Problem Solving Session

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Logic

1. Negation

The **negation** $\neg S$ of a statement S is a statement whose truth value is opposite of S.

Examples

S : 'X is a boy', $\neg S$:'X is not a boy'
Q : 'All apples are red',
$\neg Q$: 'Not all apples are red' or 'At least one apple is not red'

Note that 'all apples are not red' is not the negation of Q.

Exercise

Negate the statement

$$\text{For all } x > 0,\ f(x) \geq 1.$$

2. AND and OR

The **conjunction** of statements A, B is a statement which gives a truth value of 'true' if both A and B are true, and gives 'false' otherwise. We normally say the conjunction of A, B as A AND B.

The **disjunction** of statements A, B is a statement which gives a truth value of 'true' if any of A or B is true, and gives 'false' otherwise. We normally say the disjunction of A, B as A OR B.

3. Implication

Say A and B are two statements. When we have

'If statement A is true, then statement B is true',

we say that 'A **implies** B', or 'A is **sufficient** for B', or 'B is **necessary** for A', denoted by $A \Rightarrow B$. However, this does NOT mean $B \Rightarrow A$. For example, an apple is a fruit, however a fruit might not be an apple. Note that an implication of two statements is also a statement.

Examples

$$x = y \ \Rightarrow \ x^2 = y^2$$

$$a > b \ \Rightarrow \ a + 1 > b + 1$$

4. Equivalence

Say A and B are two statements. When we have

$$\text{`}A \Rightarrow B\text{' AND `}B \Rightarrow A\text{'}$$

we say that 'A is **equivalent** to B', or 'A **if and only if (iff)** B', or 'A is **sufficient and necessary** for B', denoted as $A \Leftrightarrow B$.

Examples

X is an equilateral triangle $\Leftrightarrow$ X is a triangle with angles 60° only

$$a = b \;\Leftrightarrow\; a + c = b + c$$

Logic

Contrapositive Property

$(A \Rightarrow B) \;\Leftrightarrow\; (\neg B \Rightarrow \neg A)$

Proofs

A typical problem looks like "Given A, prove B."

1. Direct Proofs ($A \Rightarrow B$)

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1. Direct Proofs ($A \Rightarrow B$)

OMK2017B

Let ABC be a triangle. The altitudes from A, B, C are denoted h_A, h_B, h_C respectively. Prove that

$$\frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C}$$

A typical problem looks like "Given A, prove B."

1. Direct Proofs ($A \Rightarrow B$)

OMK2017B

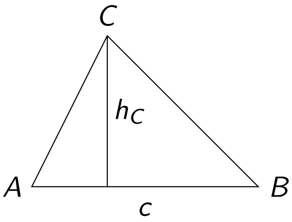
Let ABC be a triangle. The altitudes from A, B, C are denoted h_A, h_B, h_C respectively. Prove that

$$\frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C}$$

First thought: This looks like the triangle inequality $a + b > c$ (?)

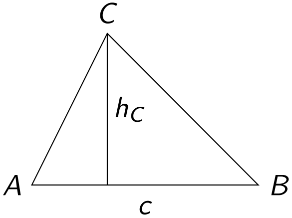
Proofs

Let's try to relate a with h_A, b with h_B, and c with h_C. What's something that relates these lengths?



Proofs

Let's try to relate a with h_A, b with h_B, and c with h_C. What's something that relates these lengths?



Area! $S = [ABC] = \frac{a \cdot h_A}{2} = \frac{b \cdot h_B}{2} = \frac{c \cdot h_C}{2}$

Proofs

Solution 1

Denote $a = BC, b = AC, c = AB$. By the triangle inequality,

$$a + b > c$$

Since the area of ABC is $S = [ABC] = \frac{a \cdot h_A}{2} = \frac{b \cdot h_B}{2} = \frac{c \cdot h_C}{2}$,

$$\frac{2S}{h_A} + \frac{2S}{h_B} > \frac{2S}{h_C}$$
$$\Rightarrow \frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C}.$$

