Linear Algebra Problems

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1 Similar Matrices

1. Let A and X be $n \times n$ invertible matrices such that $XAX^{-1} = A^2$. Prove that there exists a positive integer m such that all the eigenvalues of A are m-th roots of unity.

Solution:
Recall that similar matrices have the same eigenvalues. Let λ be an eigenvalue of A, then

$$Av = \lambda v \Rightarrow A^2 v = \lambda A v = \lambda^2 v$$

hence λ^2 is an eigenvalue of $A^2 = XAX^{-1} \Rightarrow \lambda^2$ is an eigenvalue of A. Inductively, λ^{2^n} is an eigenvalue of A for any $n \in \mathbb{N}$. Since there are finitely many eigenvalues, $\lambda^i = \lambda^j$ for some $i \neq j$, hence λ is a root of unity, say $\lambda^{f(\lambda)} = 1$. Repeat for all eigenvalues of A and take m as the LCM of all $f(\lambda)$. □

2 Olympiad

2. Let $A_1, A_2, \ldots, A_{n+1}$ be non-empty subsets of $\{1, 2, \ldots, n\}$. Prove that there exists nonempty disjoint subsets $I, J \in \{1, 2, \ldots, n+1\}$ such that

$$\bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k$$

Solution:
Let v_i be the incidence vector of A_i. Since $v_1, \ldots, v_{n+1}$ are linearly dependent in n-dimensions, there exists $c_1, \ldots, c_{n+1} \in \mathbb{R}$, not all zero, such that

$$c_1 v_1 + \cdots + c_{n+1} v_{n+1} = 0.$$

Let $I = \{k : c_k > 0\}$ and $J = \{k : c_k < 0\}$. Then

$$\sum_{k \in I} |c_k| v_k = \sum_{k \in J} |c_k| v_k$$
$$\therefore \bigcup_{k \in I} A_k = \bigcup_{k \in J} A_k$$

3. Let $A_1, A_2, \dots, A_{n+2}$ be non-empty subsets of $\{1, 2, \dots, n\}$. Prove that there exists nonempty disjoint subsets $I, J \in \{1, 2, \dots, n+2\}$ such that

$$\bigcup_{k\in I} A_k = \bigcup_{k\in J} A_k \qquad \text{and} \qquad \bigcap_{k\in I} A_k = \bigcap_{k\in J} A_k$$

Solution:
Let v_i be the incidence vector of A_i.

Lemma. There exists $c_1, \dots, c_{n+2} \in \mathbb{R}$, not all zero, such that

$$c_1 v_1 + \cdots + c_{n+2} v_{n+2} = 0 \qquad \text{and} \qquad c_1 + \cdots + c_{n+2} = 0.$$

Proof. Assume the contrary. Let $p_1, \dots, p_{n+1} \in \mathbb{R}$, not all zero, such that

$$\begin{cases} p_1 v_1 + \cdots + p_{n+1} v_{n+1} = 0 \\ p_1 + \cdots + p_{n+1} = 1. \end{cases}$$

Then there must exist a positive p_i, WLOG $p_1 > 0$. Now let $q_2, \dots, q_{n+2} \in \mathbb{R}$, not all zero, such that

$$\begin{cases} q_2 v_2 + \cdots + q_{n+2} v_{n+2} = 0 \\ q_2 + \cdots + q_{n+2} = 1. \end{cases}$$

By substraction we have

$$p_1 v_1 + (p_2 - q_2) v_2 + \cdots + (p_{n+1} - q_{n+1}) v_{n+1} + (-q_{n+2}) v_{n+2} = 0$$

and hence we can let $c_1 = p_1 \neq 0,\ c_{n+2} = -q_{n+2}$, and $c_i = p_i - q_i$ $(i = 2, \dots, n+1)$. □

Let $I = \{k : c_k > 0\}$ and $J = \{k : c_k < 0\}$. Then

$$\sum_{k\in I} |c_k| v_k = \sum_{k\in J} |c_k| v_k$$
$$\therefore \bigcup_{k\in I} A_k = \bigcup_{k\in J} A_k$$

We now prove the second property. Assume $x \in \bigcap_{k\in I} A_k$, so the x-th component of all v_k $(k \in I)$ is 1. Then the x-th component of $\sum_{k\in I} |c_k| v_k$ is equal to $\sum_{k\in I} |c_k| = \sum_{k\in J} |c_k|$. However, in the x-th component in $\sum_{k\in I} |c_k| v_k$, the sum $\sum_{k\in J} |c_k|$ is maximal and can only be achieved when the x-th component of all v_k $(k \in J)$ is 1, hence $x \in \bigcap_{k\in J} A_k$. Similarly, $x \in \bigcap_{k\in J} A_k \Rightarrow x \in \bigcap_{k\in I} A_k$. Therefore, $\bigcap_{k\in I} A_k = \bigcap_{k\in J} A_k$. □

