Functional Equations

Algebra Handout

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1 Techniques

I will elaborate what these mean in the lecture:

1. Plug in special values for some or all of the variables: $x = 0$; $x = 1$; $y = x$; $y = f(x)$ etc.
2. Guess the answer! $f(x) = x^2$? $f(x) = cx$? $f(x) = 1$?
3. Make substitutions like $g(x) = f(x+1)$ or $g(x) = f(x) - f(0)$.
4. Consider special values of the function, such as $f(0), f(1)$ and $f(-1)$.
5. Force cancellation.
6. Injective? Surjective?
7. Odd? Even?
8. Study where $f(a) = 0$ or $f(a) = a$ or any other relevant condition holds.
9. Beware of the Pointwise Trap!

2 Studying Cauchy's Equation

2.1 Warm Up

Find all $f : \mathbb{Q} \to \mathbb{Q}$ where $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.

Exercise 1. Focus on the integers. What can you say about $f(n)$ for $n \in \mathbb{Z}$?

Exercise 2. Since rational numbers are just integers divided by some other integer, what can you say about $f(1/n)$ for some $n \in \mathbb{Z} \setminus \{0\}$?

Exercise 3. And thus $f(m/n)$ for some $m, n \in \mathbb{Z} \setminus \{0\}$? Conclude your answer.

2.2 Extending to the Reals

Here is the main study of this section. What if $f : \mathbb{R} \to \mathbb{R}$ instead? First of all, repeating exercises 1 to 3, $f(q)$ is still easy to characterise when $q \in \mathbb{Q}$. You should (spoiler alert) get $f(q) = cq$ for some fixed $c \in \mathbb{R}$. Now how about the irrational numbers?

Exercise 4. Prove that if q_1, q_2 are rational, then $f(q_1 x_1 + q_2 x_2) = q_1 f(x_1) + q_2 f(x_2)$ for any $x_1, x_2 \in \mathbb{R}$.

Exercise 5. More generally, if $q_1, \cdots, q_n$ are rational, then $f(\sum q_i x_i) = \sum q_i f(x_i)$ for any $x_i \in \mathbb{R}$.

Therefore f preserves linear combinations with rational coefficients! Now, to answer the question of how f should behave on the irrational numbers, you might first suggest the very sound suggestion that simply $f(x) = cx$ for all $x \in \mathbb{R}$. And indeed, there are some cases that allow us to say that.

2.3 A Few Sufficient (but not necessary) Conditions to Say $f(x) = cx$

- Case: f is continuous.

In Olympiad level, this condition is usually only given, instead of requiring to be shown. It means for any infinite sequence $(x_1, x_2, \cdots)$ converging to x, the sequence $(f(x_1), f(x_2), \cdots)$ converges to $f(x)$. Now indeed, given any irrational x, there is always a rational number arbitrarily close to x (You can prove this by a Pigeonhole argument). Therefore, let $(x_1, x_2, \cdots)$ be a sequence of rational numbers converging to x, then $(f(x_1), f(x_2), \cdots) = (cx_1, cx_2, \cdots)$ converges to $f(x)$. But $(cx_1, cx_2, \cdots)$ converges to cx, and thus $f(x) = cx$. □

- Case: f is bounded either above or below on some interval $[a, b]$ where $a < b$.

This is a harder case. We prove the contrapositive. If $f(x) = cx$ for all $x \in \mathbb{Q}$ but $f(r) \neq cr$ for some fixed r, then there are arbitrarily large positive and negative images on $[a, b]$. We will use the result from Exercise 4.

First, given any $q \in \mathbb{Q}$, there exists at least a $q' \in \mathbb{Q}$ such that $qr + q'$ is in $[a, b]$. This is due to $\mathbb{Q}$ being dense in $\mathbb{R}$ again which can be proven using Pigeonhole. Note that

$$\begin{aligned} f(qr + q') &= qf(r) + q'f(1) \\ &= qf(r) + cq' \\ &= c(qr + q') + q(f(r) - cr) \end{aligned}$$

can achieve arbitarily large positive and negative values by blowing up q because $c(qr + q')$ is strictly bounded between ca and cb, and $f(r) - cr \neq 0$. □

- Case: f is monotone on some interval $[a, b]$ where $a < b$.

This can be included in the previous case because

- If f is increasing on $[a, b]$, then $f(x) \leq f(b)$ on the interval, hence bounded above.
- If f is decreasing on $[a, b]$, then $f(x) \geq f(b)$ on the interval, hence bounded below. □

2.4 * The Full Solution without any Conditions

You may wonder what the general solution is if we are given neither of the conditions in the previous section. Evan Chen's *Monsters* handout explains it pretty well. We will give a brief description here. You may need a fair amount of knowledge in linear algebra for the last part of the solution.

According to Exercise 5, f preserves linear combinations with rational coefficients. Therefore, if we look at a subset of the reals, say $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$, we can get

$$f(a + b\sqrt{2}) = af(1) + bf(\sqrt{2})$$

where a, b are rational. In fact, if we assign any value to each of the images $f(1), f(\sqrt{2})$ (e.g $(f(1), f(\sqrt{2})) = (0, 1)$, or (π, π^π), or whatever), Cauchy's Equation still holds.

In fact, given a basis $\{b_\alpha\}$ of $\mathbb{R}$ over $\mathbb{Q}$, we can assign any value to each of the $f(b_\alpha)$. This is the general solution. Informally, it is like

$$f(a_1 + a_2e + a_3\sqrt{2} + a_4\pi + \cdots) = a_1f(1) + a_2f(e) + a_3f(\sqrt{2}) + a_4f(\pi) + \cdots$$

and then we assign any number to those images $f(1), f(e), f(\sqrt{2}), f(\pi)$. This isn't a very correct way because it turns out that a basis of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable, and hence not listable in this order (That's also why I used $\{b_\alpha\}$ instead of $\{b_1, b_2, \cdots\}$).

3 Exercises

1. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y) = f(x)f(y)$$

   for all x, y.

2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(m + n) = f(m) + f(n) + mn$ for all $m, n \in \mathbb{N}$.

3. Solve Jensen's Equation: Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ where

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}$$

   for any $x, y \in \mathbb{Q}$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(y) + x) = xy + f(x)$$

for all $x, y \in \mathbb{R}$.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + y) + f(y + z) + f(x + z) \geq 3f(x + 2y + 3z)$$

for all x, y, z.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) \leq x$ and $f(x + y) \leq f(x) + f(y)$ for all x, y.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(xf(x) + f(y)) = f(x)^2 + y$$

for all x, y.

8. Find all $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y) = f(x)^2 + f(y)$$

for all $x, y \in \mathbb{R}$.

9. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all x, y.

10. Construct a function from the set of positive rational numbers into itself such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y where $y \neq 0$.

11. Determine all $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that

$$f(xf(y)) = yf(x)$$

for all $x, y > 0$ and $f(x) \to 0$ as $x \to \infty$.

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