■

Proofs

Solution 2

Denote $a = BC, b = AC, c = AB$.

$$\begin{aligned} &\frac{1}{h_A} + \frac{1}{h_B} > \frac{1}{h_C} \\ \Leftrightarrow &\frac{2[ABC]}{h_A} + \frac{2[ABC]}{h_B} > \frac{2[ABC]}{h_C} \\ \Leftrightarrow &\, a + b > c \end{aligned}$$

The last line is the triangle inequality, so the first line is true. ■

Proofs

2. Proof by Contradiction (A and $\neg B \Rightarrow$ contradiction)

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Santos pg 9

Let $a_1, a_2, \cdots, a_n$ be an arbitrary permutation of the numbers $1, 2, \cdots, n$, where n is an odd number. Prove that the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is even.

Proofs

2. Proof by Contradiction (A and $\neg B \Rightarrow$ contradiction)

Santos pg 9

Let $a_1, a_2, \cdots, a_n$ be an arbitrary permutation of the numbers $1, 2, \cdots, n$, where n is an odd number. Prove that the product

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is even.

First thought: Hard to prove how at least one of $a_1 - 1, a_2 - 2, \cdots, a_n - n$ is even.

Proofs

2. Proof by Contradiction (A and $\neg B \Rightarrow$ contradiction)

Solution

Assume the contrary that $(a_1 - 1)(a_2 - 2)\cdots(a_n - n)$ is odd, then all

$$a_1 - 1, a_2 - 2, \cdots, a_n - n$$

are odd, but then

$$(a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = (a_1 + \cdots + a_n) - (1 + \cdots + n) = 0$$

which is a contradiction since an odd number of odd numbers must sum up to an odd number! ■

3. Proof by Construction

OMK2017B

An integer is called an autobiographical number if the first digit is equal to the number of digits 0, the second digit is equal to the number of digits 1, the third digit is equal to the number of digits 2, the fourth digit is equal to the number of digits 3, and so on until the last digit. Two examples of autobiographical numbers are 42101000 and 6210001000. (a) Find two autobiographical numbers with 4 digits. (b) Find one autobiographical number with 5 digits.

3. Proof by Construction

You might do a lot of trial and error on a piece of blank paper, and find out two answers 1210 and 2020 for part (a), and an answer 21200 for part (b). You might think you have to write down how you got those answers. But no! The answers you got are easily verifiable with the problem statement, so they are automatically *correct*.

Solution

(a) 1210, 2020. (b) 21200.

If the problem were 'find *all* autobiographical numbers with 5 digits', then we have to show why 21200 is the only one, and that would be quite tedious to write out.

Proofs

3. Proof by Construction

Unknown

What is the maximum number of trailing 4s a perfect square can have?
E.g. $6\underline{4}$ has one trailing 4.

3. Proof by Construction

Unknown

What is the maximum number of trailing 4s a perfect square can have?
E.g. 6<u>4</u> has one trailing 4.

It's easy to find perfect squares with one or two trailing 4s, such as $12^2 = 144$. Now finding three trailing 4s is hard.

3. Proof by Construction

Turns out there are a few: $38^2 = 1444, 462^2 = 213444, \cdots$. The mere existence of one of them is enough to say that the maximum is at least 3. But how you found them does not matter. In fact, three trailing zeros is the best we can do, and we have to prove this by explaining why four trailing 4s can't work.

Proofs

3. Proof by Construction

Solution

Answer: 3.

