Sets, Counting, Pigeons

Combinatorics Handout (B)

13 Nov 2022

1 Preliminaries and Terminology

Logic

Given statements P and Q,

- $P \Rightarrow Q$ means 'if P is true then Q is true'. It does *not* contain information of whether P is actually true or not. E.g.
  - X is an equilateral triangle $\Rightarrow$ X is an isosceles triangle.
  - $1+1=3 \;\Rightarrow\; 0=10^{10}$. (Why?)
  - $x=y \;\Rightarrow\; x^2=y^2$
- $Q \Rightarrow P$ is called the *converse* of $P \Rightarrow Q$. The converse of which of the following statements above is/are true?
- $(\text{not } Q) \Rightarrow (\text{not } P)$ is called the *contrapositive* of $P \Rightarrow Q$. The **contrapositive law** says whenever $P \Rightarrow Q$ is true, its contrapositive is also true.
- $P \Leftrightarrow Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$. We say 'P if and only if Q', or that 'P and Q are equivalent'.
  - $x^2+y^2 \geq 2xy \;\Leftrightarrow\; (x-y)^2 \geq 0$.
  - The triangle X has equal interior angles $\Leftrightarrow$ The triangle X has equal sides.
- $\forall$ is short for 'for all'. $\exists$ is short for 'there exists'. $\exists!$ is short for 'there uniquely exists'.

Definition of a Set

A set is a collection of objects[1]. If an object x belongs to a set S, we write $x \in S$, otherwise $x \notin S$. Note that sets are objects too. A set is uniquely defined in terms of membership, so for example $\{x, x\} = \{x\}$ as both sets have and only have x as an element.

Definition of Cardinality

Given a set S, we denote $|S|$ or $n(S)$ as the *number of elements in* S, or the *cardinality of* S. E.g. $|\{1,2,3\}| = 3$ and $|\{1,1\}| = 1$. For infinite sets, we say they have *infinite cardinality*.

[1] A set is actually defined using the Zermelo-Fraenkel-Choice (ZFC) Axioms, but we will not delve into such formalities in Olympiad level.

Definition of a Subset
Given a set S, we say A is a subset of S ($A \subseteq S$ or $A \subset S$) when

$$\forall x : \ x \in A \Rightarrow x \in S$$

Definition of Union
Given a nonempty collection $\mathcal{A}$ of sets, we can define the union

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in \mathcal{A} : x \in A\}$$

In particular, if $\mathcal{A} = \{A, B\}$, then the union is just written as $A \cup B$.

Definition of Intersection
Given a nonempty collection $\mathcal{A}$ of sets, we can define the intersection

$$\bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}$$

In particular, if $\mathcal{A} = \{A, B\}$, then the intersection is just written as $A \cap B$.

Definition of Complement
Given two sets A and B, we define the complement $A \setminus B$ as

$$A \setminus B = \{x \mid x \in A, x \notin B\}$$

Definition of an Ordered Set
Given n objects $a_1, \cdots, a_n$, the ordered set $(a_1, \cdots, a_n)$ is an object such that

$$(a_1, \cdots, a_n) = (b_1, \cdots, b_m) \Leftrightarrow \begin{cases} m = n \\ a_i = b_i \ \forall i \end{cases}$$

For $n = 2, 3, 4$, they are usually called *pairs, triples, quadruples* etc. For a general n, it can be called an n-tuple. Note that an ordered set is uniquely defined not just in terms of membership, but also the number of times an element occurs and the order those elements are written in. E.g. $(2, 1)$ is not the same as $(1, 2)$, and both are not the same as $(1, 2, 2)$.

We allow (countably) infinite-tuples $(a_1, a_2, \cdots)$ too. E.g. $(2, 4, 6, \cdots)$.

Definition of Cartesian Product
Given two sets A and B, we define the cartesian product $A \times B$ as

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

More generally, given $A_1, \cdots, A_n$,

$$\prod_{i=1}^{n} = A_1 \times \cdots \times A_n = \{(a_1, \cdots, a_n) \mid a_i \in A_i \ \forall i\}$$

For example,

$$\{a, b, 2\} \times \{3, 2\} = \{(a, 3), (a, 2), (b, 3), (b, 2), (2, 3), (2, 2)\}$$

Of course, we also allow infinite Cartesian products.

