1. Find the maximal number of regions a circle can be divided in by segments joining n points on the boundary of the circle.

Answer. $\binom{n}{4} + \binom{n}{2} + 1.$

Solution. There are $\binom{n}{4}$ intersection points inside the circle, and n intersection points on the circle. For every internal intersection point, there are 4 segments joining it, whereas for every external intersection point, there are $n+1$ segments joining it. Hence there are $(4\binom{n}{4} + n(n+1))/2$ edges. By Euler's Characteristic Formula,

$$\begin{aligned} F + V &= E + 2 \\ F + \binom{n}{4} + n &= 2\binom{n}{4} + \frac{n(n+1)}{2} + 2 \\ F &= \binom{n}{4} + \binom{n}{2} + 2 \end{aligned}$$

Excluding the outer face, we get the desired answer. $\square$

2. An alphabet consists of n letters. What is the maximal length of a word if we know that any two consecutive letters a, b of the word are different and that the word cannot be reduced to a word of the kind *abab* with $a \neq b$ by removing letters.

Answer. $2n - 1$.

Solution. We prove by induction. When $n = 1$, the answer is $1 = 2(1) - 1$. Assume the answer for $i \leq n$ letters is $2i - 1$. For $n+1$ letters, first we have some facts:

Fact 1. The first and last letter should be the same since otherwise we can append another letter at the back which is the same as the first letter.

Say the first and last letter is k. Then k divides the word into several segments. We observe that the same letter can appear in different segments otherwise *abab* is formed, which leads to:

Fact 2. Every segment is its own entity, i.e. a sub-word that satisfies the conditions.

Now assume there are m k's. Then there will be $m - 1$ sub-words, i.e. the maximum word must have

$$m + (2a_1 - 1) + \cdots + (2a_{m-1} - 1) = 2a_1 + \cdots + 2a_{m-1} + 1 = 2(n+1) - 1.$$

whereas the word $w_1 \ldots w_{n+1} \ldots w_1$ is valid. $\square$

3. Three players A, B, C play a game with three cards and on each of these 3 cards it is written a positive integer, all 3 numbers are different. A game consists of shuffling the cards, giving each player a card and each player is attributed a number of points equal to the number written on the card and then they give the cards back. After a number (≥ 2) of games we find out that A has 20 points, B has 10 points and C has 9 points. We also know that in the last game B had the card with the biggest number. Who had in the first game the card with the second value (this means the middle card concerning its value).

Answer. C.

Solution. Let the cards be of value $a < b < c$. If this supposed number of games is n, then

$$\begin{aligned} 20 &\leq nc \\ c &\leq 10 - (n-1) \\ \therefore \frac{20}{n} &\leq 11 - n \\ \therefore n &\geq 3 \\ n(a+b+c) &= 39 \\ \therefore n &= 3, a+b+c = 13 \end{aligned}$$

Hence

$$\begin{aligned} 20 &\leq 3c \\ c &\leq 10 - 1 - 1 \\ \therefore c &= 7, 8 \end{aligned}$$

If $c = 7$,

$$(a, b, c) = (1, 2, 7)$$

which is a contradiction. If $c = 8$,

$$\begin{aligned} (a, b, c) &= (1, 4, 8) \\ 20 &= 8 + 8 + 4 \\ 10 &= 8 + 1 + 1 \\ 9 &= 4 + 4 + 1 \end{aligned}$$

And the shuffling $(8, 1, 4), (8, 1, 4), (4, 8, 1)$ is valid. $\square$

4. Find the least natural number n such that, if the set $\{1, 2, ..., n\}$ is arbitrarily divided into two non-intersecting subsets, then one of the subsets contains 3 distinct numbers such that the product of two of them equals the third.

Answer. 96.

Solution. For $n = 95$, the sets

$$\begin{aligned} A &= \{2, 3, 4, 5, 7, 48, 54, 60, 66, 72, 78, 80, 88, 90\} \\ B &= \{1, \dots, 95\} \setminus A \end{aligned}$$

form a valid configuration since the elements of B pairwise multiply to $\{48, 54, 60, 66, 72, 78, 80, 88, 90, 96, \dots\}$ which are clearly either in A or more than 95. For $n = 96$, first we mention the obvious fact that if $x > y$ are in the same set, then $xy, x/y$ must be in the other set if they are integers. We have a few cases WLOG:

Case 1. $\underline{2, 3, 4 \in A} \Rightarrow 6, 8, 12 \in B \Rightarrow 48, 72, 96 \in A \Rightarrow \underline{2, 48, 96 \in A}$.