$38^2 = 1444$, so it suffices to prove that four trailing 4s cannot. Assume such a number x^2 exists, then

$$10000 \mid x^2 - 4444 \Rightarrow x \text{ is even, } x = 2k$$
$$\Rightarrow 2500 \mid k^2 - 1111 \Rightarrow 4 \mid k^2 - 1111 \Rightarrow 4 \mid k^2 - 3$$

However,

$$(2N)^2 = 4N^2, \qquad (2N+1)^2 = 4N^2 + 4N + 1$$

shows us that there cannot be a perfect square that is 3 more than a multiple of 4. ∎

4. Proof by Induction ((True for $n = 1$) and (True for $n = k - 1 \Rightarrow$ True for $n = k$) $\Rightarrow$ True for all $n \in \mathbb{N}$)

OMK2017S

Given a positive integer n. Consider all subsets of $\{1, 2, 3, \cdots, n\}$ except the empty set. For each subset, consider a fraction $1/d$, where d is the product of all elements in the subset. Let S_n be the sum of such fractions taken over all subsets. Example: For $n = 3$, the nonempty subsets of $\{1, 2, 3\}$ are $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$. Therefore,

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} = 3.$$

Prove that $S_n = n$.

Proofs

4. Proof by Induction

Solution 1

Denote $[n] = \{1, \cdots, n\}$.
For $n = 1$, $S_1 = \frac{1}{1} = 1$. Assume $S_{k-1} = k - 1$ for $k \geq 2$, then

$$\begin{aligned} S_k &= \sum_{\varnothing \neq A \subseteq [k]} \frac{1}{\prod A} \\ &= \sum_{\varnothing \neq A \subseteq [k-1]} \frac{1}{\prod A} + \sum_{\varnothing \neq A \subseteq [k-1]} \frac{1}{k \prod A} + \frac{1}{k} \\ &= (k-1) + \frac{1}{k}(k-1) + \frac{1}{k} = k \end{aligned}$$

and hence $S_n = n$ for all positive integers n. ■

4. Proof by Induction

Solution 2

Notice that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right)$$

expands to become $1 + S_n$, so

$$S_n = \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n} - 1 = n. \qquad \blacksquare$$

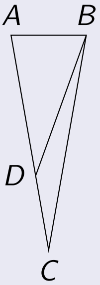
Proofs

Some common techniques

1. Guess an answer then prove why it is correct instead

APMOPS

ABC is a triangle with $AC = BC$ and $\angle BAC = 80^\circ$. Given that $AB = CD$, find $\angle BDC$.



Solution 1

Answer: 150°. (← Good practice to write the answer first)

Proof: Let $\angle ADB = \theta$, then sine rule gives

$$\frac{\sin(\theta - 20^\circ)}{\sin 20^\circ} = \frac{CD}{BD} = \frac{AB}{BD} = \frac{\sin\theta}{\sin 80^\circ}$$
$$\cos 20^\circ - \cot\theta \sin 20^\circ = \frac{\sin 20^\circ}{\sin 80^\circ}$$

Hence there is only one possible $0 < \theta < 180^\circ$. Note that $\theta = 30^\circ$ works:

$$\frac{\sin(30^\circ - 20^\circ)}{\sin 20^\circ} = \frac{\sin 30^\circ}{\sin 80^\circ} \iff 2\sin 10^\circ \sin 80^\circ = \sin 20^\circ$$

which is true since $\sin 80^\circ = \cos 10^\circ$. $\therefore \angle ADB = 30^\circ$. ■

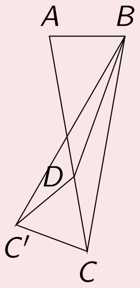
Solution 2

Answer: 150°.

Proof: Erect an eq. triangle CDC' as shown.

$$\begin{cases} CC' = BA \\ CB = CB \\ \angle C'CB = \angle ABC = 80^\circ \end{cases} \Rightarrow \Delta BCC' \cong \Delta CBA$$

and therefore BCC' is isosceles too, with BD being the symmetry line, and hence $\angle BDC = 180^\circ - \frac{60^\circ}{2} = 150^\circ$.



Some common techniques

2. Write in claims

OMK2021M

Let $f(x)$ be a function defined on the set of real numbers satisfying $f(1) = 2$ and for any real number x,

$$f(x+7) \geq f(x) + 7 \text{ and } f(x+1) \leq f(x) + 1.$$

If $g(x) = f(x) + 7 - x$, find the value of $g(2021)$.

Proofs

Some common techniques

2. Write in claims

Solution

Claim: $f(x+1) = f(x) + 1$ for all real x.

