A Semester Course on Finite-Dimensional Lie Algebras

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Lectures

Lecture 1

Basic Definitions

Lie algebras arose as the main instrument in the study of Lie groups in the work of Sophus Lie (a Norwegian mathematician) in the second half of the 19th century.

However there are many other sources of Lie algebras, and by now it has become clear that the notion of a Lie algebra is more fundamental. It gave birth to many other mathematical theories which play an important role in mathematics and physics.

Definition 1.1. An ***algebra*** is a vector space A over a field $\mathbb{F}$, endowed with a binary operation $A \times A \to A$, also called a product, defined by $(a, b) \mapsto ab$, which is *bilinear*: this means that for all $a, b, c \in A$ and $\lambda, \mu \in \mathbb{F}$,

$$a(\lambda b + \mu c) = \lambda ab + \mu ac \qquad \text{and} \qquad (\lambda b + \mu c)a = \lambda ba + \mu ca.$$

Examples 1.1.

1. Given a vector space V, the space of all endomorphisms (i.e. linear operators) of V, denoted $\mathrm{End}(V)$, with the composition operation is an associative algebra: $(ab)c = a(bc)$ for all $a, b, c \in \mathrm{End}(V)$.
2. Special case: The set of all n-by-n matrices with entries in $\mathbb{F}$, denoted $\mathrm{Mat}_{n\times n}(\mathbb{F})$, with matrix multiplication, is an associative algebra.

Definition 1.2. A ***subalgebra*** B of an algebra A is a subspace of A that is closed under the binary operation: $ab \in B$ for all $a, b \in B$.

Definition 1.3. A ***Lie algebra*** is an algebra with a bilinear binary operation denoted $[a, b]$ (instead of ab), called ***bracket***, which satisfies the following two axioms:

- $[a, a] = 0$ (skew-commutativity)
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ (Jacobi identity)

Remarks.

1. Skew-commutativity implies that $[a, b] = -[b, a]$ because

$$0 = [a + b, a + b] = [a, a] + [b, b] + [a, b] + [b, a] = [a, b] + [b, a].$$

The converse is true if $\mathrm{char}\, \mathbb{F} \neq 2$.

2. One usually writes the Jacobi identity as

$$[a,[b,c]] + \text{cycl} = 0$$

where cycl stands for the two clockwise cyclic permutations: $a \to b \to c$ and $b \to c \to a$.

Examples 1.2.

1. The vector space $\mathfrak{g}$ with bracket $[a,b] = 0$ for all $a, b \in \mathfrak{g}$. This is called an ***abelian*** Lie algebra, and is denoted $\mathfrak{ab}_n$ if $\dim \mathfrak{g} = n$.

2. $\mathfrak{g} = \mathbb{R}^3$ with $[a,b] = a \times b$ (cross product). It was Jacobi who first proved that it satisfies the Jacobi identity.

3. Let A be an algebra with a product ab. Denote by A_- the vector space A with the bracket $[a,b] = ab - ba$. Then A_- is a Lie algebra, if the algebra A is associative.

Exercise 1.1. Given an algebra A, show that the Jacobi identity in A_- holds in the following four situations (of course, skew-commutativity automatically holds):

1. 2-member identity: $(ab)c = a(bc)$, i.e. A is an associative algebra.

2. Two 3-member identities: $(ab)c + \text{cycl} = 0$ and $a(bc) + \text{cycl} = 0$.

3. 4-member identity: $a(bc) - (ab)c = a \leftrightarrow b$, where the RHS means that a and b are permuted in the LHS. This is called a ***left symmetric algebra***. The same claim holds for right symmetric algebras, when RHS $= b \leftrightarrow c$ (instead of $a \leftrightarrow b$).

4. 6-member identity: $[a, bc] + \text{cycl} = 0$.

Examples 1.3.

1. $\mathfrak{gl}_V = \text{End}(V)_-$, called the ***general linear*** Lie algebra. In the case $V = \mathbb{F}^n$, one denotes $\mathfrak{gl}_V = \mathfrak{gl}_n(\mathbb{F})$, the set of all n-by-n matrices with entries in $\mathbb{F}$ with the bracket $[a,b] = ab - ba$.

2. $\mathfrak{sl}_n(\mathbb{F}) = \{a \in \mathfrak{gl}_n(\mathbb{F}) \mid \text{tr}(a) = 0\}$.

3. Let B be a bilinear form on the vector space V, and consider the following subalgebra $\mathfrak{o}_{V,B} = \{a \in \mathfrak{gl}_V \mid B(au,v) = -B(u,av)\ \forall u,v \in V\}$.

Remark. A subalgebra of a Lie algebra is a Lie algebra.

Exercise 1.2. Show that $\mathrm{tr}[a,b] = 0$ for all $a, b \in \mathrm{Mat}_{n\times n}(\mathbb{F})$. This implies that $\mathfrak{sl}_n(\mathbb{F})$ is a subalgebra of $\mathfrak{gl}_n(\mathbb{F})$. It is called the ***special linear*** Lie algebra.

Exercise 1.3. Show that $\mathfrak{o}_{V,B}$ is a subalgebra of the Lie algebra $\mathfrak{gl}_V$.

Exercise 1.4. Let $V = \mathbb{F}^n$ and let B be the matrix of a bilinear form in the standard basis of $\mathbb{F}^n$. Show that

$$\mathfrak{o}_{\mathbb{F}^n,B} = \left\{a \in \mathfrak{gl}_n(\mathbb{F}) \mid a^\top B + Ba = 0\right\}$$

where $a^\top$ denotes the transpose matrix of a. Special cases of $\mathfrak{o}_{\mathbb{F}^n,B}$ are the following:

- $\mathfrak{so}_{n,B}(\mathbb{F})$ if B is a non-singular symmetric n-by-n matrix. This is called the ***orthogonal*** Lie algebra.
- $\mathfrak{sp}_{n,B}(\mathbb{F})$ if B is a non-singular skew-symmetric n-by-n matrix. This is called the ***symplectic*** Lie algebra.

The series of Lie algebras $\mathfrak{gl}_n(\mathbb{F}), \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_{n,B}(\mathbb{F}), \mathfrak{sp}_{n,B}(\mathbb{F})$ are the most important examples called the ***classical Lie algebras***.

Exercise 1.5. Let $f : \mathrm{Mat}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be a linear function, such that $f([a,b]) = 0$ for all a, b. Show that $f(a) = \lambda\, \mathrm{tr}(a)$ for some $\lambda \in \mathbb{F}$, independent of a.

Important notation: If X, Y are subspaces of a Lie algebra $\mathfrak{g}$, then $[X, Y]$ denotes the span of all vectors $[x, y]$ where $x \in X, y \in Y$.

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra. In the above notation, a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. A subspace $\mathfrak{h}$ is called an ***ideal*** of $\mathfrak{g}$ if $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$; an ideal is obviously a subalgebra of $\mathfrak{g}$.

Definition 1.5. A ***derived subalgebra*** of a Lie algebra $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$.

Proposition 1.1. $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of a Lie algebra $\mathfrak{g}$, such that the factor algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an abelian Lie algebra.

Proof. If $a \in \mathfrak{g}$ and $b \in [\mathfrak{g}, \mathfrak{g}]$, then $[a, b] \in [\mathfrak{g}, \mathfrak{g}]$, and thus $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$. The fact that $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian is obvious. ■

Classification of all Lie algebras of dimension $\leqslant 2$

Dimension 1

$\mathfrak{g} = \mathbb{F}a$, and $[a, a] = 0$ by the skew-commutativity axiom, hence $\mathfrak{g} = \mathfrak{ab}_1$.

Dimension 2

$\mathfrak{g} = \mathbb{F}x + \mathbb{F}y$, where x, y is a basis of $\mathfrak{g}$. Then, clearly, $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}[x, y]$ is at most 1-dimensional (by skew-symmetry). Hence either $\mathfrak{g} = \mathfrak{ab}_2$, or else $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}b$ where $b \neq 0$. In that case, take $a \in \mathfrak{g} \backslash \mathbb{F}b$, so that a, b is a basis of $\mathfrak{g}$. Then $[a, b] \in [\mathfrak{g}, \mathfrak{g}] = \mathbb{F}b$, so that $[a, b] = \lambda b$ for some nonzero $\lambda \in \mathbb{F}$. Replacing a by $\lambda^{-1}a$, we get $[a, b] = b$. This Lie algebra is isomorphic to the subalgebra with zero second row in $\mathfrak{gl}_2(\mathbb{F})$, since taking

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we get $[a, b] = b$. This is the only 2-dimensional non-abelian Lie algebra, up to isomorphism.

Lecture 2

Some Sources of Lie Algebras

Recall that an algebra is a vector space A over a field $\mathbb{F}$ with a bilinear product ab, i.e. for $a, b, c \in A$ and $\lambda, \mu \in \mathbb{F}$:

$$a(\lambda b + \mu c) = \lambda ab + \mu ac \qquad \text{and} \qquad (\lambda b + \mu c)a = \lambda ba + \mu ca.$$

(More generally, there can be several products.)

Recall that a Lie algebra is an algebra with product denoted $[a, b]$, and called bracket, satisfying the skew-commutativity and the Jacobi identity axioms.

(a) From associative or more general algebras

Given an algebra A with product ab, we can form a new algebra A_- with the bilinear binary operation the commutator $[a, b] = ab - ba$. Then, as was discussed in Lecture 1, A_- is a Lie algebra if A is associative, or if it satisfies a variety of other conditions.

(b) As algebra of derivations of an algebra

Definition 2.1. For any algebra A over a field $\mathbb{F}$, a ***derivation*** of A is an endomorphism D of A, viewed as a vector space over $\mathbb{F}$, satisfying

$$D(ab) = D(a)b + aD(b).$$

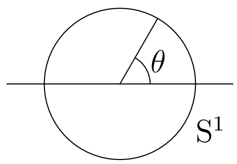
Let $\mathrm{Der}(A) \subseteq \mathfrak{gl}_A$ be the vector space over $\mathbb{F}$ of all derivations of the algebra A.

Exercise 2.1. Prove that $\mathrm{Der}(A)$ is a subalgebra of the Lie algebra $\mathfrak{gl}_A$, and thus it is a Lie algebra.

Geometric Picture. Let $\mathcal{F}$ be the algebra of smooth functions on a manifold, then $\mathrm{Der}(\mathcal{F})$ is the Lie algebra of all vector fields on this manifold. For example, on $\mathbb{R}^2$, all vector fields are

$$P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}.$$

A very important example for string theory: complex valued vector fields on the circle defined by $f(\theta)\dfrac{\mathrm{d}}{\mathrm{d}\theta}$, where $f(\theta)$ is a complex valued function on S^1:



Letting $L_n = -e^{2\pi i n\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}$, we see that

$$[L_m, L_n] = (m-n)L_{m+n} \qquad (m, n \in \mathbb{Z}).$$

This Lie algebra has a remarkable central extension, called the ***Virasoro algebra***:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}C \qquad \text{where } [C, L_m] = 0, \quad m \in \mathbb{Z}.$$

Bonus Problem. Prove that this is a Lie algebra.

For an element a of a Lie algebra $\mathfrak{g}$, define a map

$$\text{ad } a : \mathfrak{g} \to \mathfrak{g}, \ b \mapsto [a, b].$$

This map is referred to as the ***adjoint operator***. Rewriting the Jacobi identity as

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]],$$

we see that ad a is a derivation of the Lie algebra $\mathfrak{g}$. Derivations of this form are referred as ***inner*** derivations of $\mathfrak{g}$.

Proposition 2.1. Inner derivations form an ideal of $\mathrm{Der}(\mathfrak{g})$. More precisely,

$$[D, \text{ad } a] = \text{ad } D(a) \qquad \text{for all } D \in \mathrm{Der}(\mathfrak{g}), \ a \in \mathfrak{g}.$$

Proof. Apply both sides to $b \in \mathfrak{g}$:

$$[D, \text{ad } a]\, b = D[a, b] - [a, Db] \overset{(*)}{=} [Da, b] = (\text{ad } D(a))\, b$$

where $(*)$ is true since D is a derivation of $\mathfrak{g}$. ■

Bonus Problem. If A is an associative algebra, then ad a, defined by $(\text{ad } a)(b) = ab - ba$, is a derivation of A. Prove that for $A = \mathrm{End}(V)$, where V is a finite-dimensional vector space, these are all derivations.

(c) From Poisson brackets

Exercise 2.2. Let A be the algebra of smooth functions in $x_1, \cdots, x_n$. Define a ***Poisson bracket*** on A by

$$\{f, g\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \{x_i, x_j\} \qquad \text{for given choices } \{x_i, x_j\} \in A.$$

Show that this bracket satisfies the axioms of a Lie algebra if and only if

$$\{x_i, x_i\} = 0 \text{ for all } i, \qquad \{x_i, x_j\} = -\{x_j, x_i\} \text{ for all } i, j,$$
$$\text{and } x_i, x_j, x_k \text{ satisfy the Jacobi identity for all } i, j, k.$$

Example 2.1. Let A be the algebra of smooth functions in $p_1, \cdots, p_n, q_1, \cdots, q_n$ and let $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{p_i, q_j\} = -\{q_i, p_j\} = \delta_{ij}$. Then the conditions of Exercise 2.2 obviously hold, and we get

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

which is a Poisson bracket in classical mechanics.

