part is $\{\{1,\cdots,n\}\}$.
- $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1$ because the only set partition with 1 part is $\{\{1\},\cdots,\{n\}\}$.
- $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$ for $k \le 0$ or $k > n$.

Proposition 2.1. For any integers n, k, $\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}$.

Proof. For any set partition of $[n]$ with k parts, the part that contains n can contain one element (i.e. $\{n\}$), or more than one element. In the former case, those partitions have a one-to-one correspondence with set partitions of $[n-1]$ with $k-1$ parts. In the latter case, those partitions have a one-to-k correspondence with set partitions σ' of $[n-1]$ with k parts because n can be in any of the k parts of σ'. □

Proposition 2.2. $x^n = \sum_k \left\{ n \atop k \right\} x^{(k)}$.

Proof. For $n = 1$ it is true. Now,

$$\begin{aligned}
\sum_k \left\{ n \atop k \right\} x^{(k)} = x^n = x \sum_k \left\{ n-1 \atop k \right\} x^{(k)} \\
= \sum_k \left\{ n-1 \atop k \right\} (x - k + k) x^{(k)} \\
= \sum_k \left(k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\} \right) x^{(k)}
\end{aligned}$$

so by induction it is true for all n. □

Proposition 2.3. If S and s are infinite matrices where $S_{ij} = \left[i \atop j \right]$ and $s_{ij} = \left\{ i \atop j \right\}$, then $Ss = I$.

Proof. Since $\{x^n\}$ and $\{x^{(n)}\}$ are bases for the vector space $\mathbb{R}[x]$, proposition 1.3 and 2.2 shows that S and s are just matrices of basis change between $\{x^n\}$ and $\{x^{(n)}\}$, so they are inverses of each other. □

3 Examples

Example 1. Find $\left[n+1 \atop n \right]$.

Solution. $\left[n+1 \atop n \right] = \left[n \atop n-1 \right] + n \left[n \atop n \right] = \left[n \atop n-1 \right] + n$, so

$$\left[n+1 \atop n \right] = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

Example 2. Find $\left[n+2 \atop n \right]$.

Solution. $\left[n+2 \atop n \right] = \left[n+1 \atop n-1 \right] + (n+1) \left[n+1 \atop n \right] = \left[n+1 \atop n-1 \right] + \frac{n(n+1)^2}{2}$, so

$$\left[n+2 \atop n \right] = \sum_{k=1}^{n} \frac{k(k+1)^2}{2} = \frac{n(n+1)(n+2)(3n+5)}{24}.$$

Example 3. Find $\left\{ n+1 \atop n \right\}$.

Solution. $\left\{ {n+1 \atop n} \right\} = \left\{ {n \atop n-1} \right\} + n \left\{ {n \atop n} \right\} = \left\{ {n \atop n-1} \right\} + n$, so

$$\left\{ {n+1 \atop n} \right\} = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

Example 3. Find $\left\{ {n+2 \atop n} \right\}$.

Solution. $\left\{ {n+2 \atop n} \right\} = \left\{ {n+1 \atop n-1} \right\} + n \left\{ {n+1 \atop n} \right\} = \left\{ {n \atop n-1} \right\} + \frac{n^2(n+1)}{2}$, so

$$\left\{ {n+2 \atop n} \right\} = \sum_{k=1}^{n} \frac{k^2(k+1)}{2} = \frac{n(n+1)(n+2)(3n+1)}{24}.$$

Example 4. Find $\left\{ {n+3 \atop n} \right\}$.

Solution. $\left\{ {n+3 \atop n} \right\} = \left\{ {n+2 \atop n-1} \right\} + n \left\{ {n+2 \atop n} \right\} = \left\{ {n \atop n-1} \right\} + \frac{n^2(n+1)(n+2)(3n+1)}{24}$, so

$$\left\{ {n+1 \atop n} \right\} = \sum_{k=1}^{n} \frac{k^2(k+1)(k+2)(3k+1)}{24} = \frac{n^2(n+1)^2(n+2)(n+3)}{48}.$$

4 Finite Difference and Finite Integration

This section requires knowledge from my previous article, *Finite Differences*.

Example 1. Find the value of $\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+1}$.

Solution. We basically want $(I-E)^n x^{n+1} = (-1)^n \Delta^n x^{n+1}$ evaluated at $x = 0$.

$$\begin{aligned} \Delta^n(x^{n+1}) &= \Delta^n \left(x^{(n+1)} + \left\{ {n+1 \atop n} \right\} x^{(n)} + \cdots \right) \\ &= (n+1)^{(n)} x^{(1)} + n! \left\{ {n+1 \atop n} \right\}, \end{aligned}$$

so the answer is

$$(-1)^n n! \left\{ {n+1 \atop n} \right\} = (-1)^n n! \cdot \frac{n(n+1)}{2} = (-1)^n \cdot \frac{(n+1)!n}{2}.$$ □

Example 2. Find the value of $\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+2}$.

Solution. We basically want $(I-E)^n x^{n+2} = (-1)^n \Delta^n x^{n+2}$ evaluated at $x = 0$.

$$\begin{aligned} \Delta^n(x^{n+2}) &= \Delta^n \left(x^{(n+2)} + \left\{ {n+2 \atop n+1} \right\} x^{(n+1)} + \left\{ {n+2 \atop n} \right\} x^{(n)} + \cdots \right) \\ &= (n+2)^{(n)} x^{(2)} + (n+1)^{(n)} \left\{ {n+2 \atop n+1} \right\} x^{(1)} + n! \left\{ {n+2 \atop n} \right\}, \end{aligned}$$

so the answer is

$$(-1)^n n! \left\{ {n+2 \atop n} \right\} = (-1)^n n! \cdot \frac{n(n+1)(n+2)(3n+1)}{24} = (-1)^n \cdot \frac{(n+2)!(3n+1)n}{24}. \qquad \square$$

Example 3. Find the value of $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^{n+3}$.

Solution. We basically want $(I-E)^n x^{n+3} = (-1)^n \Delta^n x^{n+3}$ evaluated at $x=0$.

$$\begin{aligned}\Delta^n(x^{n+3}) &= \Delta^n \left(x^{(n+3)} + \left\{ {n+3 \atop n+2} \right\} x^{(n+2)} + \left\{ {n+3 \atop n+1} \right\} x^{(n+1)} + \left\{ {n+3 \atop n} \right\} x^{(n)} + \cdots \right) \\ &= (n+3)^{(n)} x^{(3)} + (n+2)^{(n)} \left\{ {n+3 \atop n+2} \right\} x^{(2)} + (n+1)^{(n)} \left\{ {n+3 \atop n+1} \right\} x^{(1)} + n! \left\{ {n+3 \atop n} \right\},\end{aligned}$$

so the answer is

$$(-1)^n n! \left\{ {n+3 \atop n} \right\} = (-1)^n n! \cdot \frac{n^2(n+1)^2(n+2)(n+3)}{48} = (-1)^n \cdot \frac{(n+2)!(n+1)(n+3)n^2}{24}. \qquad \square$$

References

[1] Stirling Numbers of the First Kind
`https://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind`

[2] Stirling Numbers of the Second Kind
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