4. Compute the determinant of

$$\mathbf{M} = \begin{pmatrix} a_1^2+k & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2+k & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2+k & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2+k \end{pmatrix}$$

Solution 1:
Note that $\text{rank}(\mathbf{M}-k\mathbf{I}) = 1 \Rightarrow \text{null}(\mathbf{M}-k\mathbf{I}) = n-1$. Therefore there exists $n-1$ mutually orthonormal vectors in the null space of $\mathbf{M}-k\mathbf{I}$, i.e. there are $n-1$ mutually orthonormal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ of $\mathbf{M}$ with eigenvalue k. Now we see

$$\mathbf{v} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}^{\mathrm{T}} \Rightarrow \mathbf{M}\mathbf{v} = (a_1^2 + a_2^2 + \cdots + a_n^2 + k)\mathbf{v}$$

and hence $\mathbf{v}$ is another eigenvector of $\mathbf{M}$, with eigenvalue $h = a_1^2 + a_2^2 + \cdots + a_n^2 + k$. We claim that $\mathbf{v}$ is orthogonal to any of the $\mathbf{v}_i$ $(i = 1, \ldots, n-1)$. Indeed,

$$k\mathbf{v}^{\mathrm{T}}\mathbf{v}_i = \mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{v}_i = (\mathbf{M}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{v}_i = (\mathbf{M}\mathbf{v})^{\mathrm{T}}\mathbf{v}_i = h\mathbf{v}^{\mathrm{T}}\mathbf{v}_i \Rightarrow \mathbf{v}^{\mathrm{T}}\mathbf{v}_i = 0.$$

Therefore $\widehat{\mathbf{v}}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}$ are mutually orthonormal eigenvectors of $\mathbf{M}$! In that case,

$$\mathbf{Q} = \begin{pmatrix} \widehat{\mathbf{v}} & \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_{n-1} \end{pmatrix} \Rightarrow \mathbf{M} = \mathbf{Q} \begin{pmatrix} h & 0 & 0 & \cdots & 0 \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{pmatrix} \mathbf{Q}^{-1}$$

which implies $\det(M) = k^{n-1}h = k^{n-1}(a_1^2 + a_2^2 + \cdots + a_n^2 + k)$.

Solution 2:
We prove by simple induction,

$$\begin{aligned} \det(M) &= \begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2+k & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2+k & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2+k \end{vmatrix} + \begin{vmatrix} k & 0 & 0 & \cdots & 0 \\ a_2a_1 & a_2^2+k & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2+k & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2+k \end{vmatrix} \\ &= \begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ 0 & k & 0 & \cdots & 0 \\ 0 & 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k \end{vmatrix} + k \begin{vmatrix} a_2^2+k & a_2a_3 & \cdots & a_2a_n \\ a_3a_2 & a_3^2+k & \cdots & a_3a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_2 & a_na_3 & \cdots & a_n^2+k \end{vmatrix} \\ &= a_1^2k^{n-1} + k^{n-1}(a_2^2 + \cdots + a_n^2) = k^{n-1}(a_1^2 + a_2^2 + \cdots + a_n^2 + k). \end{aligned}$$

5. Let $\mathbf{A}$ and $\mathbf{B}$ be equally sized matrices. Prove that $\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{B}) \geq \text{rank}(\mathbf{A} + \mathbf{B})$.

Solution:
Let $r = \text{rank}(\mathbf{A}), s = \text{rank}(\mathbf{B}), t = \text{rank}(\mathbf{A} + \mathbf{B})$. Pick $u_1, \ldots, u_r$ as a basis of $C(\mathbf{A})$ whereas $v_1, \ldots, v_s$ as a basis of $C(\mathbf{B})$. Let $\vec{x}$ be any vector in $C(\mathbf{A} + \mathbf{B})$,

$$\begin{aligned} \vec{x} &= (\mathbf{A} + \mathbf{B})\vec{y} \\ &= \mathbf{A}\vec{y} + \mathbf{B}\vec{y} \end{aligned}$$

for some vector $\vec{y}$. Since $\mathbf{A}\vec{y}$ and $\mathbf{B}\vec{y}$ are in $C(\mathbf{A})$ and $C(\mathbf{B})$ respectively, they can be expressed as a linear combination of u_i $(1 \leq i \leq r)$ and v_i $(1 \leq i \leq s)$ respectively.

Therefore **any vector in $C(\mathbf{A}+\mathbf{B})$ can be expressed as a linear combination of $u_1, \ldots, u_r, v_1, \ldots, v_s$, which is at most $r + s$ vectors (some of them might be equal).**

Choose a basis B of the space spanned by $S = \{u_1, \ldots, u_r, v_1, \ldots, v_s\}$. Since any vector in $C(\mathbf{A} + \mathbf{B})$ can be written as a linear combination of B which is linearly independent, we conclude that B is also a basis of $C(\mathbf{A} + \mathbf{B})$. Because $B \subseteq S$,

$$t = |B| \leq |S| = r + s$$

which completes the proof. $\square$