Case 2. $\underline{2, 3 \in A, 4 \in B} \Rightarrow 6 \in B \Rightarrow 24 \in A \Rightarrow 48 \in B \Rightarrow 12 \in A \Rightarrow \underline{2, 12, 24 \in A}$.

Case 3. $\underline{2, 4 \in A, 3 \in B} \Rightarrow 8 \in A \Rightarrow 24 \in A \Rightarrow 6, 12 \in B \Rightarrow 18, 36 \in A \Rightarrow \underline{2, 18, 36 \in A}$.

Case 4. $\underline{3, 4 \in A, 2 \in B} \Rightarrow 12 \in B \Rightarrow 6 \in A \Rightarrow 24 \in B \Rightarrow \underline{2, 12, 24 \in B}$.

All of which are contradictions. $\square$

5. Around a circular table an even number of persons have a discussion. After a break they sit again around the circular table in a different order. Prove that there are at least two people such that the number of participants sitting between them before and after a break is the same.

Solution. Let there be $2n$ people and let σ be a permutation of $\{1, \ldots, 2n\}$. The problem wants $i < j$ such that

$$\begin{aligned} &\sigma(j) - \sigma(i) \equiv j - i \pmod{2n} \\ \Leftrightarrow\ &\sigma(j) - j \equiv \sigma(i) - i \pmod{2n}. \end{aligned}$$

Assume otherwise, then $\sigma(i) - i (i = 1, \ldots, 2n)$ forms a complete residue system mod $2n$,

$$\begin{aligned} 0 &= (\sigma(1) + \cdots + \sigma(2n)) - (1 + \cdots + 2n) \\ &= (\sigma(1) - 1) + \cdots + (\sigma(2n) - 2n) \\ &\equiv 1 + \cdots + 2n \\ &= n(2n+1) \not\equiv 0 \pmod{2n} \end{aligned}$$

which is a contradiction. □

6. Let L denote the set of all lattice points of the plane (points with integral coordinates). Show that for any three 4 points A, B, C of L there is a fourth point D, different from A, B, C, such that the interiors of the segments AD, BD, CD contain no points of L. Is the statement true if one considers four points of L instead of three?

Solution. The points $(0, 0), (1, 0), (0, 1), (1, 1)$ clearly ensures some pair of distances to be both even, hence the second statement.

For 3 points, it wants a lattice (x, y) such that $\gcd(x - x_i, y - y_i) = 1$ for $1 \le i \le 3$. Let (a, b) be a pair of residue classes mod 2 that doesn't appear among the A_i. Since there are 4 possible pairs of residue classes and only three points A_i, there must be such a vacant residue class. Similarly, we can find a pair (c, d) of residue classes mod 3 that is also not represented. Fix x to be any integer congruent to a modulo 2 and to c modulo 3.

We consider the set P of all primes dividing any of the numbers $x - x_i$. For each $p \in P$, find some $y(p)$ such that $(x - x_i, y(p) - y_i)$ is not congruent to $(0, 0)$ mod p. This can be done for $p = 2, 3$ by our choice of x, and for $p \ge 5$ simply because there are p options for $y(p)$ and only 3 constraints. Alas we can use CRT. □

7. A rectangular array of numbers is given. In each row and each column, the sum of all numbers is an integer. Prove that each non-integral number x in the array can be changed into either $\lfloor x \rfloor$ or $\lceil x \rceil$ so that the row-sums and column-sums remain unchanged.

Solution. Complete for the column-sums easily. Now assume we are not done. Let S be the sum of absolute values of differences of the each current row-sum from its previous row-sum. Clearly S is even. A row is said to be *lower* if the current sum is lower than its original sum, and *higher* if it is higher than its original sum. Now a tile is labelled $+$ if the ceiling is used; and $-$ if the floor is used. A row R_1 is said to be *accessible* to another row R_2 (denoted by $R_1 \to R_2$) if there exists a column C such that $R_1 \cap C$ is $-$ and $R_2 \cap C$ is $+$. WLOG R is a lower row. We have the following lemma:

Lemma. There exists a sequence $R \to R_1 \to \cdots \to R_k$ such that R_k is higher.

