Classical Inequalities

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April 20, 2018

1 The Trivial Inequality

The trivial inequality states that

$$x^2 \geq 0, x \in \mathbb{R}$$

which is quite straightforward.

2 The AM-GM Inequality

The AM-GM inequality is very commonly used in inequality problems. First make sure you are familiar with the definitions of Arithmetic Means and Geometric Means. It states that if $a_1, a_2, ..., a_n$ are positive reals,

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1 a_2 ... a_n)^{1/n}$$

and equality holds if and only if a_i are all equal $(i = 1, 2, ..., n)$.
Proof. This is an inductive proof. For $n = 1$ it is obvious. For $n = 2$,

$$\begin{aligned} & \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} && (1) \\ \Leftrightarrow \quad & \frac{a_1^2 + 2a_1 a_2 + a_2^2}{4} \geq a_1 a_2 \\ \Leftrightarrow \quad & a_1^2 - 2a_1 a_2 + a_2^2 \geq 0 \\ \Leftrightarrow \quad & (a_1 - a_2)^2 \geq 0 \end{aligned}$$

which is true.
Now we claim that the statement is true for all $n = 2^k$ where k is any positive integer.
Assume the statement is true for $n = 2^{k-1}$. Then,

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1 a_2 ... a_n)^{1/n}$$

Because (1) is true,

$$\begin{aligned} \frac{\frac{a_1+a_2+...+a_n}{n} + \frac{a_{n+1}+a_{n+2}+...+a_{2n}}{n}}{2} &\geq \sqrt{\frac{a_1 + a_2 + ... + a_n}{n} \frac{a_{n+1} + a_{n+2} + ... + a_{2n}}{n}} \\ &\geq \sqrt{(a_1 a_2 ... a_n)^{1/n} (a_{n+1} a_{n+2} ... a_{2n})^{1/n}} \\ &= \sqrt{(a_1 a_2 ... a_{2n})^{1/n}} \\ &= (a_1 a_2 ... a_{2n})^{1/(2n)} \end{aligned}$$

and thus $2n = 2^k$ is true as well. So by induction this claim is proven.
Now we work backwards to prove for n is not a power of two. Assume our statement is true for n, then

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq (a_1 a_2 ... a_n)^{1/n}$$

Let $a_n = \frac{a_1+a_2+...+a_{n-1}}{n-1}$,

$$\frac{a_1+a_2+...+a_{n-1}+\frac{a_1+a_2+...+a_{n-1}}{n-1}}{n} \geq (a_1a_2...a_{n-1}\frac{a_1+a_2+...+a_{n-1}}{n-1})^{1/n}$$
$$\frac{\frac{n(a_1+a_2+...+a_{n-1})}{n-1}}{n} \geq (a_1a_2...a_{n-1}\frac{a_1+a_2+...+a_{n-1}}{n-1})^{1/n}$$
$$(\frac{a_1+a_2+...+a_{n-1}}{n-1})^n \geq a_1a_2...a_{n-1}\frac{a_1+a_2+...+a_{n-1}}{n-1}$$
$$(\frac{a_1+a_2+...+a_{n-1}}{n-1})^{n-1} \geq a_1a_2...a_{n-1}$$
$$\frac{a_1+a_2+...+a_{n-1}}{n-1} \geq (a_1a_2...a_{n-1})^{1/(n-1)}$$

and therefore we have proved that the statement is true even if n is not a power of two. In conclusion we have proved that the statement is true for all $n \in \mathbb{N}$. (Q.E.D)

3 The Cauchy-Schwarz Inequality

The formal definition of Cauchy-Schwarz is in the form of vectors. However, we can also write it in a more understandable way: If $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ are all real numbers,

$$(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2) \geq (a_1b_1+a_2b_2+...a_nb_n)^2$$

and equality holds if and only if $\frac{a_i}{b_i}$ are all equal $(i = 1, 2, ..., n)$.
The proof can be done by simple induction, so it is left as an exercise for the reader. We hereby present another proof:
Proof. Consider the polynomial:

$$\begin{aligned} P(x) &= (a_1x+b_1)^2+(a_2x+b_2)^2+...+(a_nx+b_n)^2 \\ &= (a_1^2+a_2^2+...+a_n^2)x^2+2(a_1b_1+a_2b_2+...+a_nb_n)x+(b_1^2+b_2^2+...+b_n^2) \end{aligned}$$

Since $P(x) \geq 0$ and $P(x)$ is a quadratic equation, therefore $\Delta = B^2 - 4AC \leq 0$.