Proof:
$f(x+7) \leq f(x+6) + 1 \leq f(x+5) + 2 \leq \cdots \leq f(x) + 7 \leq f(x+7)$.
Therefore all the terms above are equal, in particular,
$f(x+1) + 6 = f(x) + 7$, i.e. $f(x+1) = f(x) + 1$. □

This means $f(1) = 2 \Rightarrow f(2) = 3 \Rightarrow \cdots \Rightarrow f(2021) = 2022$, and thus

$$g(2021) = f(2021) + 7 - 2021 = 8.$$

■

Some common techniques

3. Use generalised forms

OMK2017M

Find

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2016^2}+\frac{1}{2017^2}}.$$

Proofs

Solution

$$\sum_{k=1}^{2016}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}} = \sum_{k=1}^{2016}\sqrt{\frac{k^4+2k^3+3k^2+2k+1}{k^2(k+1)^2}}$$

$$= \sum_{k=1}^{2016}\sqrt{\frac{(k^2+k+1)^2}{k^2(k+1)^2}}$$

$$= \sum_{k=1}^{2016}\frac{k^2+k+1}{k(k+1)}$$

$$= \sum_{k=1}^{2016}\left(1+\frac{1}{k(k+1)}\right)$$

$$= 2016\frac{2016}{2017}$$

Proofs

Some common techniques

3. Use generalised forms

OMK2019S

We call a sequence of five numbers a *good* sequence if it is an arithmetic progression that contains the terms 20 and 19. For example, these two sequences are good sequences:

$$18, 19, 20, 21, 22;$$

$$20, 19\frac{2}{3}, 19\frac{1}{3}, 19, 18\frac{2}{3}.$$

For each good sequence, we take the sum of all terms in the sequence. Then we add the sums over all possible good sequences. What will be the result?

Solution (pg 1)

Answer: 1950.

For an AP $a_1, \cdots, a_5$, the sum is just 5 times a_3. Thus we just have to sum over all a_3's. Assume $a_i = 19, a_j = 20$ where $\{i, j\} \subseteq \{1, 2, 3, 4, 5\}$. Then the common difference is $d = \frac{20-19}{j-i}$ and $a_1 = a_i - (i-1)d$, so

$$a_3 = a_1 + 2d = a_i + (i-3)d = 19 + \frac{i-3}{j-i}$$

and so it remains to find

$$5 \sum_{\{i,j\} \subseteq \{1,2,3,4,5\}} \left(19 + \frac{i-3}{j-i} \right)$$

Proofs

Solution (pg 2)

$$= 5\left(\sum_{i,j} 19 + \sum_{i,j} \frac{i-3}{j-i}\right)$$

The first term in the brackets is just $19 \times 5 \times 4 = 380$. The second sum can be computed by pairing swaps:

$$2\sum_{i,j} \frac{i-3}{j-i} = \sum_{i,j}\left(\frac{i-3}{j-i} + \frac{j-3}{i-j}\right) = \sum_{i,j} 1 = 5 \times 4$$

so the answer is

$$5(380 + 10) = 1950. \qquad \blacksquare$$

Exercise

OMK2017M

Given an odd integer $N \geq 5$. Show that $N^2 + 5$ can be written as the sum of four different positive perfect squares.

OMK2018S

For any positive integer k, denote by g_k the largest odd factor of k. For example, $g_8 = 1, g_9 = 9$ and $g_{10} = 5$.

(a) Prove that $g_{n+1} + g_{n+2} + \cdots + g_{2n} = n^2$ for all positive integers n.

(b) Find the value of $g_1 + g_2 + g_3 + \cdots + g_{512}$.

Exercise

OMK2019S (modified)

Given three distinct positive reals a, b, c, prove that

$$\left(\frac{1}{x+a}-\frac{1}{x}\right)+\left(\frac{1}{x+b}-\frac{1}{x}\right)+\left(\frac{1}{x+c}-\frac{1}{x}\right)=0$$

has a real root.