Definition of a Function
Given two sets A and B, a *function* $f(x)$ from A to B (written $f : A \to B$) is a rule of assignment where element in A is assigned exactly one element in B. There is no restriction of what elements in B should be assigned to (i.e. whether an element in B can be assigned to multiple elements in A, or even be assigned at all). For example, if $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$ then

$$f(a) = 1, \qquad f(b) = 2, \qquad f(c) = 1, \qquad f(d) = 1$$

is a valid function from A to B. Furthermore, A is said to be the *domain* of f, and B is said to be the codomain of f. The subset of B of elements that are images of f is called the *range* or *image set* of f. E.g. in the above example, the domain is A, the codomain is $\{1, 2, 3\}$ and the range is $\{1, 2\}$

If a function satisfies $f(x) \neq f(y)$ for all $x \neq y$ (i.e. by contrapositive $f(x) = f(y) \Rightarrow x = y$), we say f is *injective* or *one-to-one*.

If a function's codomain is equal to its range, it is said to be *surjective* or *onto*.

If a function happens to be both injective and surjective, it is said to be *bijective*. Only bijective functions have inverses.

A Few Important Sets

- $\mathbb{N} = \{1, 2, 3, \cdots\}$ is the set of natural numbers.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$.
- $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is the set of integers.
- $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in Z, q \neq 0\}$ is the set of rational numbers.
- $\mathbb{R}$ is the set of real numbers[2]
- $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$ is the set of complex numbers.
- For any set S, $\mathcal{P}(S) = \{X \mid X \subseteq S\}$ is called the *power set* of S.

2 Counting

From now on, denote $[n]$ to be the set $\{1, 2, \cdots, n\}$. A *permutation* of $[n]$ is a bijection from $[n]$ to $[n]$. A *permutation of k numbers from* $[n]$ is an injection from $[k]$ to $[n]$.

[2]This is hard to define formally. It is the completion of $\mathbb{Q}$ with respect to the standard metric, but you can now think of it as the set of all possible decimals, including bizarre numbers like π, $\sqrt{2}$ etc.

Exercise 1. There are $n!$ permutations of $[n]$.

Exercise 2. There are ${}^n\mathrm{P}_k = \dfrac{n!}{(n-k)!}$ permutations of k numbers from $[n]$.

Exercise 3. There are $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ subsets of $[n]$ of size k.

3 Bijections

The theory is simple: If A and B are finite sets and there exists a bijection between A and B, then $|A| = |B|$.

Exercise 1. How many (a, b, c) are there such that $a, b, c \in [100]$ and $a < b < c$ form an arithmetic progression? What if we change 100 to n?

Exercise 2. How many $(a, b, c, d) \in \mathbb{N}^4$ are there such that $a + b + c + d = 20$?

Exercise 3. How many $(a, b, c, d) \in \mathbb{N}_0^4$ are there such that $a + b + c + d = 20$?

Exercise 4. Generalise the previous 2 examples to any number of variables and any sum.

Exercise 5. Find the number of $(a, b, c) \in \mathbb{N}^3$ such that $a \le b \le c$ and $a + b + c = n$.

Exercise 6. You have an $n \times n$ checkerboard and every square is to be coloured black or white. Your angry lecturer demands that every row must have an even number of black squares and every column too. How many ways are there to colour the checkerboard?

Exercise 7. A nonempty subset S of $[n]$ is called *nice* if

- The sum of elements of S is a multiple of three.
- S cannot have **both** 1 and 2 in it, though it is allowed to have one or neither of them.

How many nice subsets are there?

Exercise 8. A spider is to climb the following web from point A to point B. The only direction it can move is up and right. How many ways are there? What if we changed the web to a 100×60 instead of 5×3? Or $m \times n$?

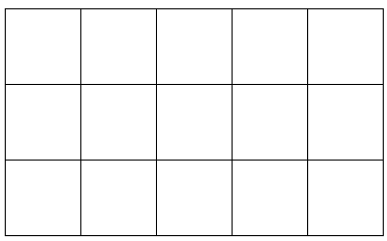

Figure 1: Spiderweb

Exercise 9. You start off with 50$ in a game. Every round, you toss a coin. If you obtain heads, you gain 1$. Otherwise, you lose 1$. How many possible scenarios are there so that you tossed 100 coins and end up with 50$ again, but throughout the game you never dropped below 20$?

Exercise 10. Let $A = [n]$, A subset P of A is called *nice* if $P = \{a_1, a_2, a_3\}$ and $a_1 + 6 \le a_2 + 4 \le a_3$. How many nice subsets does the set A have?

4 Inclusion-Exclusion Principle

Exercise 11. Verify **De Morgan's Laws** by using Venn Diagrams:

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ X \setminus (A \cap B) &= (X \setminus A) \cup (X \setminus B) \\ X \setminus (A \cup B) &= (X \setminus A) \cap (X \setminus B) \end{aligned}$$

Given two sets A and B, we know

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This is the **Inclusion-Exclusion Principle** (PIE) for 2 sets. For 3 sets, we show that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

One way we can verify this is by looking at the number of times an element x is counted:

- If $x \in A$ but not in B, C, then in the RHS, x is counted in the term $|A|$, and hence counted once.