Proof. Let $\mathbf{R}$ be the set of rows accessible from R directly or indirectly and $\overline{\mathbf{R}}$ be the remaining rows. We want to prove that there is a higher row in $\mathbf{R}$. Assume otherwise. Pick any column C. If $\mathbf{R} \cap C$ has a $-$, since $\forall r \in \overline{\mathbf{R}}$ is inaccessible by rows in $\mathbf{R}$, the labelled signs of every tile in $\overline{\mathbf{R}} \cap C$ is $-$, therefore the sum of all tiles in $\mathbf{R} \cap C$ must be higher then its original sum in order to preserve the constancy of column-sums. If $\mathbf{R} \cap C$ has all $+$, then the sum of all tiles in $\mathbf{R} \cap C$ must be obviously higher. In conclusion, we can sum up all $\mathbf{R} \cap C$ and conclude that the sum of rows in $\mathbf{R}$ is higher than its original sum, hence there must be a higher row. $\square$

Now we can swap in order the labelled signs of $(R \cap C, R_1 \cap C), (R_1 \cap C, R_2 \cap C), \ldots, (R_{k-1} \cap C, R_k \cap C)$ and we get to decrease S by 2. Since S is even, we must hit 0 eventually. $\square$

8. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

Solution. Let $C_1, \ldots, C_m$ and $J_1, \ldots, J_n$ be the candidates and judges respectively. We count the number of elements in $T = \{(C_i, \{J_j, J_k\}) : C_i \text{ is graded equally by } J_j \text{ and } J_k.\}$. Let r_i be the number of judges that grade fail on C_i. On one hand,

$$|T| \leq \binom{n}{2} \cdot k$$

Since each pair of judges agrees on at most k candidates. On the other hand,

$$|T| = \sum_{i=1}^{m} \binom{r_i}{2} + \binom{n-r_i}{2} \geq m\left(\binom{\frac{n-1}{2}}{2} + \binom{\frac{n+1}{2}}{2}\right)$$

Therefore

$$\begin{aligned} m\left(\binom{\frac{n-1}{2}}{2} + \binom{\frac{n+1}{2}}{2}\right) &\leq \binom{n}{2} \cdot k \\ m(n-1)^2 &\leq 2kn(n-1) \\ \frac{k}{m} &\geq \frac{n-1}{2n} \end{aligned}$$

and we are done. $\square$

9. Let A be a set of N residues $\pmod{N^2}$. Prove that there exists a set B of of N residues $\pmod{N^2}$ such that $A + B = \{a + b | a \in A, b \in B\}$ contains at least half of all the residues $\pmod{N^2}$.

Solution. Work in $\mathbb{Z}/N^2\mathbb{Z}$. Let $[N^2] = 0, \ldots, N^2 - 1$. Below we compute (the condition under the summation is omitted after it is mentioned):

$$\begin{aligned}
&\sum_{0 \le i_1 < \cdots < i_N \le N^2-1} \left| \bigcup_{j=1}^{N} (A + \{i_j\}) \right| \\
&= \sum \left| N^2 - \bigcap_{j=1}^{N} \overline{(A + \{i_j\})} \right| \\
&= \binom{N^2}{N} N^2 - \sum \left| \bigcap_{j=1}^{N} \overline{(A + \{i_j\})} \right| \\
&= \binom{N^2}{N} N^2 - \sum \left| \{x \in [N^2] : x - i_j \notin A\} \right| \\
&= \binom{N^2}{N} N^2 - \sum_{x \in [N^2]} \left| \{\{i_1, \ldots, i_N\} : x - i_j \notin A\} \right| \\
&= \binom{N^2}{N} N^2 - \sum_{x \in [N^2]} \binom{N^2 - N}{N} \\
&= \left(\binom{N^2}{N} - \binom{N^2 - N}{N} \right) N^2
\end{aligned}$$

therefore by Pigeonhole Principle there must exist $\{i_1, \ldots, i_N\}$ such that

$$\begin{aligned}
\left| \bigcup_{j=1}^{N} (A + \{i_j\}) \right| &\ge \frac{\binom{N^2}{N} - \binom{N^2-N}{N}}{\binom{N^2}{N}} \cdot N^2 \\
&= \left(1 - \prod_{k=1}^{N} \left(1 + \frac{N}{N^2 - 2N + k} \right)^{-1} \right) N^2 \\
&= \left(1 - \left(1 + \frac{1}{N-1} \right)^{-N} \right) N^2 \\
&> (1 - e^{-1}) N^2 \\
&> \frac{1}{2} N^2
\end{aligned}$$

and we are done. $\square$