OMK2018S

Let $\{a_1, a_2, a_3, \cdots\}$ be the set that consists of all integers that can be expressed as a sum of four distinct positive fourth powers. Assume that $a_1 < a_2 < a_3 < \cdots$. If $a_i = 2018$, find the value of i.

Note: A positive fourth power is a number in the form k^4, where k is a positive integer.

Topics

1. Algebra

Topics

1. Algebra
2. Combinatorics

Topics

1. Algebra
2. Combinatorics
3. Geometry

Topics

1. Algebra
2. Combinatorics
3. Geometry
4. Number Theory

1. Inequalities
2. Functional Equations
3. Recursion
4. Polynomials

Algebra (Inequalities)

All inequalities are based on

$$x^2 \geq 0, \quad x \in \mathbb{R}$$

where equality holds if and only if $x = 0$.

Examples

$\frac{a+b}{2} \geq \sqrt{ab}$ for all $a, b > 0$.
Proof: Equivalent to $(\sqrt{a} - \sqrt{b})^2 \geq 0$. ■

Examples

$x^2 + y^2 + z^2 \geq xy + yz + xz$ for all $x, y, z \in \mathbb{R}$.
Proof: Equivalent to $(x - y)^2 + (y - z)^2 + (x - z)^2 \geq 0$. ■

Examples

$\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ for all $x, y, z > 0$.
Proof: Equivalent to
$(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}) \left[(\sqrt[3]{x} - \sqrt[3]{y})^2 + (\sqrt[3]{y} - \sqrt[3]{z})^2 + (\sqrt[3]{x} - \sqrt[3]{z})^2\right] \geq 0.$ ■

That's a bit of a stretch! We can do better by knowing some well-known inequalities. You can quote them in contests.

AM-GM Inequality

For any $a_1, a_2, \cdots, a_n > 0$,

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Cauchy-Schwarz's Inequality

For any $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in \mathbb{R}$,

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) \geq \left(\sum a_i b_i\right)^2$$

Equality holds if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$ (where $\frac{0}{0}$ is any number).

Examples

$x^2 + y^2 + z^2 \geq xy + yz + xz$ for all $x, y, z \in \mathbb{R}$.

Proof: By Cauchy,

$$(x^2 + y^2 + z^2)(y^2 + z^2 + x^2) \geq (xy + yz + zx)^2$$

$$\therefore x^2 + y^2 + z^2 \geq |xy + yz + zx| \geq xy + yz + xz. \qquad \blacksquare$$

OMK2021S

The polynomial $x^4 + ax^3 + 2x^2 + bx + 1$ has a real solution. Find the minimum value of $a^2 + b^2$.

OMK2021S

Prove that for all real numbers x, y, z, the following inequality holds:

$$x^2 + y^2 + z^2 - xy - yz - xz \geq \frac{3}{4}(x - y)^2$$

Combinatorics

1. Bijections
2. Pigeonhole Principle
3. Combinatorial Sums
4. Graph Theory
5. and more

Combinatorics (Pigeonhole Principle)

100 pigeons distributed in 99 boxes. What can you say about the number of pigeons in each box?

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Pigeonhole Principle

- If $n + 1$ objects is distributed in n boxes, there must be one box with at least 2 objects.
- (Generalised) If N objects is distributed in n boxes, there must be one box with at least $\left\lceil \frac{N}{m} \right\rceil$ objects.
- If an infinite number of objects is distributed in n boxes, there must be one box with an infinite number of objects.

Combinatorics (Pigeonhole Principle)

Examples

At most how many elements can you take from the set $\{1, 2, \cdots, 1000\}$ so that no two elements add up to 1000?

Answer: 501.

Proof. Consider the 501 sets

$$\{1, 999\}, \{2, 998\}, \cdots, \{499, 501\}, \{500\}, \{1000\}$$

If 502 elements were taken, by PP there must be 2 in a same set, which means they add up to 1000, a contradiction.

501 is possible: Take $1, 2, \cdots, 500$ and 1000. ∎

Combinatorics (Pigeonhole Principle)

?