These are special cases of the notion of a ***Poisson algebra***, which is a commutative associative algebra, endowed with a bracket $\{a, b\}$, satisfying the axioms of a Lie algebra, and the ***Leibniz rule***:

$$\{a, bc\} = \{a, b\} c + b \{a, c\}.$$

Given a family of associative algebras A_h, depending on a parameter h (i.e. an algebra over $\mathbb{F}[h]$, such that multiplication by h has trivial kernel), then hA_h is its ideal, and if $A = A_h / hA_h$ is commutative, it gets a well defined Poisson bracket, given by

$$\{a, b\} = \lim_{h \to 0} \frac{\left\{\widetilde{a}, \widetilde{b}\right\}}{h},$$

where $\widetilde{a}, \widetilde{b}$ are some preimages of a and b respectively under the canonical map $A_h \to A$.

Recovering A_h from the Poisson algebra A is called ***quantization***.

(d) Via structure constants

Given a basis $e_1, e_2, \cdots$ of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, the bracket is determined by the structure constants $c_{ij}^k \in \mathbb{F}$, defined by

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

The skew-commutativity axiom means

$$c_{ii}^k = 0, \qquad c_{ij}^k = -c_{ji}^k,$$

and a more complicated quadratic condition corresponds to the Jacobi identity.

However, changing basis changes the structure constants dramatically, so it is difficult to see from the structure constants whether we have isomorphic Lie algebras.

General Remark. Algebraic objects are considered up to isomorphism. Isomorphic objects are indistinguishable.

Given two algebras $\mathfrak{g}_1, \mathfrak{g}_2$ (with one or more products), a ***homomorphism*** $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a linear map over $\mathbb{F}$, which preserves these products:

$$\varphi([a, b]) = [\varphi(a), \varphi(b)] \qquad \text{for Lie algebras.}$$

The homomorphism φ is called an ***isomorphism*** if φ is bijective.

If there exists an isomorphism φ, we say that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ***isomorphic***, written $\mathfrak{g}_1 \simeq \mathfrak{g}_2$. For example, we proved last time that any 2-dimensional Lie algebra is isomorphic either to the abelian Lie algebra $\mathfrak{ab}_2$, or to the Lie algebra of 2-by-2 matrices with zero second row.

Exercise 2.3. Let $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of algebras. Then

(a) $\mathrm{Ker}(\varphi)$ is an ideal of $\mathfrak{g}_1$.

(b) $\mathrm{Im}(\varphi)$ is a subalgebra of $\mathfrak{g}_2$.

(c) $\mathrm{Im}(\varphi) \simeq \mathfrak{g}_1/\mathrm{Ker}(\varphi)$ (the Fundamental Homomorphism Theorem).

(e) As the Lie algebra of an algebraic group

Definition 2.2. A (linear) ***algebraic group*** G over a field $\mathbb{F}$ is a collection $\{P_\alpha\}_{\alpha\in I}$ of polynomials on the space of matrices $\mathrm{Mat}_{n\times n}(\mathbb{F})$, such that for any unital commutative associative algebra A over $\mathbb{F}$, the set

$$G(A) = \{g \in \mathrm{Mat}_{n\times n}(A) \mid g \text{ is invertible, and } P_\alpha(g) = 0 \text{ for all } \alpha \in I\}$$

is a group under matrix multiplication.

Examples 2.2.

1. The general linear group GL_n is defined by the empty set of polynomials, so that $GL_n(A)$ is the set of all invertible n-by-n matrices with entries in A. This is a group for any A, so that GL_n is an algebraic group.
2. The special linear group SL_n corresponds to $\{P_\alpha\} = \{\det(x_{ij}) - 1\}$, so that $SL_n(A)$ is a subgroup of $GL_n(A)$ of n-by-n matrices with determinant 1.

Exercise 2.4. Given $B \in \mathrm{Mat}_{n\times n}(\mathbb{F})$, let

$$O_{n,B}(A) = \left\{g \in GL_n(A) \mid g^\top B g = B\right\}.$$

Show that $O_{n,B}$ is an algebraic group.

Definition 2.3. Define the ***algebra of dual numbers*** D over a field $\mathbb{F}$ by

$$D = \mathbb{F}[\varepsilon]/(\varepsilon^2) = \left\{a + b\varepsilon \mid a, b \in \mathbb{F},\ \varepsilon^2 = 0\right\}.$$

The ***Lie algebra*** Lie G ***of an algebraic group*** G is

$$\mathrm{Lie}\, G = \{X \in \mathfrak{gl}_n(\mathbb{F}) \mid I_n + \varepsilon X \in G(D)\},$$

where I_n is the n-by-n identity matrix.

Example 2.3. Lie $GL_n = \mathfrak{gl}_n(\mathbb{F})$, since $(I_n + \varepsilon X)^{-1} = I_n - \varepsilon X$ (Intuitively, $I_n - \varepsilon X$ approximates the inverse to order 2, which is ignored over D).

Exercise 2.5. Prove that

(a) Lie $SL_n = \mathfrak{sl}_n(\mathbb{F})$,

(b) Lie $O_{n,B} = \mathfrak{o}_{\mathbb{F}^n,B}$.

Theorem 2.1. Lie G is a subalgebra of the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$.

Proof. We first show that Lie G is a subspace of $\mathfrak{gl}_n(\mathbb{F})$. Indeed, $X = (x_{ij}) \in$ Lie G if and only if $P_\alpha(I_n + \varepsilon X) = 0$ for all α.

Since $\varepsilon^2 = 0$, the Taylor expansion is

$$P_\alpha(I_n + \varepsilon X) = P_\alpha(I_n) + \varepsilon \sum_{i,j=1}^{n} \frac{\partial P_\alpha}{\partial x_{ij}}(I_n)x_{ij}.$$

But $P_\alpha(I_n) = 0$ since every group contains the identity. Hence $P_\alpha(I_n + \varepsilon X)$ is linear in (x_{ij}), so that Lie G is a subspace of $\mathfrak{gl}_n(\mathbb{F})$.

Next, suppose $X, Y \in$ Lie G. We wish to prove $[X, Y] = XY - YX \in$ Lie G. We have

$$I_n + \varepsilon X \in G\left(\mathbb{F}[\varepsilon]/(\varepsilon^2)\right) \quad \text{and} \quad I_n + \varepsilon' Y \in G\left(\mathbb{F}[\varepsilon']/(\varepsilon'^2)\right).$$

Viewing these as elements of $G\left(\mathbb{F}[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2)\right)$, we have

$$\begin{aligned}
&(I_n + \varepsilon X)(I_n + \varepsilon' Y)(I_n + \varepsilon X)^{-1}(I_n + \varepsilon' Y)^{-1} \\
&= (I_n + \varepsilon X)(I_n + \varepsilon' Y)(I_n - \varepsilon X)(I_n - \varepsilon' Y) \\
&= I_n + \varepsilon\varepsilon'(XY - YX) \\
&\in G\left(\mathbb{F}[\varepsilon\varepsilon']/((\varepsilon\varepsilon')^2)\right) \subseteq G\left(\mathbb{F}[\varepsilon, \varepsilon']/(\varepsilon^2, \varepsilon'^2)\right).
\end{aligned}$$

Since $\mathbb{F}[\varepsilon\varepsilon']/((\varepsilon\varepsilon')^2) \simeq D$, we see that $XY - YX \in$ Lie G. ■

(f) From quantum field theory

A ***vertex operator algebra*** is a vector space V with the vacuum vector 1 and bilinear products $a_n b$ for each $n \in \mathbb{Z}$ such that $a_n b = 0$ for $n \gg 0$, subject to the following axioms:

- $1_n a = \delta_{n,-1} a$, and $a_{-1} 1 = a$, (vacuum axiom)
- $\displaystyle\sum_{j=0}^{\infty} \binom{m}{j} (a_{n+j}b)_{m+k-j}\, c = \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} \left(a_{m+n-j}(b_{k+j}c) - (-1)^n b_{n+k-j}(a_{m+j}c)\right)$

for all $m, n, k \in \mathbb{Z}$. (Borcherds identity).

Bonus Problem. Let $TV = \{a_{-2}1 \mid a \in V\}$. Then $[a, b] = a_0 b$ is a well-defined Lie algebra bracket on V/TV.

Bonus Problem. $[a, b] = a_0 b$ and $ab = a_{-1}b$ give a well-defined Poisson algebra structure on $V/(TV)$, where (TV) denotes the 2-sided ideal generated by TV.

Lecture 3

Engel's Theorem

The notion of a representation is very important in the study of an algebraic structure. The proof of Engel's Theorem is a nice demonstration of this principle.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let V be a vector space over $\mathbb{F}$. A ***representation*** of $\mathfrak{g}$ in V is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}_V, \qquad a \mapsto \pi(a).$$

In other words, it is a linear map $a \mapsto \pi(a)$ from $\mathfrak{g}$ to End V, such that

$$\pi\left([a, b]\right) = \pi(a)\pi(b) - \pi(b)\pi(a).$$

Examples 3.1.

1. ***Trivial representation*** of $\mathfrak{g}$ in V:

$$\pi(a) = 0 \quad \text{for all } a \in \mathfrak{g}.$$

2. ***Adjoint representation*** ad of $\mathfrak{g}$ in $\mathfrak{g}$:

$$a \mapsto \text{ad } a, \quad a \in \mathfrak{g}, \quad \text{where } (\text{ad } a)\, b = [a, b].$$

In order to check that it is a representation, we need to show that, for $a, b \in \mathfrak{g}$,

$$\text{ad}\,[a, b] = (\text{ad } a)(\text{ad } b) - (\text{ad } b)(\text{ad } a).$$

Applying both sides to $c \in \mathfrak{g}$, we get

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]]$$

which is just the Jacobi identity by skew-commutativity of the bracket. (Another proof is immediate by Proposition 2.1.)

3. ***Tautological representation*** of $\mathfrak{gl}_V$ in V and $\mathfrak{o}_{V,B}$ in V:

$$\pi(a) = a.$$

Definition 3.2. The ***center*** of a Lie algebra $\mathfrak{g}$ is $Z(\mathfrak{g}) = \{a \in \mathfrak{g} \mid [a, \mathfrak{g}] = 0\}$. Clearly, $Z(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$.

Exercise 3.1. Show that $Z(\mathfrak{gl}_n(\mathbb{F})) = \mathbb{F}I_n$ and

$$Z(\mathfrak{sl}_n(\mathbb{F})) = \begin{cases} 0 & \text{if } \operatorname{char} \mathbb{F} \nmid n, \\ \mathbb{F}I_n & \text{otherwise.} \end{cases}$$

Proposition 3.1. The adjoint representation defines an embedding of the Lie algebra $\mathfrak{g}/Z(\mathfrak{g})$ in $\mathfrak{gl}_{\mathfrak{g}}$.

Proof. $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ is a homomorphism with kernel $Z(\mathfrak{g})$. The proposition follows from the Fundamental Homomorphism Theorem. ■

Ado's Theorem. Any finite-dimensional Lie algebra over $\mathbb{F}$ embeds in $\mathfrak{gl}_n(\mathbb{F})$ for some n.

We present this theorem without proof. Proposition 3.1 proves it when $Z(\mathfrak{g}) = 0$. ■

Define the ***Heisenberg Lie algebra*** $\mathfrak{heis}_{2n+1}$ to be the $(2n+1)$-dimensional Lie algebra with basis $\{p_1, \cdots, p_n, q_1, \cdots, q_n, c\}$ and all non-zero brackets

$$[p_i, q_i] = -[q_i, p_i] = c, \qquad i = 1, \cdots, n.$$

It is very important in quantum mechanics.

Exercise 3.2. Let $n = \dim \mathfrak{g}$. Prove that $\dim Z(\mathfrak{g}) \neq n-1$.

Exercise 3.3. Prove that any n-dimensional Lie algebra, for which $\dim Z(g) = n-2$, is isomorphic either to $\mathfrak{ab}_{n-3} \oplus \mathfrak{heis}_3$ or to $\mathfrak{ab}_{n-2} \oplus \mathfrak{h}$, where $\oplus$ denotes the direct sum of Lie algebras; $\mathfrak{ab}_j$ denotes the j-dimensional abelian Lie algebra; and $\mathfrak{h}$ the 2-dimensional non-abelian Lie algebra.

Construction of representations from given ones

(a) Direct sum of representations

Given representations π_i $(i = 1, \cdots, k)$ of $\mathfrak{g}$ in vector spaces V_i, we have their direct sum

$$(\pi_1 \oplus \cdots \oplus \pi_k)(a) = \pi_1(a) \oplus \cdots \oplus \pi_k(a), \qquad a \in \mathfrak{g},$$

in the vector space $V_1 \oplus \cdots \oplus V_k$.

(b) Subrepresentations and factor representations

Given a representation π of $\mathfrak{g}$ in V, if a subspace U in V is invariant with respect to all operators $\pi(a)$, $a \in \mathfrak{g}$, we have the subrepresentation π_U of $\mathfrak{g}$ in U:

$$a \mapsto \pi(a)\mid_U,$$

and the factor representation of $\mathfrak{g}$ in V/U:

$$a \mapsto \pi(a)\mid_{V/U}.$$

(c) Restriction of a representation of $\mathfrak{g}$ in V to a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$

Definition 3.3. A linear operator A on a vector space V is called ***nilpotent*** if $A^N = 0$ for some positive integer N.

Exercise 3.4. Show that if $\dim V < \infty$, then A is a nilpotent operator on V if and only if all its eigenvalues are 0.