- If $x \in A, B$ but not in C, then in the RHS, x is counted in the terms $|A|, |B|, |A \cap B|$, and hence counted $1 + 1 - 1 = 1$ time.

- If $x \in A, B, C$, then in the RHS, x is counted in all the terms, and hence counted $1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$ time.

By logical symmetry these are all the cases.

Another way we can verify the identity is by applying the 2-set version twice:

$$\begin{aligned} |A \cup (B \cup C)| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B \cup C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B)| - |(A \cap C)| + |(A \cap B) \cap (A \cap C)| \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

Let's crank this up and look for the n-set version. But first we look at the 3-set version's pattern: We first add the sizes of the single sets, then subtract the sizes of the pairwise intersections, then add the size of the triple intersection. It would be reasonable to guess

$$|A_1 \cup \cdots \cup A_n| = \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots \tag{$*$}$$

and it fact it is true! Let's again verify it using the two methods that we used to verify the 3-set version.

Method 1. Say $x \in A_1, \cdots, A_k$ for some $1 \leq k \leq n$. Then in $(*)$, x in counted k times in the first sum, $\binom{k}{2}$ in the second sum, $\binom{k}{3}$ times in the third sum and so on. The total number of times x is counted is

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots + (-1)^{k-1}\binom{k}{k}.$$

This is an example of a **combinatorial sum**, which is a huge topic on its own. We will give a way to compute this sum:

$$\begin{aligned}
\sum_{i=1}^{n}(-1)^{i-1}\binom{k}{i} &= \binom{k}{0} - \sum_{i=1}^{n}(-1)^{i}\binom{k}{i} \\
&= \binom{k}{0} - [1+(-1)]^k && \text{(Binomial Theorem)} \\
&= 1
\end{aligned}$$

and hence x is counted once. □

Method 2. We proceed by induction. Say $(*)$ is true for some n. Then

$$\begin{aligned}
|A_1 \cup \cdots \cup A_{n+1}| &= |A_1 \cup \cdots \cup (A_n \cup A_{n+1})| \\
&= \sum_{1 \leq i \leq n-1} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots \\
&\quad + |A_n \cup A_{n+1}| - \sum_{1 \leq i \leq n-1} |A_i \cap (A_n \cup A_{n+1})| + \cdots \\
&= \sum_{1 \leq i \leq n-1} |A_i| - \sum_{1 \leq i < j \leq n-1} |A_i \cap A_j| + \cdots \\
&\quad + (|A_n| + |A_{n+1}| - |A_n \cap A_{n+1}|) \\
&\quad - \sum_{1 \leq i \leq n-1} |(A_i \cap A_n) \cup (A_i \cap A_{n+1})| + \cdots
\end{aligned}$$

then apply de Morgan's Law on the remaining sums. This is all very messy notation (and it fact begs a more rigorous and formal one that avoids the '$\cdots$'), but at least it works. You can verify the last step yourself as I want to avoid writing the mess, so I will just (forgive me, and don't do this in actual tests) sweep it under the rug. □

Exercise 12. A derangement of $[n]$ is a permutation where all elements are no longer in its original position. How many derangements of $[n]$ are there?

5 Pigeonhole Principle

Pigeonhole Principle 1. Given n pigeonholes and $n+1$ pigeons in those holes, at least one of the holes has at least two pigeons.

Pigeonhole Principle 2. Given n pigeonholes and m pigeons in those holes, at least one of the holes has at least $\lceil m/n \rceil$ pigeons.

Pigeonhole Principle 3. Given n pigeonholes and infinitely many pigeons in those holes, at least one of the holes has infinitely many pigeons.

Exercise 13. Let n be any odd number not divisible by 5. Prove that one of $1, 11, 111, \cdots$ is divisible by n.

Exercise 14. Among any 7 perfect squares there exist two whose difference is divisible by 10.

Exercise 15. Let S be a set of n integers. Prove that S contains a subset with sum of elements divisible by n.

Exercise 16. Given any 10 points within an equilateral triangle of side length 1, there are two whose distance apart is at most $1/3$.

Exercise 17. From ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

Exercise 18. Prove that among any $n+1$ numbers from the set $\{1, 2, \cdots, 2n\}$ there is one that is divisible by another.

Exercise 19. Dirichlet's Theorem: Let α be any irrational number and N be any positive integer. There must exist $m, n \in \mathbb{Z}$ such that $|m\alpha - n| < 1/N$.

Exercise 20. Erdös-Szekeres Theorem: Every sequence of $mn+1$ real numbers contains either a nondecreasing subsequence of length of $m+1$ or a nonincreasing subsequence of length $n+1$.