What is the maximum number of elements you could pick from $\{1, 2, 3, \cdots, 2024\}$ so that no two distinct elements a, b are picked with a dividing b?

?

Let a be a fixed positive integer. Prove that for any positive integer n, there exists a power of a that ends with $0 \cdots 01$ (n zeros). Note: a power of a is a^N for some positive integer N.

OMK2019M

A group of students collected 200 seashells at a beach. What would be the maximum possible number of students in the group, if every student collected at least one seashell, and all students collected different numbers of seashells?

OMK2019S

Consider the set $A = \{1, 2, 3, 4, 5, \cdots, 100\}$. For a positive integer k, let $f(k)$ represents the maximum size of a subset of A such that no two elements in that subset differ by k. Determine the number of possible values of k that fulfill the condition $f(k) = 50$.

Geometry

1. Angle and length chasing
2. Cyclic quadrilaterals
3. Various centres of a triangle
4. Trigonometry
5. Menelaus's Theorem
6. 'Bashing'
7. and more

OMK2019M

Given a square $ABCD$. A point P is chosen such that $\angle PAB = 15^\circ, \angle PBD = 90^\circ$ and $\angle PBC < 90^\circ$. Prove that ACP is an isosceles triangle.

OMK2019M

Let PQR be a triangle in which $PQ = PR$ and I be its incenter. Given that $QR = PQ + PI$. Let S be a point on the line QP extended beyond P such that $PS = PI$. Prove that $SPIR$ is a cyclic quadrilateral. (The incenter of triangle PQR is the point of intersection of the three internal angle bisectors)

OMK2019S

Let ABC be an acute triangle. Let D be the reflection of point B with respect to the line AC. Let E be the reflection of point C with respect to the line AB. Let Γ_1 be the circle that passes through A, B, and D. Let Γ_2 be the circle that passes through A, C, and E. Let P be the intersection of Γ_1 and Γ_2, other than A. Let Γ be the circle that passes through A, B, and C. Show that the center of Γ lies on line AP.

OMK2019M

Given PA and PB are two tangent lines of a circle from a point P outside the circle, and A, B are the contact points. PD is a secant line, and it intersects the circle at C and D. BF is parallel to PA and meets the lines AC and AD at E and F respectively. Given the length of BF is 8, find the length of BE.

Number Theory

1. Prime numbers
2. Modular arithmetic
3. Diophantine equations (tricks etc)
4. and more

OMK2019M

Let a, b, and c be integers such that $7a + 4b - 3c = 0$. Prove that $(a+b)(b+c)(c+a)$ is divisible by 42.

OMK2018M

Let a and b be positive integers such that

1. both a and b have at least two digits;
2. $a + b$ is divisible by 10;
3. a can be changed into b by changing its last digit.

Prove that the hundreds digit of the product ab is even.

OMK2018M

Let n be an integer greater than 1, such that $3n + 1$ is a perfect square. Prove that $n + 1$ can be expressed as a sum of three perfect squares.

OMK2021S

Find all positive integers k and n satisfying the following equation:

$$\underbrace{1\cdots1}_{k}\underbrace{0\cdots0}_{2k+3}+\underbrace{7\cdots7}_{k+1}\underbrace{0\cdots0}_{k+1}+\underbrace{1\cdots1}_{k+2}=3n^3$$

OMK2021M

Find all pairs of positive integers (m, n) such that $m^2n^3 = 10^{10}$.

OMK2017S (modified)

Prove that there exist 2017 positive integers $a_1, a_2, a_3, \cdots, a_{2017}$ such that each of the following numbers is a perfect square:

$a_1^2, \quad a_1^2 + a_2^2, \quad \cdots \quad a_1^2 + a_2^2 + \cdots + a_{2017}^2$

1. https://aops.com/community/c13_contests
2. *Junior Problem Seminar* by Santos
3. *Euclidean Geometry in Mathematical Olympiads* by Evan Chen
4. *Techniques for High School Mathematics Contests*

Visit https://tristanchaang.github.io/