Lemma 3.1. Let A be a nilpotent operator on a vector space V. Then

(a) There exists a non-zero $v \in V$, such that $Av = 0$.

(b) ad A is a nilpotent operator on $\mathfrak{gl}_V$.

Proof. (a) Consider the minimal positive integer N such that $A^N = 0$, then $A^{N-1} \neq 0$. Choose a non-zero vector $v \in A^{N-1}V$. Then $Av = 0$.

(b) Note that

$$\text{ad } A = L_A - R_A, \tag{1}$$

where $L_A(B) = AB$ and $R_A(B) = BA$. Furthermore $L_A R_B = R_B L_A$ due to associativity of product of operators:

$$L_A R_B(C) = A(CB) = (AC)B = R_B L_A(C).$$

Hence we may apply the binomial formula to (1):

$$(\text{ad } A)^M = \sum_{j=0}^{M} \binom{M}{j} L_A^j R_A^{M-j}.$$

Apply both sides to B with $M = 2N$:

$$(\text{ad } A)^{2N} B = \sum_{j=0}^{2N} \binom{2N}{j} A^j B A^{2N-j},$$

which is zero, since either $j \geqslant N$ or $2N - j \geqslant N$. ■

Theorem 3.1. *(Engel's Theorem)* Let V be a non-zero vector space (not necessarily finite-dimensional) and let $\mathfrak{g} \subseteq \mathfrak{gl}_V$ be a finite-dimensional subalgebra, consisting of nilpotent operators. Then there exists a non-zero vector $v \in V$ such that

$$Av = 0 \qquad \text{for all } A \in \mathfrak{g}.$$

Remark. If we assume $\dim V < \infty$, then $\dim \mathfrak{g} \leqslant (\dim V)^2$ is automatically finite.

Proof of Engel's Theorem. We use induction on $\dim \mathfrak{g}$.

If $\dim \mathfrak{g} = 1$, then $\mathfrak{g} = \mathbb{F}A$ for $A \in \mathfrak{gl}_V$ and, by Lemma (a), Engel's Theorem holds.

Henceforth assume that $\dim \mathfrak{g} \geqslant 2$, and let $\mathfrak{h}$ be a maximal proper subalgebra of $\mathfrak{g}$. Since $[a, a] = 0$, we have that $\dim \mathfrak{h} \geqslant 1$.

Step 1. $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$.

Consider the adjoint representation of $\mathfrak{g}$ (on itself), and its restriction to $\mathfrak{h}$, so we have $\mathfrak{h}$ is an invariant subspace of $\mathfrak{g}$ for this representation of $\mathfrak{h}$ on $\mathfrak{g}$ (since $\mathfrak{h}$ is a subalgebra). Hence we may consider the factor representation π of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$. Then $\pi(\mathfrak{h}) \subseteq \mathfrak{gl}_{\mathfrak{g}/\mathfrak{h}}$ and $\dim \pi(\mathfrak{h}) \leqslant \dim \mathfrak{h} < \dim \mathfrak{g}$. But by Lemma 3.1(b), $\pi(\mathfrak{h})$ consists of nilpotent operators on $\mathfrak{g}/\mathfrak{h}$.

Hence we may apply the inductive assumption: there exists a non-zero vector $\overline{a} \in \mathfrak{g}/\mathfrak{h}$, such that $\pi(\mathfrak{h})\overline{a} = 0$ for all $h \in \mathfrak{h}$. If $a \in \mathfrak{g}$ is an arbitrary preimage of $\overline{a}$ under the map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, we get that

$$[\mathfrak{h}, a] \subseteq \mathfrak{h}, \tag{2}$$

and $a \notin \mathfrak{h}$ since $\overline{a} \neq 0$. Hence $\mathfrak{h} + \mathbb{F}a$ is a subalgebra of $\mathfrak{g}$, larger than $\mathfrak{h}$. Since $\mathfrak{h}$ is a (maximal) proper subalgebra of $\mathfrak{g}$, we conclude that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}a, \tag{3}$$

and (2) and (3) show that $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$.

Step 2. By the inductive assumption, there exists a non-zero vector $v \in V$ such that $Av = 0$ for all $A \in \mathfrak{h}$. Let V_0 denote the subspace of all vectors, satisfying $Av = 0$ for all

$A \in \mathfrak{h}$. It is a non-zero subspace.

We claim that $aV_0 \subseteq V_0$. Indeed, $V_0 = \{v \in V \mid \mathfrak{h}(v) = 0\}$. So, if $v \in V_0$, then we have

$$h(av) = [h, a]v + ah(v) = 0 + 0 = 0,$$

since $[h, a] \in \mathfrak{h}$. By Lemma 3.1(a) there exists a non-zero vector $v \in V_0$, annihilated by a. Since v is also annihilated by $\mathfrak{h}$, we conclude, by (3), that v is annihilated by $\mathfrak{g}$. ■

Corollary. Let $\pi : \mathfrak{g} \to \mathfrak{gl}_V$ be a representation of a Lie algebra $\mathfrak{g}$ in a finite-dimensional vector space V, such that $\pi(a)$ is a nilpotent operator for all $a \in \mathfrak{g}$. Then there exists a basis of V, with respect to which all operators $\pi(a)$ $(a \in \mathfrak{g})$ have strictly upper triangular matrices. In particular, any subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}_V$, where $\dim V < \infty$, consisting of nilpotent operators, is a subalgebra of the Lie algebra of strictly upper triangular matrices in some basis of V.

Proof. By induction on $\dim V$. By Engel's Theorem, there exists a non-zero vector e_1 such that $\pi(a)e_1 = 0$ for all $a \in \mathfrak{g}$. Since $\mathbb{F}e_1$ is a $\mathfrak{g}$-invariant subspace of V, we may consider the factor representation of $\mathfrak{g}$ in $V/\mathbb{F}e_1$.

Applying the inductive assumption, we get a basis $\overline{e}_2, \cdots, \overline{e}_n$ of $V/\mathbb{F}e_1$ in which al matrices of $\pi_{V/\mathbb{F}e_1}$ are strictly upper triangular.

Take arbitrary preimages $e_2, \cdots, e_n$ of $\overline{e}_2, \cdots, \overline{e}_n$. Then in the basis $e_1, \cdots, e_n$ of V, all matrices of the operators in $\pi(\mathfrak{g})$ are strictly upper triangular. ■

Exercise 3.5. Construct in $\mathfrak{sl}_3(\mathbb{F})$ a 2-dimensional subspace, consisting of nilpotent matrices, which do not have a common eigenvector. (Hence the assumption in Engel's theorem that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}_V$ is essential)

Hint: Consider the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Bonus problem. *(Very difficult)* If $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ are matrices for which all their linear combinations are diagonalizable, then $[A, B] = 0$. This is called the *Motzkin-Taussky theorem.* Hence, the nilpotent case is dramatically different from the diagonalizable case.

Lecture 4

Nilpotent and Solvable Lie Algebras

A ***flag*** in a d-dimensional vector space V is a sequence of subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_d = V, \qquad \text{where } \dim V_j = j.$$

To such a flag we may associate two associative subalgebras of End V:

$$B_d = \{A \in \text{End } V \mid AV_j \subseteq V_j \text{ for all } j \geqslant 0\},$$
$$N_d = \{A \in \text{End } V \mid AV_j \subseteq V_{j-1} \text{ for all } j \geqslant 1\}.$$

Note that the product of any d operators from N_d is 0, i.e N_d is a nilpotent (associative) subalgebra of End V. Note that $N_d \subseteq B_d$.

Choosing a basis $e_1, \cdots, e_d$ of V, we can construct a flag by letting

$$0 = V_0 \subset V_1 = \mathbb{F}e_1 \subset V_2 = \mathbb{F}e_1 + \mathbb{F}e_2 \subset \cdots \subset V_d = \mathbb{F}^d.$$

In this basis B_d consists of upper triangular matrices, and N_d of strictly upper triangular matrices. Then $(B_d)_- = \mathfrak{b}_d$ and $(N_d)_- = \mathfrak{n}_d$ are subalgebras of the Lie algebra $\mathfrak{gl}_d(\mathbb{F})$.

Exercise 4.1. Show that $\mathfrak{n}_d = [\mathfrak{b}_d, \mathfrak{b}_d]$.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The ***central series*** of $\mathfrak{g}$ is the descending chain of subspaces

$$\mathfrak{g}^1 := \mathfrak{g} \supseteq \mathfrak{g}^2 := \left[\mathfrak{g}, \mathfrak{g}^1\right] \supseteq \mathfrak{g}^3 := \left[\mathfrak{g}, \mathfrak{g}^2\right] \supseteq \cdots \supseteq \mathfrak{g}^n := \left[\mathfrak{g}, \mathfrak{g}^{n-1}\right] \supseteq \cdots,$$

while the ***derived series*** of $\mathfrak{g}$ is

$$\mathfrak{g}^{(0)} := \mathfrak{g} \supseteq \mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^{(2)} := \left[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right] \supseteq \cdots \supseteq \mathfrak{g}^{(n)} := \left[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}\right] \supseteq \cdots.$$

Note that $\mathfrak{g}^2 = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ is the derived subalgebra of $\mathfrak{g}$, and that

(1) $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^{n+1}$ for $n \geqslant 1$ by induction on n.

(2) All $\mathfrak{g}^{(n)}$ and $\mathfrak{g}^n$ are ideals of $\mathfrak{g}$.

Definition 4.2. A Lie algebra $\mathfrak{g}$ is called ***nilpotent*** (resp. ***solvable***) if $\mathfrak{g}^n = 0$ (resp. $\mathfrak{g}^{(n)} = 0$) for some $n > 0$.

Note that we have for Lie algebras

$$\{\text{abelian}\} \subsetneq \{\text{nilpotent}\} \subsetneq \{\text{solvable}\}$$

Examples 4.1.

1. $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$, $[a,b] = b$. Then $\mathfrak{g}^{(1)} = \mathfrak{g}^2 = [\mathfrak{g},\mathfrak{g}] = \mathbb{F}b$, $\mathfrak{g}^3 = \mathfrak{g}^4 = \cdots = \mathbb{F}b$, while $\mathfrak{g}^{(2)} = 0$. Hence $\mathfrak{g}$ is a solvable, but not nilpotent, Lie algebra.

2. $\mathfrak{heis}_{2n+1}$ for $n \geqslant 1$ is a nilpotent Lie algebra since its derived subalgebra is central, hence $\mathfrak{g}^3 = 0$.

Exercise 4.2. $\mathfrak{b}_d$ is a solvable (but not nilpotent for $d \geqslant 2$) Lie algebra, and $\mathfrak{n}_d$ is a nilpotent Lie algebra.

Obviously any subalgebra of a nilpotent (resp. solvable) Lie algebra is nilpotent (resp. solvable) Lie algebra and the same holds for factor algebras by ideals.

Exercise 4.3. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ its ideal. Prove that if $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable Lie algebras, then $\mathfrak{g}$ is solvable too.

Example 4.2. $\mathfrak{g} = \mathbb{F}a + \mathbb{F}b$, $[a,b] = b$. Then $\mathbb{F}b$ is an abelian (hence nilpotent) ideal and the factor algebra $\mathfrak{g}/\mathbb{F}\mathfrak{b}$ is abelian. But $\mathfrak{g}$ is not a nilpotent Lie algebra.

Theorem 4.1.

(a) If $\mathfrak{g}$ is a non-zero nilpotent Lie algebra, then its center $Z(\mathfrak{g})$ is non-zero.

(b) If $\mathfrak{g}$ is a Lie algebra, such that $\mathfrak{g}/Z(\mathfrak{g})$ is a nilpotent Lie algebra, then $\mathfrak{g}$ is nilpotent.

Proof.

(a) Take the minimal positive integer N such that $\mathfrak{g}^N = 0$. Since $\mathfrak{g} \neq 0$, $N \geqslant 2$, but then $\mathfrak{g}^{N-1} \neq 0$ and $\left[\mathfrak{g}, \mathfrak{g}^{N-1}\right] = \mathfrak{g}^N = 0$, so $\mathfrak{g}^{N-1} \subseteq Z(\mathfrak{g})$.

(b) The Lie algebra $\overline{\mathfrak{g}} = \mathfrak{g}/Z(\mathfrak{g})$ nilpotent means that $\overline{\mathfrak{g}}^n = 0$ for some $n \geqslant 1$, hence $\mathfrak{g}^n \subseteq Z(\mathfrak{g})$ and $\mathfrak{g}^{n+1} = 0$. ∎

Engel's characterization of nilpotent Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is a nilpotent Lie algebra if and only if the operator $\operatorname{ad} a$ is nilpotent for each $a \in \mathfrak{g}$.

Proof. If $\mathfrak{g}$ is a nilpotent Lie algebra, then $\mathfrak{g}^{n+1} = 0$ for some positive integer n. In particular, $(\text{ad } a)^n b = 0$ for all $a, b \in \mathfrak{g}$.

Conversely, the adjoint representation embeds $\mathfrak{g}/Z(\mathfrak{g})$ in $\mathfrak{gl}_{\mathfrak{g}}$, and by the assumption, it consists of nilpotent operators.