$$4(a_1b_1+...+a_nb_n)^2 - 4(a_1^2+...+a_n^2)(b_1^2+...+b_n^2) \leq 0$$

and the result follows. (Q.E.D)

4 The Rearrangement Inequality

Let's consider a real-life example: You have the opportunity to receive notes of RM100, RM20 and RM5 such that the number of notes is 8, 5, 2 (any order of this). How would you take to get the maximum amount of money? How about the minimum?
Do you get the idea now? This is precisely the rearrangement inequality, which, in formal definitions, is: If $(a_1, a_2, ...a_n)$ and $(b_1, b_2, ...b_n)$ are both increasing sequences,

$$\begin{aligned} &a_1b_1+a_2b_2+...+a_nb_n \\ &\geq a_1b_{\sigma_1}+a_2b_{\sigma_2}+...+a_nb_{\sigma_n} \\ &\geq a_1b_n+a_2b_{n-1}+...+a_nb_1 \end{aligned}$$

where $(\sigma_1, \sigma_2, ..., \sigma_n)$ is any permutation of $(1, 2, ..., n)$.

5 IMO shortlist problem

Let a, b, c be positive reals and $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(a+c)}+\frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

Solution 1.
Let the original equation be A. Then

$$A = \frac{b^2c^2}{a(b+c)}+\frac{a^2c^2}{b(a+c)}+\frac{a^2b^2}{c(a+b)}$$

$$\begin{aligned}(a(b+c)+b(c+a)+c(a+b))\cdot A &\geq (\sqrt{a(b+c)}\frac{bc}{\sqrt{a(b+c)}}+\sqrt{b(c+a)}\frac{ca}{\sqrt{b(c+a)}}+\sqrt{c(a+b)}\frac{ab}{\sqrt{c(a+b)}})^2\\ &= (bc+ca+ab)^2\\ &\geq (bc+ca+ab)\cdot 3(bc\cdot ca\cdot ab)^{1/3}\\ &= 3(bc+ca+ab)\end{aligned}$$

i.e. $2(bc+ca+ab).A \geq 3(bc+ca+ab)$ and the conclusion follows. (Q.E.D)

Solution 2.
By Titu's Lemma,

$$\frac{x_1^2}{y_1}+\frac{x_2^2}{y_2}+...+\frac{x_n^2}{y_n} \geq \frac{(x_1+x_2+...+x_n)^2}{y_1+y_2+...+y_n}$$

Now

$$\begin{aligned}A &= \frac{b^2c^2}{a(b+c)}+\frac{a^2c^2}{b(a+c)}+\frac{a^2b^2}{c(a+b)}\\ &\geq \frac{(bc+ca+ab)^2}{2(bc+ca+ab)}\\ &= \frac{1}{2}(bc+ca+ab)\\ &\geq \frac{1}{2}(3(bc\cdot ca\cdot ab)^{1/3})\\ &= \frac{3}{2}\end{aligned}$$

(Q.E.D)

Solution 3.
From the AM-GM inequality,

$$\begin{aligned}a^2+b^2 &\geq 2ab\\ \frac{a^2}{b} &\geq 2a-b\end{aligned}$$

Now

$$\begin{aligned}A &= \frac{1}{4}(\frac{4b^2c^2}{a(b+c)}+\frac{4a^2c^2}{b(a+c)}+\frac{4a^2b^2}{c(a+b)})\\ &\geq \frac{1}{4}(4bc-(ab+ac)+4ca-(ab+bc)+4ab-(ac+bc))\\ &= \frac{1}{2}(bc+ca+ab)\\ &\geq \frac{1}{2}(3(bc\cdot ca\cdot ab)^{1/3})\\ &= \frac{3}{2}\end{aligned}$$

(Q.E.D)

6 Real Problems

Assume in all of these problems, the unknowns are all positive reals.

1. Prove the extended form of the AM-GM inequality, which is the QM-AM-GM-HM inequality:

$$\sqrt{\frac{a_1^2+...+a_n^2}{n}} \geq \frac{a_1+...+a_n}{n} \geq (a_1...a_n)^{1/n} \geq \frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}}$$

   (You just need to prove the left and right hand side.)

2. If $3a+b=1$, what is the minimum value of a^2+b^2? (2017/HuaLuoGeng)

3. Prove that $a^2+b^2+c^2 \geq ab+bc+ca$.

4. If $a+b+c=6$, what is the maximum value of a^3b^2c? (2017/ChenJingRun)

5. If $a+b \leq 1$, what is the minimum value of $ab+\frac{1}{ab}$? (BIMO Junior Test 2018)