Hence, by Corollary of Engel's theorem, there is a basis of $\mathfrak{g}$ in which all operators from $\mathfrak{g}/Z(\mathfrak{g})$ are strictly upper triangular. Therefore, $\mathfrak{g}/Z(\mathfrak{g})$ is a nilpotent Lie algebra, and, by Theorem 4.1(b), $\mathfrak{g}$ is nilpotent too. ■

The meaning of this theorem is as follows. Let $(\text{ad } a_1) \cdots (\text{ad } a_{n-1}) a_n$ be a commutator of length n. For example, $(\text{ad } a)^{n-1} b$. Note that in a d-dimensional nilpotent Lie algebra $\mathfrak{g}$ any commutator of length $d+1$ is zero, since the central series of $\mathfrak{g}$ is strictly decreasing.

Thus, Engel's characterization theorem says that if all the commutators of length $d+1$ of the form $(\text{ad } a)^d b$ are zero in a nilpotent Lie algebra of dimension d, then all commutators of length $d+1$ are zero.

The famous Zelmanov's theorem says that in any, possible infinite-dimensional, Lie algebra, $(\text{ad } a)^d = 0$ for all a implies that all commutators of length $d+1$ are 0.

Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra. Recall that, by Theorem 4.1(a), $Z(\mathfrak{g}) \neq 0$, so that $\dim \mathfrak{g}/Z(\mathfrak{g}) < \dim \mathfrak{g}$.

Definition 4.3. Define inductively by $\dim \mathfrak{g}$ the notion of a ***n-step nilpotent Lie algebra*** $\mathfrak{g}$:

- 1-step nilpotent $\mathfrak{g}$ is abelian.
- 2-step nilpotent $\mathfrak{g}$ if $\mathfrak{g}/Z(\mathfrak{g})$ is abelian.
- k-step nilpotent $\mathfrak{g}$ if $\mathfrak{g}/Z(\mathfrak{g})$ is $(k-1)$-step nilpotent $(k \geqslant 2)$.

Exercise 4.4. Prove that any 2-step finite-dimensional nilpotent Lie algebra with 1-dimensional center is isomorphic to $\mathfrak{heis}_{2n+1}$ for some integer $n \geqslant 1$.

How to classify 2-step finite dimensional nilpotent Lie algebras?

Let $V = \mathfrak{g}/Z(\mathfrak{g})$; it is an abelian Lie algebra. Consider the bilinear form on V with values in $Z(\mathfrak{g})$:

$$B : V \times V \to Z(\mathfrak{g}), \quad (a, b) \mapsto \left[\widetilde{a}, \widetilde{b}\right],$$

where $\widetilde{a}, \widetilde{b}$ are some preimages of $a, b \in V$ respectively under the map $\mathfrak{g} \to V$.

Note that B is a well-defined alternating bilinear form, i.e. $B(x,x) = 0$ for all $x \in V$. Moreover, B is non-singular in the sense that $B(a,V) = 0$ implies that $a = 0$; otherwise $Z(\mathfrak{g})$ is larger.

Conversely, given a non-singualr alternating bilinear form B on a vector space V with values in a vector space Z, we can construct a 2-step nilpotent Lie algebra

$$\mathfrak{g}_B = V \oplus Z \qquad \text{(direct sum of vector spaces)},$$

for which $Z = Z(\mathfrak{g})$ and the bracket on V is defind by $[a,b] = B(a,b)$.

Exercise 4.5. Prove that $\mathfrak{g}_B$ is a Lie algebra and that $\mathfrak{g}_B \simeq \mathfrak{g}_{B_1}$ if and only if B and B_1 are isomorphic bilinear forms.

The classification of 2-step finite-dimensional nilpotent Lie algebras is equivalent to the classification of alternating non-singular bilinear forms on a finite-dimensional vector space V with values in a finite-dimensional vector space Z.

If $\dim Z = 1$ there is only one such bilinear form, up to isomorphism, which leads to Exercise 4.4. In the case $\dim Z = 2$ the problem was solved by G. Belitskii, R. Lipyanski and V. Sergeichuk in 2005. However it was proved in the same paper that the case dim $Z = 3$ is *impossible* (a 'wild' problem of linear algebra).

Remark. A problem of linear algebra is called ***wild*** if it contains the problem of classification of pairs of linear operators in a finite-dimensional vector space as a subproblem, otherwise the problem is called ***tame***. An example of a tame problem is the classification of linear operators in a finite-dimensional vector space (Jordan form). An example of a wild problem is classification of triples of linear maps A, B, C from U to V, where U and V are finite-dimensional vector spaces. Indeed, take $\dim U = \dim V$, and $C = I$. Then we get the problem of classification of pairs of linear operators on V. Classification of pairs of linear maps $A, B : U \to V$ is a tame problem.

Another example is classification of m-tuples of subspaces in a finite-dimensional vector space. This problem is tame for $m \leqslant 4$, but wild for $m \geqslant 5$. For example $U \subseteq V$ is determined by $\dim U$; 2-tuple $U_1, U_2 \subseteq V$ is determined by $\dim U_1, \dim U_2$ and $\dim U_1 \cap U_2$. For $m = 3$ it is a little more complicated, but the problem is still finite. For $m = 4$ the problem is tame, but infinite.

Lecture 5

Lie's Theorem

Definition 5.1. Let $\mathfrak{h}$ be a Lie algebra over a field $\mathbb{F}$, let $\pi : \mathfrak{h} \to \mathfrak{gl}_V$ be a representation of $\mathfrak{h}$ in a vector space V over $\mathbb{F}$, and let $\lambda : \mathfrak{h} \to \mathbb{F}$ be a linear function. We define the ***weight subspace*** of V for $\mathfrak{h}$, attached to λ, as

$$V_\lambda^{\mathfrak{h}} := \{v \in V \mid \pi(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda^{\mathfrak{h}} \neq 0$, we say that λ is a ***weight*** of the representation π.

Lie's Lemma. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic 0, and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{gl}_V$ be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V over $\mathbb{F}$. Then each weight space $V_\lambda^{\mathfrak{h}}$ for $\pi|_{\mathfrak{h}}$, where λ is a linear function on $\mathfrak{h}$, is invariant under $\pi(\mathfrak{g})$.

Proof. $\pi(a)v \in V_\lambda^{\mathfrak{h}}$ for $a \in \mathfrak{g}$ means

$$\pi(h)\left(\pi(a)v\right) = \lambda(h)\pi(a)v \quad \text{for all } h \in \mathfrak{h}.$$

We have, using that $[a, \mathfrak{h}] \subseteq \mathfrak{h}$,

$$\begin{aligned} \text{LHS} &= [\pi(h), \pi(a)]\, v + \pi(a)\pi(h)v \\ &= \pi([h,a])v + \lambda(h)\left(\pi(a)v\right) \\ &= \lambda([h,a])v + \lambda(h)\left(\pi(a)v\right). \end{aligned}$$

Hence it suffices to prove that

$$\lambda\left([h,a]\right) = 0 \quad \text{for all } h \in \mathfrak{h},\ a \in \mathfrak{g}.$$

Fix $a \in \mathfrak{g}$ and pick a non-zero $v \in V_\lambda^{\mathfrak{h}}$ (we may assume that $V_\lambda^{\mathfrak{h}} \neq 0$). Let $W_{-1} = 0$ and

$$W_m = \operatorname{span}\{v, \pi(a)v, \cdots, \pi(a)^m v\} \text{ if } m \geqslant 0.$$

Take the maximal integer N such that the vectors $v, \pi(a)v, \cdots, \pi(a)^N v$ are linearly independent (recall that $\dim V < \infty$). Then we have

$$W_{-1} \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_N = W_{N+1} = \cdots,$$

hence

$$\pi(a)W_N \subseteq W_N. \tag{1}$$

We shall prove by induction on $m \geqslant 0$ the following two properties

- $(2)_m : \pi(h)\pi(a)^m(v) - \lambda(h)\pi(a)^m(v) \in W_{m-1}$ for all $h \in \mathfrak{h}$,
- $(3)_m : W_m$ is $\pi(\mathfrak{h})$-invariant.

This is true for $m = 0$ since $v \in V_\lambda^{\mathfrak{h}}$. Suppose $m \geqslant 1$ and (2) and (3) hold for $m - 1$:

- $(2)_{m-1} : \pi(h)\pi(a)^{m-1}(v) - \lambda(h)\pi(a)^{m-1}(v) \in W_{m-2}$ for all $h \in \mathfrak{h}$,
- $(3)_{m-1} : W_{m-1}$ is $\pi(\mathfrak{h})$-invariant.

Let us prove $(2)_m$:

$$\begin{aligned}
&\pi(h)\pi(a)^m v - \lambda(h)\pi(a)^m v \\
&= \pi(h)\pi(a)\pi(a)^{m-1}v - \lambda(h)\pi(a)^m v \\
&= \pi\left([h,a]\right)\pi(a)^{m-1}v + \pi(a)\left(\pi(h)\pi(a)^{m-1}v - \lambda(h)\pi(a)^{m-1}v\right).
\end{aligned}$$

The first summand lies in W_{m-1} by $(3)_{m-1}$, and the second lies in W_{m-1} by $(2)_{m-1}$.

Let us prove $(3)_m$. By $(3)_{m-1}$, it suffices to check that $\pi(h)\pi(a)^m v \in W_m$. But, as above, it is equal to

$$\pi\left([h,a]\right)\pi(a)^{m-1}v + \pi(a)\left(\pi(h)\pi(a)^{m-1}v\right).$$

Both summands lie in W_m by $(3)_{m-1}$.

Now, by (1) and $(3)_N$, the space W_N is $\pi(a)$-invariant and $\pi(\mathfrak{h})$-invariant, and, by $(2)_N$, the matrix of $\pi(h)$ for $h \in \mathfrak{h}$ is upper triangular with $\lambda(h)$ on the diagonal. Hence, for $h \in \mathfrak{h}$ we have

$$\mathrm{tr}_{W_N}\pi\left([h,a]\right) = \mathrm{tr}_{W_N}\left[\pi(h),\pi(a)\right] = N\lambda\left([h,a]\right).$$

Since the trace of the commutator of two operators in a finite-dimensional vector space is 0, and char $\mathbb{F} = 0$, we conclude that $\lambda\left([h,a]\right) = 0$. ■

Exercise 5.1. Show that Lie's Lemma holds if char $\mathbb{F} > \dim V$.

Lie's Theorem. Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and let $\pi : \mathfrak{g} \to \mathfrak{gl}_V$ be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V over $\mathbb{F}$. Then there exists a linear function λ on $\mathfrak{g}$, such that $V_\lambda^{\mathfrak{g}} \neq 0$. In other words, there exists a common eigenvector v in V for all $\pi(a), a \in \mathfrak{g}$, such that the eigenvalue is linear in a.

Proof. We may assume that $\dim \mathfrak{g} < \infty$, replacing $\mathfrak{g}$ by $\pi(\mathfrak{g}) \subseteq \mathrm{End}\, V$. As $\mathfrak{g}$ is solvable, $\pi(\mathfrak{g})$ is solvable as well.

We shall prove Lie's Theorem by induction on $\dim \mathfrak{g}$. The case $\dim \mathfrak{g} = 0$ is trivial. Suppose that $d = \dim \mathfrak{g} \geqslant 1$ and we have proved Lie's Theorem for $\dim \mathfrak{g} = d - 1$, and we want to show that it holds for $\dim \mathfrak{g} = d$.

Since $\mathfrak{g}$ is a solvable Lie algebra of positive dimension, $\mathfrak{g}$ properly includes $[\mathfrak{g}, \mathfrak{g}]$, let $\mathfrak{h}$ be a subspace of codimension 1 in $\mathfrak{g}$, containing $[\mathfrak{g}, \mathfrak{g}]$. It is an ideal of $\mathfrak{g}$, hence we have

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}a \quad \text{(direct sum of vector spaces)}, \quad [a, \mathfrak{h}] \subseteq \mathfrak{h}.$$

Since $\mathfrak{h}$ is solvable of dimension $d - 1$, by the inductive hypothesis, $V^{\mathfrak{h}}_{\lambda'} \neq 0$ for some linear function λ' on $\mathfrak{h}$.

By Lie's Lemma, $V^{\mathfrak{h}}_{\lambda'}$ is $\mathfrak{g}$-invariant. In particular $aV^{\mathfrak{h}}_{\lambda'} \subseteq V^{\mathfrak{h}}_{\lambda'}$. Since $\mathbb{F}$ is algebraically closed, there exists a non-zero v in $V^{\mathfrak{h}}_{\lambda'}$, such that $av = \ell v$ for some $\ell \in \mathbb{F}$. Define a linear function λ on $\mathfrak{g}$, letting

$$\lambda(h + \mu a) = \lambda(h) + \mu \ell.$$

Then $v \in V^{\mathfrak{g}}_{\lambda}$, and the proof is complete. ■

Exercise 5.2. Consider the following representation of $\mathfrak{heis}_3 = \{p, q, c\}$ in $\mathbb{F}[x]$:

$$c \mapsto I_{\mathbb{F}[x]}, \quad p \mapsto \frac{\mathrm{d}}{\mathrm{d}x}, \quad q \mapsto \text{multiplication by } x.$$

Show that $x^p\mathbb{F}[x]$ is an invariant subspace with respect to $\mathfrak{heis}_3$ if $\operatorname{char} \mathbb{F} = p$, and that $\mathfrak{heis}_3$ has no weight in $V = \mathbb{F}[x]/x^p\mathbb{F}[x]$. This shows that Lie's Theorem fails if $\operatorname{char} \mathbb{F} = p$. Explain why this example also shows that Lie's Lemma fails over fields of characteristic p.

Exercise 5.3. Prove the following two corollaries of Lie's Theorem:

(a) For any representation π of a solvable Lie algebra $\mathfrak{g}$ in a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$ of characteristic 0 there exists a basis of V for which the matrices of $\pi(\mathfrak{g})$ are upper triangular.

(b) Under the same assumption on V and $\mathbb{F}$, a subalgebra of $\mathfrak{gl}_V$ is solvable if and only if it is contained in a subalgebra of upper triangular matrices for some basis of V.

Proposition 5.1. Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field of characteristic 0. Then $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra.

Proof. Recall that $\mathfrak{g}/Z(\mathfrak{g})$ is a subalgebra of $\mathfrak{gl}_{\mathfrak{g}}$, and that it is solvable since $\mathfrak{g}$ is. By

Exercise 5.3(b) this subalgebra is contained in the subalgebra of upper triangular matrices in some basis of $\mathfrak{g}$. Hence $[\mathfrak{g}/Z(\mathfrak{g}), \mathfrak{g}/Z(\mathfrak{g})]$ is a nilpotent Lie algebra since it consists of strictly upper triangular matrices. The proposition follows, using the following exercise. ■

Exercise 5.4. Prove that $[\mathfrak{g}, \mathfrak{g}]$ is a nilpotent Lie algebra if $[\mathfrak{g}/Z(\mathfrak{g}), \mathfrak{g}/Z(\mathfrak{g})]$ is (for any Lie algebra $\mathfrak{g}$).

Remark. Not any nilpotent subalgebra of $\mathfrak{gl}_V$ is a subalgebra of strictly upper triangular matrices in some basis of V. For example, the Lie algebra of diagonal matrices in $\mathfrak{gl}_n(\mathbb{F})$ is abelian, hence nilpotent.

Note that, due to Exercise 5.3(a), the subspace $\mathfrak{n}$ of a solvable Lie algebra $\mathfrak{b}$ over a field $\mathbb{F}$ of characteristic 0, consisting of ad-nilpotent elements, is the maximal nilpotent subalgebra in $\mathfrak{b}$.

Another application of Lie's Lemma and Lie's Theorem is the following proposition.

Proposition 5.2. Let $\mathfrak{b}$ be a finite-dimensional solvable Lie algebra over an algebraically closed field $\mathbb{F}$ of a characteristic 0, and let $\mathfrak{n}$ be its maximal nilpotent subalgebra. Then for any derivation D of $\mathfrak{b}$ we have $D(\mathfrak{b}) \subseteq \mathfrak{n}$.

Proof. It is by induction on $\dim \mathfrak{b}$. By Lie's Theorem there exists a linear function λ on $\mathfrak{b}$, such that the weight space $\mathfrak{b}_\lambda$ for the adjoint representation is non-zero. Consider the Lie algebra $\mathfrak{g} = \mathbb{F}D \oplus \mathfrak{b}$ (direct sum of vector spaces) with $\mathfrak{b}$ an ideal and $[D, b] = D(b)$ for $b \in \mathfrak{b}$. Then, by Lie's Lemma, $[\mathfrak{g}, \mathfrak{b}_\lambda] \subseteq \mathfrak{b}_\lambda$. In particular,

$$[D, \mathfrak{b}_\lambda] \subseteq \mathfrak{b}_\lambda, \quad \text{and} \quad [\mathfrak{b}, \mathfrak{b}_\lambda] \subseteq \mathfrak{b}_\lambda, \quad \text{and} \quad [[D, a], \mathfrak{b}_\lambda] \subseteq \mathfrak{b}_\lambda \text{ for all } a \in \mathfrak{b}.$$

Since all eigenvalues of $\operatorname{ad}[D, a]$ in $\mathfrak{b}_\lambda$ are equal to $\lambda([D, a])$, we conclude that

$$0 = \operatorname{tr}_{\mathfrak{b}_\lambda}[\operatorname{ad} D, \operatorname{ad} a] = \operatorname{tr}_{\mathfrak{b}_\lambda} \operatorname{ad}[D, a] = \lambda([D, a]) \dim \mathfrak{b}_\lambda.$$

Hence $\operatorname{ad}[D, a]$ is a nilpotent operator on $\mathfrak{b}_\lambda$.

Applying the inductive assumption to $\overline{\mathfrak{b}} = \mathfrak{b}/\mathfrak{b}_\lambda$, we see that $D\left(\overline{\mathfrak{b}}\right) \subseteq \overline{\mathfrak{n}}$ and therefore $D(\mathfrak{b}) \subseteq \mathfrak{n}$. ■

Exercise 5.5. By going to the algebraic closure of $\mathbb{F}$, remove the condition that $\mathbb{F}$ is algebraically closed in Proposition 5.2.

Remark. Lie's Theorem and its corollary in Exercise 5.3(a) is important for ***differential Galois theory***.

In the usual Galois theory one associates to a polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$, where $a_i \in \mathbb{C}$, the field extension $\mathbb{E} \supseteq \mathbb{F}$ where $\mathbb{F} \subseteq \mathbb{C}$ is the field generated by $a_1, \cdots, a_n$ (the minimal subfield of $\mathbb{C}$ containing $a_1, \cdots, a_n$), and $\mathbb{E} = \mathbb{F}[\alpha_1, \cdots, \alpha_n]$ where α_i are all the roots of $P(x)$. The Galois group $\mathrm{Gal}(\mathbb{E}, \mathbb{F})$ is the (finite) group of all automorphisms of the field $\mathbb{E}$ that fix all elements of $\mathbb{F}$.

Galois' Theorem says that the roots of $P(x)$ can be expressed in terms of radicals (solved in radicals) of elements of $\mathbb{F}$ if and only if the group $\mathrm{Gal}(\mathbb{E}, \mathbb{F})$ is solvable.

Similarly, in differential Galois theory, one considers a linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x) = 0, \tag{$*$}$$

where $a_1(x), \cdots, a_n(x)$ are "nice" functions in x, for example, polynomials with coefficients in $\mathbb{C}$. Let $\mathbb{F}$ be the field, genereated over $\mathbb{C}$ by $a_1(x), \cdots, a_n(x)$, and $\mathbb{E} = \mathbb{F}(\alpha_1(x), \cdots, \alpha_n(x))$, where $\alpha_1(x), \cdots, \alpha_n(x)$ is a basis of solutions of the equation $(*)$. The Galois group

$$G = \mathrm{Gal}(\mathbb{E}, \mathbb{F})$$

is the algebraic group of automorphisms of the field $\mathbb{E}$, fixing all elements of $\mathbb{F}$.

Then $\mathbb{E}$ can be obtained from $\mathbb{F}$ by adding $\int a,\ e^{\int a}$ with $a \in \mathbb{F}$ or algebraic functions over $\mathbb{F}$ if and only if Lie G is a solvable Lie algebra (such extensions are called Liouville extensions).

Example. $y'' - ay = 0$, where $a \in \mathbb{C}[x]$. Then

(a) $G = \{1\}$ if and only if $a = 0$,

(b) $G \cong \{\text{diagonal invertible } 2 \times 2 \text{ matrices}\}$ if and only if $a \in \mathbb{C} \setminus \{0\}$,

(c) $G \cong B_2$ (2×2 upper triangular invertible matrices) if and only if $a = b' + b^2$ for some $b \in \mathbb{C}[x] \setminus \mathbb{C}$,

(d) $G \cong SL_2$ if and only if $a \neq b' + b^2$ for any $b \in \mathbb{C}[x]$.

For example, the differential equation

$$y'' - xy = 0 \qquad \text{(Airy equation)}$$

is not solvable in the above sense, since its differential Galois group is SL_2, hence its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is not solvable.

Lecture 6

Representation Theory of Nilpotent Lie Algebras and Generalized Weight Spaces

Definition 6.1. Let A be a linear operator on a vector space V over a field $\mathbb{F}$ and let $\lambda \in \mathbb{F}$. Then the subspace

$$V_{(\lambda)} = \left\{ v \in V \mid (A - \lambda I_V)^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0} \right\}$$

is called a ***generalized eigenspace*** of A with eigenvalue λ. Note that the eigenspace V_λ of A in V with eigenvalue λ is a subspace of $V_{(\lambda)}$.

Example 6.1. A is a nilpotent operator on V if and only if $V = V_{(0)}$.

Recall from linear algebra:

Proposition 6.1. Let A be a linear operator on a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$, and let $\lambda_1, \cdots, \lambda_s$ be all eigenvalues of A with multiplicities $m_1, \cdots, m_s$ respectively. Then one has the generalized eigenspace decomposition

$$V = \bigoplus_{j=1}^{s} V_{(\lambda_j)}, \quad \text{where } \dim V_{(\lambda_j)} = m_j. \tag{1}$$

Each $V_{(\lambda_j)}$ is A-invariant, and $A \mid_{V_{(\lambda_j)}} = \lambda_j I_{m_j} + N_j$ where N_j is a nilpotent operator on $V_{(\lambda_j)}$.

From Proposition 6.1 we obtain the following decomposition of the linear operator A, called the classical Jordan decomposition:

$$A = A_s + A_n, \tag{2}$$

where $A_s \mid_{V_{(\lambda_j)}} = \lambda_j I_{m_j}$ and $A_n \mid_{V_{(\lambda_j)}} = N_j$. It has the following three properties:

(i) A_s is a diagonalizable operator (usually called semisimple),

(ii) A_n is a nilpotent operator,

(iii) $A_s A_n = A_n A_s$.

Indeed, (i) and (ii) are obvious, while (iii) holds since each $V_{(\lambda_j)}$ is A-invariant, and $A_s \mid_{V_{(\lambda_j)}} = \lambda_j I_{m_j}$.

Definition 6.2. A decomposition (2) of a linear operator A with properties (i), (ii), (iii) is called a ***Jordan decomposition*** of A.

We established its existence, provided that $\dim V < \infty$ and $\mathbb{F}$ is an algebraically closed field.

Proposition 6.2. Jordan decomposition is unique if V is a finite-dimensional vector space over an algebraically closed field.

The proof of this proposition uses

Lemma 6.1. Let A and B be commuting operators on a vector space V, i.e. $AB = BA$. Then

(a) All generalized eigenspaces of A are B-invariant.

(b) If $A = A_s + A_n$ is the classical Jordan decomposition, then B commutes with both A_s and A_n.

Proof. (a) is immediate from the definition of a generalized eigenspace. (b) follows from (a) since each $V_{(\lambda_j)}$ is B-invariant and $A_s |_{V_{(\lambda_j)}} = \lambda_j I_{m_j}$, therefore B and A_s commute on each $V_{(\lambda_j)}$, hence commute on V. □

Proof of Proposition 6.2 on uniqueness of a Jordan decomposition.
Consider a Jordan decomposition $A = A'_s + A'_n$, and let $A = A_s + A_n$ be the classical Jordan decomposition. Taking the difference, we get:

$$A_s - A'_s = A'_n - A_n. \tag{3}$$

But A'_s commutes with A'_n and itself, hence with A. Hence, by taking $B = A'_s$ in Lemma 6.1(b), we conclude that A'_s commutes with A_s and A_n. Therefore $A'_n = A - A'_s$ also commutes with A_s and A_n. So in (3) we have differences of commuting operators on both sides. Hence LHS of (3) is a diagonalizable, and RHS of (3) is a nilpotent operator. Hence both sides of (3) are 0. ■

Bonus Problem. Prove that a Jordan decomposition of a linear operator A in any vector space is unique (if it exists).

Exercise 6.1. Show that any non-abelian 3-dimensional nilpotent Lie algebra is isomorphic to $\mathfrak{heis}_3$.

After this digression to linear algebra, we turn to representation theory.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and π its representation in a finite-dimensional vector space V, both over an algebraically closed field $\mathbb{F}$ of characteristic 0. We have the following generalized eigenspace decomposition for fixed $a \in \mathfrak{g}$:

$$V = \bigoplus_{\lambda \in \mathbb{F}} V^a_{(\lambda)}, \quad \text{where } V^a_{(\lambda)} = \left\{ v \in V \mid (\pi(a) - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0} \right\}.$$

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{F}} \mathfrak{g}^a_{(\alpha)}, \quad \text{where } \mathfrak{g}^a_{(\alpha)} = \left\{ g \in \mathfrak{g} \mid (\mathrm{ad}(a) - \alpha I)^N g = 0 \text{ for some } N \in \mathbb{Z}_{>0} \right\}.$$

We shall prove the following theorem.

Theorem 6.1. $\pi\left(\mathfrak{g}^a_{(\alpha)}\right) V^a_{(\lambda)} \subseteq V^a_{(\lambda+\alpha)}$, for $\alpha, \lambda \in \mathbb{F}$.

In order to prove this theorem, we need a lemma whose proof is similar to that of Lemma 3.1(b).

Lemma 6.2. Let $\mathfrak{U}$ be an associative unital algebra over a field $\mathbb{F}$, and let $a, b \in \mathfrak{U}$, $\lambda, \alpha \in \mathbb{F}$. Then

$$(a - \alpha - \lambda)^N b = \sum_{j=0}^{N} \binom{N}{j} \left((\mathrm{ad}\ a - \alpha I)^j b \right) (a - \lambda)^{N-j} \tag{4}$$

Proof. Let L_a (resp. R_a) be the operator of left (resp. right) multiplication by a. They commute by associativity of $\mathfrak{U}$. We have

$$L_{a-\alpha-\lambda} = L_a - \alpha I - \lambda I = (\mathrm{ad}\ a) + R_a - \alpha I - \lambda I = (\mathrm{ad}\ a - \alpha I) + R_{a-\lambda}. \tag{5}$$

Since the operators in the RHS of (5) commute, (4) follows from the binomial theorem by rising both sides of (5) to the N-th power. □

Proof of Theorem 6.1. Applying Lemma 6.2 to $\mathfrak{U} = \mathrm{End}\, V$, $\pi(a)$ and $\pi(g)$, we have:

$$(\pi(a) - \alpha - \lambda)^N \pi(g) = \sum_{j=0}^{N} \binom{N}{j} \left((\mathrm{ad}\ \pi(a) - \alpha I)^j \pi(g) \right) (\pi(a) - \lambda)^{N-j} \tag{6}$$

where $g \in \mathfrak{g}_\alpha$. Apply both sides to $v \in V^a_{(\lambda)}$ with $N > \dim V^a_{(\lambda)} + \dim \mathfrak{g}^a_{(\alpha)}$. Then either $j > \dim \mathfrak{g}^a_{(\alpha)}$ or $N - j > \dim V^a_{(\lambda)}$.

If $j > \dim \mathfrak{g}^a_{(\alpha)}$, then $(\mathrm{ad}\ \pi(a) - \alpha I)^j \pi(g) = 0$ since $g \in \mathfrak{g}^a_{(\alpha)}$. Otherwise $N - j > \dim V^a_{(\lambda)}$, so $(\pi(a) - \lambda I)^{N-j} v = 0$ since $v \in V^a_{(\lambda)}$.

This makes the RHS of (6) zero, when applied to $v \in V^a_{(\lambda)}$, so $(\pi(a) - \alpha - \lambda)^N \pi(g) v = 0$. Since this holds for all $g \in \mathfrak{g}^a_{(\alpha)}$, $v \in V^a_{(\lambda)}$, Theorem 6.1 follows. ■

Now we turn to representation theory of nilpotent Lie algebras.

Definition 6.3. Let $\mathfrak{g}$ be a Lie algebra and let π be a representation of $\mathfrak{g}$ in a vector space V over a field $\mathbb{F}$. Let $\lambda : \mathfrak{g} \to \mathbb{F}$ be a linear function on $\mathfrak{g}$. The ***generalized weight space*** of $\mathfrak{g}$ in V, attached to λ, is

$$V^{\mathfrak{g}}_{(\lambda)} = \left\{ v \in V \mid (\pi(g) - \lambda(g) I_V)^N v = 0 \text{ for some } N \in \mathbb{Z}_{>0} \text{ depending on } g \in \mathfrak{g} \right\}.$$

Note that the notion of a weight space $V^{\mathfrak{g}}_{\lambda}$ introduced in Lecture 5 is a special case of this, and that $V^{\mathfrak{g}}_{(\lambda)} \supseteq V^{\mathfrak{g}}_{\lambda}$.

Theorem 6.2. Let $\mathfrak{h}$ be a nilpotent Lie algebra and π its representation in a finite-dimensional vector space V over an algebraically closed field of characteristic 0. Then we have the generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^{\mathfrak{h}}_{(\lambda)}. \tag{6}$$

Proof. **Case 1.** For each $a \in \mathfrak{h}$, $\pi(a)$ has only one eigenvalue. Then V is a generalized eigenspaace $V^{a}_{(\lambda(a))}$, so we just need to check the linearity of λ.

Since $\mathfrak{h}$ is a nilpotent Lie algebra, it is solvable, and we may apply Lie's Theorem, which guarantees the existence of $\lambda' \in \mathfrak{h}^*$ with a non-zero weight space $V^{\mathfrak{h}}_{(\lambda')}$. Then $\lambda'(a)$ is the eigenvalue of $\pi(a)$ on $V^{\mathfrak{h}}_{(\lambda')}$, so $\lambda = \lambda' \in \mathfrak{h}^*$.

Case 2. For some $a_0 \in \mathfrak{h}$, $\pi(a_0)$ has at least two distinct eigenvalues. Since $\mathfrak{h}$ is a nilpotent Lie algebra, ad a is a nilpotent operator on $\mathfrak{h}$ for all $a \in \mathfrak{h}$, hence $\mathfrak{h} = \mathfrak{h}^{\pi(a)}_{(0)}$. Therefore, by Theorem 6.1,

$$\pi(\mathfrak{h}) V^{\pi(a)}_{(\lambda)} \subseteq V^{\pi(a)}_{(\lambda)} \text{ for all } a \in \mathfrak{h}. \tag{7}$$

Since $\mathbb{F}$ is algebraically closed, V is a direct sum of the generalized eigenspaces for $\pi(a_0)$. Since each $V^{a_0}_{(\lambda)}$ is $\pi(\mathfrak{h})$-invariant by (7), it is also a representation of $\mathfrak{h}$. Since $0 < \dim V^{\pi(a_0)}_{(\lambda)} < \dim V$ for some $\lambda \in \mathbb{F}$, we may apply induction on $\dim V$. ∎

If $\mathfrak{g}$ is a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and $\mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}$, we may apply Theorem 6.2 to the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$, to obtain the ***generalized root space decomposition*** with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \quad \text{where } \mathfrak{g}^{\mathfrak{h}}_{(\alpha)} = \left\{ a \in \mathfrak{g} \mid (\mathrm{ad}(h) - \alpha(h) I_{\mathfrak{g}})^{\dim \mathfrak{g}} a = 0 \text{ for all } h \in \mathfrak{h} \right\}. \tag{8}$$

Theorem 6.3. Let $\mathfrak{g}$ and $\mathfrak{h}$ be as above, and let π be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V, so that we have the generalized weight space decomposition (6) with respect to $\pi(\mathfrak{h})$. Then

$$\pi\left(\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}\right) V^{\mathfrak{h}}_{(\lambda)} \subseteq V^{\mathfrak{h}}_{(\lambda+\alpha)}. \tag{9}$$

Proof. If $g \in \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}$, then $g \in \mathfrak{g}^{a}_{(\alpha(a))}$ for all $a \in \mathfrak{h}$. By Theorem 6.1, $\pi(g)V^a_{(\lambda(a))} \subseteq V^a_{(\lambda(a)+\alpha(a))}$ for all $a \in \mathfrak{h}$. Hence, if $v \in \bigcap_{a\in\mathfrak{h}} V^a_{(\lambda(a))}$, then $\pi(g)v \in \bigcap_{a\in\mathfrak{h}} V^a_{(\lambda(a)+\alpha(a))}$. Since $\bigcap_{a\in\mathfrak{h}} V^a_{(\lambda(a)+\alpha(a))} = V^{\mathfrak{h}}_{(\lambda)}$, this establishes (9). ∎

If $V = \mathfrak{g}$ and $\pi = \mathrm{ad}$, we obtain the following corollary of Theorem 6.3.

Corollary 6.1. $\left[\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \mathfrak{g}^{\mathfrak{h}}_{(\beta)}\right] \subseteq \mathfrak{g}^{\mathfrak{h}}_{(\alpha+\beta)}$.

Exercise 6.2. Let $\mathbb{F}$ be a field of characteristic 2, and $V = \mathbb{F}[x]/x^2\mathbb{F}[x]$ be the representation of $\mathfrak{heis}_3$, given by $p \mapsto \frac{\mathrm{d}}{\mathrm{d}x}, q \mapsto x, c \mapsto I_V$. Show that $V = V^{\mathfrak{heis}_3}_{(\lambda)}$, but λ is not a linear function on $\mathfrak{heis}_3$. Compute λ.

Exercise 6.3. By the example of the adjoint representation of the 2-dimensional non-abelian solvable Lie algebra, show that the generalized weight space decomposition fails for solvable Lie algebras that are not nilpotent.

Exercise 6.4. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{h} = \{\text{diagonal matrices}\}$. Find the generalized weight space decomposition for the tautological and adjoint representations with respect to $\mathfrak{h}$, and show that (9) and Corollary 6.1 hold. They are actually weight space decompositions.

Lecture 7

Regular Elements and Rank of a Finite-Dimensional Lie Algebra

For further development of the theory of Lie algebras we need the notions of a topological space and Zariski topology.

Definition 7.1. A ***topological space*** is a set X with a collection of subsets $\mathscr{F}$, called the ***closed*** subsets, satisfying the following axioms:

1. $X \in \mathscr{F}$ and $\varnothing \in \mathscr{F}$,
2. the union of a finite collection of closed subsets is a closed subset,
3. the intersection of an arbitrary collection of closed subsets is a closed subset,
4. (weak separation axiom) for any two distinct points $x, y \in X$, there exists $F \in \mathscr{F}$, such that $x \in F$, but $y \notin F$.

Given a closed subset $F \in \mathscr{F}$, its complement F^c in X is called an ***open*** subset. The axioms for a topological space can also be phrased in terms of open subsets.

Definition 7.2. Let $X = \mathbb{F}^n$ where $\mathbb{F}$ is a field. The ***Zariski topology*** on X is defined as follows: A subset in X is closed if and only if it is the set of common zeros of a (possibly infinite) collection of polynomials $\{P_\alpha(x)\}$ on $\mathbb{F}^n$.

For a collection of polynomials $S = \{P_\alpha(x)\}$, we denote $F(S)$ the set of common zeros of all polynomials from S. If S contains precisely one non-constant polynomial, then $F(S)$ is called a ***hypersurface*** in $\mathbb{F}^n$.

Exercise 7.1. Prove that the Zariski topology is indeed a topology.

Example 7.1. If $X = \mathbb{F}$, the closed subsets in Zariski topology are precisely $\varnothing, \mathbb{F}$ and finite subsets of $\mathbb{F}$.

Theorem 7.1. Let $\mathbb{F}$ be an infinite field and n a positive integer. Then

(a) The complement to a hypersurface in $\mathbb{F}^n$ is an infinite set. Consequently, the complement to $F(S)$, where S contains a non-zero polynomial, is an infinite set.

(b) Every two non-empty Zariski open subsets in $\mathbb{F}^n$ have a non-empty intersection.

(c) If a polynomial $p(x)$ vanishes on a non-empty Zariski open subset, then $p(x)$ is the zero polynomial.

Proof. We prove (a) by induction on n. When $n = 1$, since any non-zero polynomial $p(x)$ has at most $\deg p(x)$ roots, $F(p(x))^c$ is an infinite set (since $\mathbb{F}$ is). If $n > 1$, then any non-constant polynomial $p(x_1, \cdots, x_n)$ can be written, after a permutation of the indeterminates as

$$p(x_1, \cdots, x_n) = p_0(x_2, \cdots, x_n)x_1^d + p_1(x_2, \cdots, x_n)x_1^{d-1} + \cdots,$$

where $p_0(x_2, \cdots, x_n)$ is a non-zero polynomial. By the inductive hypothesis, we can find $x_2^\circ, \cdots, x_n^\circ \in \mathbb{F}$, such that $p_0(x_2^\circ, \cdots, x_n^\circ) \neq 0$. Now fixing these values, we get

$$p(x_1^\circ, \cdots, x_n^\circ) = p_0(x_2^\circ, \cdots, x_n^\circ)x_1^d + p_1(x_2^\circ, \cdots, x_n^\circ)x_1^{d-1} + \cdots,$$

so we are back to the $n = 1$ case. Hence we can find an infinite number of values $x_1^\circ \in \mathbb{F}$, such that $p(x_1^\circ, \cdots, x_n^\circ) \neq 0$.

To prove the second claim of (a), just observe that if S is a collection of polynomials containing a non-zero polynomial $p(x)$, then $F(S)^c \supset F(p(x))^c$, hence, by the first claim, $F(S)^c$ is an infinite set.

(b) Let S_1 and S_2 be two sets of polynomials, each containing a non-zero polynomial $p_1(x)$ and $p_2(x)$ respectively. It suffices to prove that $F(p_1(x))^c \cap F(p_2(x))^c$ is non-empty. But $F(p_1p_2) = F(p_1) \cup F(p_2)$, so $F(p_1p_2)^c = F(p_1)^c \cap F(p_2)^c$, which is an infinite set by (a), hence non-empty.

(c) If $p(x)$ vanishes on $F(S)^c$ for S containing a non-zero polynomial $q(x)$, then $p(x)$ vanishes on $F(q)^c$. If $p(x)$ were non-zero, then by (b), $F(q)^c \cap F(p)^c \neq \varnothing$, a contradiction. ■

Remark. The condition that $\mathbb{F}$ be infinite is essential. For example, if $\mathbb{F} = \mathbb{F}_2 = \{0, 1\}$, the polynomial $p(x) = x^2 + x$ vanishes on $\mathbb{F}$, but is not the zero polynomial, hence (c) fails.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra of dimension d over a field $\mathbb{F}$. Note that, as a vector space, it is isomorphic to $\mathbb{F}^d$. Consider the characteristic polynomial of an endomorphism ad a for some $a \in \mathfrak{g}$:

$$\det_{\mathfrak{g}}(\text{ad } a - \lambda I) = (-\lambda)^d + c_{d-1}(a)(-\lambda)^{d-1} + \cdots + \det_{\mathfrak{g}}(\text{ad } a).$$

Note that $c_d(a) = 1$, hence this polynomial has degree d, and its constant term $c_0(a) = \det_{\mathfrak{g}}(\text{ad } a)$ is a zero polynomial, since $(\text{ad } a)a = [a, a] = 0$, so that the determinant of the operator ad a on $\mathfrak{g}$ is 0.

Exercise 7.2. Show that $c_j(a)$ is a homogeneous polynomial on $\mathfrak{g}$ of degree $d - j$. (Since $\mathfrak{g} \cong \mathbb{F}^d$ as vector spaces, a polynomial $p(v)$ on $\mathfrak{g}$ is a polynomial $p(x_1, \cdots, x_d)$ where $v = x_1e_1 + \cdots + x_de_d$.)

Definition 7.3. The smallest positive integer r, such that $c_r(a)$ is a non-zero polynomiial on $\mathfrak{g}$, is called the ***rank*** of $\mathfrak{g}$. The non-zero polynomial $c_r(a)$ of degree $d - r$ is called the ***discriminant*** of the Lie algebra $\mathfrak{g}$. An element $a \in \mathfrak{g}$ is called regular if $c_r(a) \neq 0$. Note that $1 \leqslant r \leqslant d$.

Theorem 7.2. Let $\mathfrak{g}$ be a Lie algebra of dimension d and rank r, over a field $\mathbb{F}$. Then

(a) $r = d$ if and only if the Lie algebra $\mathfrak{g}$ is nilpotent.

(b) The set of regular elements coincides with $\mathfrak{g}$ if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

(c) If $\mathfrak{g}$ is not a nilpotent Lie algebra, then the set of regular elements is the complement to a hypersurface, hence it is infinite if $\mathbb{F}$ is.

Proof. (a) $r = d$ means that $\det(\text{ad}\ \lambda I) = (-\lambda)^d$ for all $a \in \mathfrak{g}$. This holds if and only if all the eigenvalues of ad a are 0, which happens if and only if ad a is a nilpotent operator for all $a \in \mathfrak{g}$. By Engel's characterization theorem this happens if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

(b) By (a), $\mathfrak{g}$ is a nilpotent Lie algebra if and only if $\det(\text{ad}\ \lambda I) = (-\lambda)^d$, i.e. $c_r(a) = c_d(a) = 1$, so all $a \in \mathfrak{g}$ are regular.

(c) If $\mathfrak{g}$ is not a nilpotent Lie algebra, then $r < d$, so that $F(c_r(x))^c$ is a complement to a hypersurface, defined by a homogeneous polynomial $c_r(x)$ of degree $d - r > 0$. By Theorem 7.1(a) the complement to a hypersurface is an infinite set if $\mathbb{F}$ is infinite. ∎

Exercise 7.3.

(a) Show that the Jordan decomposition of ad a in $\mathfrak{gl}_n(\mathbb{F})$ is given by

$$\text{ad}\ a = (\text{ad}\ a_s) + (\text{ad}\ a_n),$$

where $a = a_s + a_n$ is the Jordan decomposition of $a \in \mathfrak{gl}_n(\mathbb{F})$.

(b) If $\lambda_1, \cdots, \lambda_n$ are all the eigenvalues of a_s, then $\lambda_i - \lambda_j$ are all the eigenvalues of ad a_s.

(c) ad a_s has the same eigenvalues as ad a.

Exercise 7.4.

(a) $\operatorname{rank}(\mathfrak{gl}_n(\mathbb{F})) = n$.

(b) The discriminant of $\mathfrak{gl}_n(\mathbb{F})$ is given by

$$c_n(a) = \prod_{i \neq j} (\lambda_i - \lambda_j),$$

where λ_i's are all eigenvalues of $a \in \mathfrak{gl}_n(\mathbb{F})$ (over $\overline{\mathbb{F}}$), taken with their multiplicities.

(c) Show that the discriminant $c_2(a)$ of $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{F})$ is equal to $4 \det a - (\operatorname{tr} a)^2$.

Bonus Problem. It follows from Exercise 7.4 that for $n \times n$ matrix A over $\mathbb{F}$ the function $\prod_{i \neq j}(\lambda_i - \lambda_j)$, where the λ_i's are all eigenvalues of A taken with their multiplicities, is a polynomial of degree $n^2 - n$ over $\mathbb{F}$, in the entries of A. Find an explicit formula for this polynomial (called the discriminant of the matrix A). The corresponding hypersurface consists of the A, which have equal eigenvalues.

Lecture 8

Cartan Subalgebras

Definition 8.1. Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Then

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{a \in \mathfrak{g} \mid [a, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

is a subalgebra of $\mathfrak{g}$, called the ***normalizer*** of $\mathfrak{h}$ in $\mathfrak{g}$.

The fact that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra is immediate by Jacobi identity. Note also that $N_{\mathfrak{g}}(\mathfrak{h})$ is the maximal subalgebra of $\mathfrak{g}$, containing $\mathfrak{h}$ as an ideal.

Bonus problem. Let G be an algebraic group, and H be an algebraic subgroup of G. Let $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ be the normalizer of H in G. Prove that it is an algebraic subgroup, whose Lie algebra is $N_{\mathfrak{g}}(\mathfrak{h})$, where $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{h} = \operatorname{Lie} H$.

Lemma 8.1. Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h} \subsetneq \mathfrak{g}$ a proper subalgebra. Then $\mathfrak{h}$ is a proper subalgebra of $N_{\mathfrak{g}}(\mathfrak{h})$.

Proof. Since $\mathfrak{g}$ is a nilpotent Lie algebra, its central series has the form:

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] \supseteq \cdots \supseteq \mathfrak{g}^n = 0,$$

for some positive integer n (note that $\mathfrak{g} \neq 0$). Take j to be the maximal positive integer such that $\mathfrak{g}^j \subsetneq \mathfrak{h}$. Clearly we have that $1 < j < n$. But then, by the choice of j,

$$[\mathfrak{g}^j, \mathfrak{h}] \subseteq \mathfrak{h}.$$

Hence $\mathfrak{g}^j \subseteq N_{\mathfrak{g}}(\mathfrak{h})$, but $\mathfrak{g}^j \subsetneq \mathfrak{h}$. Therefore $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$. ∎

Definition 8.2. A ***Cartan subalgebra*** of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$, satisfying the following two conditions:

(i) $\mathfrak{h}$ is a nilpotent Lie algebra,

(ii) $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Corollary of Lemma 8.1. Any Cartan subalgebra of a Lie algebra $\mathfrak{g}$ is a maximal nilpotent subalgebra.

Exercise 8.1. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$ with char $\mathbb{F} \neq 2$. Let $\mathfrak{h} = \mathbb{F}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This is a maximal nilpotent subalgebra, but not a Cartan subalgebra.

The following simple, but important, theorem allows one to construct Cartan subalgebras.

Theorem 8.1. Let $\mathfrak{g} \subseteq \mathfrak{gl}_n(\mathbb{F})$ be a subalgebra, containing a diagonal matrix $a = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$ with distinct a_i's. Then the subalgebra $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof. Since the subalgebra $\mathfrak{h}$ is abelian, it is a nilpotent Lie algebra. It remains to check that $\mathfrak{h}$ coincides with its normalizer in $\mathfrak{g}$. Let $b = \sum_{i,j=1}^{n} b_{ij} E_{ij} \in \mathfrak{g}$, such that $[b, \mathfrak{h}] \subset \mathfrak{h}$. Here E_{ij} are the matrix units (1 at entry (i,j) and 0 otherwise). Then, in particular, $[a, b]$ is a diagonal matrix. But

$$[a, b] = \left[\sum_k a_k E_{kk}, \sum_{i,j} b_{ij} E_{ij}\right] = \sum_{i,j} (a_i - a_j) b_{ij} E_{ij},$$

which is a diagonal matrix only if $b_{ij} = 0$ for $i \neq j$. Hence $[a, b] \in \mathfrak{h}$ only if $b \in \mathfrak{h}$. ∎

Cartan's Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$. Let $a \in \mathfrak{g}$ be a regular element (which exists since $\mathbb{F}$ is infinite), and let

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{F}} \mathfrak{g}^a_{(\alpha)} \tag{1}$$

be the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ad a. Then $\mathfrak{h} = \mathfrak{g}^a_{(0)}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof. The proof uses the fact that any two non-empty Zariski open sets have a non-empty intersection (Theorem 7.1(b)). Recall also that $\left[\mathfrak{g}^a_{(\alpha)}, \mathfrak{g}^a_{(\beta)}\right] \subseteq \mathfrak{g}^a_{(\alpha+\beta)}$, and, in particular, if $\alpha = 0$,

$$\left[\mathfrak{h}, \mathfrak{g}^a_{(\mu)}\right] \subseteq \mathfrak{g}^a_{(\mu)}. \tag{2}$$

Let $V = \bigoplus_{\alpha \neq 0} \mathfrak{g}^a_{(\alpha)}$, then by (1) and (2) we have:

$$\mathfrak{g} = \mathfrak{h} \oplus V \text{ (direct sum of vector spaces)}; \quad [\mathfrak{h}, V] \subseteq V. \tag{3}$$

Consider the following two subsets of $\mathfrak{h}$:

$$U = \{h \in \mathfrak{h} \mid \text{ad } h\,|_{\mathfrak{h}} \text{ is not a nilpotent operator}\},$$
$$R = \{h \in \mathfrak{h} \mid \text{ad } h\,|_{V} \text{ is a non-singular operator}\}.$$

Both U and R are Zariski open subsets of $\mathfrak{h}$.

Now, we shall prove that $\mathfrak{h}$ is a nilpotent Lie algebra. Suppose the contrary, then by Engel's characterization theorem, there exists $h \in \mathfrak{h}$, such that ad h is not a nilpotent operator on $\mathfrak{h}$. But then $h \in U$, hence $U \neq \varnothing$. Also $a \in R$ since all eigenvectors with zero eigenvalues of ad a in $\mathfrak{g}$, lie in $\mathfrak{h}$, since a is a regular element of $\mathfrak{g}$. Hence $R \neq \varnothing$.

Therefore, $U \cap R \neq \varnothing$, and we take $b \in U \cap R$.

Then ad $b \,|_{\mathfrak{h}}$ is not a nilpotent operator, and ad $b \,|_V$ is a non-singular operator. Hence, by (3), $\mathfrak{g}^b_{(0)} \subsetneq \mathfrak{h}$, which contradicts the choice of a as a regular element (since for regular a, $\dim \mathfrak{g}^a_{(0)}$ is minimal among all $a \in \mathfrak{g}$). This contradiction completes the proof that $\mathfrak{h}$ is a nilpotent Lie algebra.

Finally, we prove that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. If $b \in N_{\mathfrak{g}}(\mathfrak{h})$, so that $[b, \mathfrak{h}] \subset \mathfrak{h}$, we have that, in particular, $[b, a] \in \mathfrak{h}$.

But since $a \in \mathfrak{h}$ and $\mathfrak{h}$ is a nilpotent Lie algebra, then ad $a \,|_{\mathfrak{h}}$ is a nilpotent operator on $\mathfrak{h}$. In particular, $0 = (\text{ad } a)^N[a, b] = (\text{ad } a)^{N+1} b$ for some positive integer N. Hence $b \in \mathfrak{g}^a_{(0)} = \mathfrak{h}$, which completes the proof of the theorem. ■

Remark. The dimension of the Cartan subalgebra of $\mathfrak{g}$, constructed in the proof of Cartan's theorem equals rank $\mathfrak{g}$, by definitions.

Proposition 8.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0, and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Consider the generalized weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}^{\mathfrak{h}}_{(\lambda)}.$$

Then $\mathfrak{g}^{\mathfrak{h}}_{(0)} = \mathfrak{h}$.

Proof. Since $\mathfrak{h}$ is a nilpotent Lie algebra, ad $h \,|_{\mathfrak{h}}$ is a nilpotent operator for all $h \in \mathfrak{h}$. Hence $\mathfrak{h} \subseteq \mathfrak{g}^{\mathfrak{h}}_{(0)}$.

But by definition of $\mathfrak{g}^{\mathfrak{h}}_{(0)}$, ad $h \,|_{\mathfrak{g}^{\mathfrak{h}}_{(0)}}$ is a nilpotent operator for all $h \in \mathfrak{h}$. Hence ad $h \,|_{\mathfrak{g}^{\mathfrak{h}}_{(0)}/\mathfrak{h}}$ is nilpotent operator for all $h \in \mathfrak{h}$. Therefore, by Engel's theorem, there exists a non-zero element $\overline{b} \in \mathfrak{g}^{\mathfrak{h}}_{(0)}/\mathfrak{h}$, which is annihilated by all ad $h \,|_{\mathfrak{g}^{\mathfrak{h}}_{(0)}/\mathfrak{h}}$, $h \in \mathfrak{h}$. Taking a preimage $b \in \mathfrak{g}$ of $\overline{b}$ under the map $\mathfrak{g}^{\mathfrak{h}}_{(0)} \to \mathfrak{g}^{\mathfrak{h}}_{(0)}/\mathfrak{h}$, this means that $[b, \mathfrak{h}] \subseteq \mathfrak{h}$. Hence $b \in N_{\mathfrak{g}}(\mathfrak{h})$, but $b \notin \mathfrak{h}$, which contradicts the fact that $\mathfrak{h}$ is a Cartan subalgebra. ■

Corollary. Under the assumptions of Proposition 8.1, we have the generalized root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \mathfrak{h}^* \\ \alpha \neq 0}} \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \quad \text{where } \mathfrak{h} = \mathfrak{g}^{\mathfrak{h}}_{(0)} \text{ and } \left[\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \mathfrak{g}^{\mathfrak{h}}_{(\beta)}\right] \subseteq \mathfrak{g}^{\mathfrak{h}}_{(\alpha+\beta)}. \tag{4}$$

Next, we use the generalized root space decomposition to classify all Lie algebras of dimension 3 over an algebraically closed field $\mathbb{F}$ of characteristic 0.

We know that for 3-dimensional $\mathfrak{g}$, rank $\mathfrak{g} = 3, 2$, or 1, and rank $\mathfrak{g} = 3$ if and only if $\mathfrak{g}$ is a nilpotent Lie algebra.

- If rank $\mathfrak{g} = 3$: then by Exercise 6.1, $\mathfrak{g} \simeq \mathfrak{ab}_3$ or $\mathfrak{heis}_3$.

- If rank $\mathfrak{g} = 2$: in this case $\dim \mathfrak{h} = 2$, and since $\mathfrak{h}$ is a nilpotent Lie algebra, it must be abelian (the only non-abelian 2-dimensional Lie algebra is not nilpotent). Hence the generalized root space decomposition (4) in this case is

  $$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{F}c, \quad \text{where } c \neq 0 \text{ and } [\mathfrak{h}, c] \subseteq \mathbb{F}c.$$

  Since $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$, then $[\mathfrak{h}, c] \neq 0$. Also $\mathbb{F}c = \mathfrak{g}^{\mathfrak{h}}_{(\lambda)}$, where λ is a non-zero linear function on $\mathfrak{h} = \mathbb{F}a \oplus \mathbb{F}b$, $[a, b] = 0$. We can choose the basis a, b of $\mathfrak{h}$, such that $\lambda(a) = 1, \lambda(b) = -1$, which means that

  $$[a, c] = c, \quad [b, c] = -c, \quad [a, b] = 0.$$

  This 3-dimensional Lie algebra is isomorphic to the Lie algebra $\mathfrak{b}_2(\mathbb{F})$ of upper-triangular 2-by-2 matrices by letting

  $$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

- If rank $\mathfrak{g} = 1$: in this case $\mathfrak{g} = \mathbb{F}h \oplus V$, where $\dim V = 2$, $[h, V] \subseteq V$.

Exercise 8.2. Show that in the last case $\mathfrak{g}$ is isomorphic to one of the following Lie algebras with basis h, a, b:

(i) $[h, a] = a, [h, b] = a + b, [a, b] = 0$,

(ii) $[h, a] = a, [h, b] = \lambda b$, where $\lambda \in \mathbb{F} \backslash \{0\}$, $[a, b] = 0$,

(iii) $[h, a] = a, [h, b] = -b, [a, b] = h$.

Exercise 8.3. Show that all Lie algebras from (i) and (ii) are solvable and find conditions of their isomorphism. Show that (iii) is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, and that it is not solvable.

Lecture 9

Chevalley Conjugacy Theorem

In the last lecture we defined Cartan subalgebras and gave a construction, using regular elements. In this lecture, we will show that this construction gives all Cartan subalgebras by proving the Chevalley conjugacy theorem on conjugacy of Cartan subalgebras.

To state the theorem, we need the notion of the exponential of a nilpotent operator.

Definition 9.1. Let A be a nilpotent operator on a vector space V over a field $\mathbb{F}$ of characteristic 0. Define

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

(as A is nilpotent, this is a finite sum).

Exercise 9.1. If A and B are commuting nilpotent operators on a vector space over a field of characteristic 0, so that $A + B$ is nilpotent as well, show that

$$e^{A+B} = e^A e^B.$$

Deduce that $e^A e^{-A} = I$, hence e^A is an invertible operator.

Exercise 9.2. Let $\mathfrak{g}$ be an arbitrary (not necessarily Lie) algebra over a field $\mathbb{F}$ of characteristic 0, and let D be a derivation of the algebra $\mathfrak{g}$. Show that e^D is an automorphism of the algebra $\mathfrak{g}$, provided that D is a nilpotent operator.

We are now ready to state the main result of the lecture.

Chevalley Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0. Denote by G the group of automorphisms of the Lie algebra $\mathfrak{g}$, generated by automorphisms of the form $e^{\operatorname{ad} a}$, for $a \in \mathfrak{g}$ such that $\operatorname{ad} a$ is a nilpotent operator on $\mathfrak{g}$. Then any two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g}$ are conjugate by G, i.e. there exists $\sigma \in G$, such that $\sigma(\mathfrak{h}_1) = \mathfrak{h}_2$.

Before proving this theorem, we give a corollary that addresses the question with which we opened the lecture.

Corollary. Let $\mathfrak{g}$ and $\mathbb{F}$ be as in the Chevalley Theorem. Then any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is of the form $\mathfrak{g}^a_{(0)}$ for some regular element $a \in \mathfrak{g}$, and, in particular, rank $\mathfrak{g} = \dim \mathfrak{h}$. Also, all such subalgebras $\mathfrak{g}^a_{(0)}$ are isomorphic.

Proof. Fix a regular element $a \in \mathfrak{g}$. By the Chevalley theorem, any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is conjugate to $\mathfrak{g}^a_{(0)}$, say, $\mathfrak{h} = \sigma\left(\mathfrak{g}^a_{(0)}\right)$ for some $\sigma \in G$. Hence $\dim \mathfrak{h} = \dim \mathfrak{g}^a_{(0)} = \operatorname{rank} \mathfrak{g}$.

Next, because σ is an automorphism of the Lie algebra $\mathfrak{g}$, it is easy to check that

$$\sigma\left(\mathfrak{g}^a_{(0)}\right) = \mathfrak{g}^{\sigma(a)}_{(0)},$$

and that $\sigma(a)$ is a regular element of $\mathfrak{g}$. Finally the last claim is immediate since conjugate subalgebras are isomorphic. ■

In order to prove Chevalley's theorem, we need two lemmas

Lemma 9.1. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and suppose there is a regular element $a \in \mathfrak{g}$, which is in $\mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{g}^a_{(0)}$.

Proof. The Lie algebra $\mathfrak{h}$ is nilpotent since it is a Cartan subalgebra. Hence $\operatorname{ad} a \mid_{\mathfrak{h}}$ is a nilpotent operator, and so $\mathfrak{h} \subseteq \mathfrak{g}^a_{(0)}$. But, being a Cartan subalgebra, $\mathfrak{g}^a_{(0)}$ is a nilpotent Lie algebra, and, being a Cartan subalgebra, $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$ (by Corollary of Lemma 8.1). Hence $\mathfrak{h} = \mathfrak{g}^a_{(0)}$. □

The next lemma is a special case of a general result from algebraic geometry.

Lemma 9.2. Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. Let $f : \mathbb{F}^m \to \mathbb{F}^m$ be a polynomial map, i.e.

$$f(x_1, \cdots, x_m) = (f_1(x_1, \cdots, x_m), \cdots, f_m(x_1, \cdots, x_m)),$$

where the f_i's are polynomials. Suppose that for some $a \in \mathbb{F}^m$ the linear map

$$(\mathrm{d}f) \mid_{x=a}: \mathbb{F}^m \to \mathbb{F}^m$$

is non-singular. Then $f(\mathbb{F}^m)$ contains a non-empty Zariski open subset of $\mathbb{F}^m$.

Exercise 9.3. Recall that

$$(\mathrm{d}f) \mid_{x=a} (b) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a+tb).$$

Show that $(\mathrm{d}f) \mid_{x=a}$ is a linear operator on $\mathbb{F}^m$ with the matrix $\left(\dfrac{\partial f_i}{\partial x_j}(a)\right)_{i,j=1}^m$.

Bonus Problem. Prove Lemma 9.2 by the following steps, using Exercise 9.3.

1. If F is a non-zero polynomial in m indeterminates, such that $F(f_1, \cdots, f_m)$ is identically zero, then $\det\left(\frac{\partial f_i}{\partial x_j}\right)$ is identically 0.

2. Given algebraically independent polynomials $y_1, \cdots, y_m$ in $\mathbb{F}[x_1, \cdots, x_m]$, show that the field extension $\mathbb{F}(x_1, \cdots, x_m) \supseteq \mathbb{F}(y_1, \cdots, y_m)$ is finite, i.e. each x_i satisfies a non-zero polynomial equation over $\mathbb{F}(y_1, \cdots, y_m)$.

3. For each $i = 1, \cdots, m$, take a polynomial equation satisfied by x_i over $\mathbb{F}(f_1, \cdots, f_m)$, clear the denominators to get a polynomial over $\mathbb{F}[f_1, \cdots, f_m]$, and let $p_i(f_1, \cdots, f_m)$ be the leading coefficient of this polynomial. Show that the set of points

$$\{y \in \mathbb{F}^m \mid p_i(y) \neq 0 \text{ for } i = 1, \cdots, m\}$$

is contained in $f(\mathbb{F}^m)$.

Example 9.1. Consider the map $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2$. Then $f(\mathbb{R}) = \mathbb{R}_{\geqslant 0}$, which doesn't contain a non-empty Zariski open subset (which is either $\mathbb{R}$ or the complement to a finite set). Thus, the algebraic closure assumption in Lemma 9.2 is essential.

However, $f(\mathbb{R})$ contains an open subset in the metric topology. In fact, for each $a \in \mathbb{R}$ with $(\mathrm{d}f)\mid_{x=a}$ non-singular, the image of f contains an open neighborhood of $f(a)$ in the metric topology for any smooth map $f : \mathbb{R}^m \to \mathbb{R}^m$ by the Inverse function theorem.

Proof of Chevalley's Theorem. Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is a nilpotent Lie algebra, we have the corresponding generalized root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \quad \text{where } \mathfrak{h} = \mathfrak{g}^{\mathfrak{h}}_{(0)}, \left[\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}, \mathfrak{g}^{\mathfrak{h}}_{(\beta)}\right] \subseteq \mathfrak{g}^{\mathfrak{h}}_{(\alpha+\beta)}. \tag{1}$$

(Recall that $\mathfrak{h} = \mathfrak{g}^{\mathfrak{h}}_{(0)}$ by Proposition 8.1). Then for any $x \in \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}$, with $\alpha \neq 0$, the operator ad x is nilpotent on $\mathfrak{g}$. Indeed, by (1), $(\operatorname{ad} x)^N \mathfrak{g}^{\mathfrak{h}}_{(\beta)} \subseteq \mathfrak{g}^{\mathfrak{h}}_{(\beta+N\alpha)}$. As $\alpha \neq 0$ and char $\mathbb{F} = 0$, $\{\beta + N\alpha \mid N \in \mathbb{Z}_{>0}\}$ is an infinite set of distinct linear functions on $\mathfrak{h}$. But there are only finitely many of them, for which the attached generalized weight space is non-zero. Hence $(\operatorname{ad} x)^N \mathfrak{g}^{\mathfrak{h}}_{(\beta)} = 0$ for N big enough.

Next, we show that there is a Zariski open subset of $\mathfrak{g}$, consisting of images of elements of $\mathfrak{h}$ under the action of the group G. Let $\Delta = \{\alpha \in \mathfrak{h}^*\}$ be the (finite) set of non-zero linear functions on $\mathfrak{h}$, such that $\mathfrak{g}^{\mathfrak{h}}_{(\alpha)} \neq 0$, and let $\{b_j\}_{j=1}^m$ be a basis of $V = \bigoplus_{\alpha\in\Delta} \mathfrak{g}^{\mathfrak{h}}_{(\alpha)}$, compatible with its decomposition. Then any element of $\mathfrak{g}$ has a unique expression of the form

$$h + \sum_{j=1}^{m} x_j b_j, \quad \text{where } h \in \mathfrak{h}, x_j \in \mathbb{F}.$$

Define a map $f : \mathfrak{g} \to \mathfrak{g}$ by

$$f\left(h + \sum_{j=1}^{m} x_j b_j\right) = e^{x_1 \operatorname{ad}(b_1)} e^{x_2 \operatorname{ad}(b_2)} \cdots e^{x_m \operatorname{ad}(b_m)}(h).$$

The map f is polynomial since all operators ad b_j are nilpotent on $\mathfrak{g}$.

Let us compute $(\mathrm{d}f)\,|_{x=a}$ for $a \in \mathfrak{h}$, applied to $b + h$, where $h \in \mathfrak{h}$, $b = \sum x_j b_j$:

$$\begin{aligned}(\mathrm{d}f)\,|_{x=a}\,(b+h) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f\left(a + t(b+h)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} e^{tx_1 \operatorname{ad}(b_1)} e^{tx_2 \operatorname{ad}(b_2)} \cdots e^{tx_m \operatorname{ad}(b_m)}(a + th).\end{aligned}$$

To compute this derivative, it suffices to expand the function that we are differentiating to the first order in t. We find

$$\begin{aligned}(\mathrm{d}f)\,|_{x=a}\,(b+h) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \prod_{j=1}^{m} \left(I_{\mathfrak{g}} + tx_j \operatorname{ad}(b_j)\right)(a + th)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(a + th + \sum_{j=1}^{m} tx_j\,[b_j, a]\right) \\ &= h + \sum_{j=1}^{m} x_j\,[b_j, a] = h + [b, a]\,.\end{aligned}$$

Thus, the linear operator $(\mathrm{d}f)\,|_{x=a}$ restricts to the identity operator on $\mathfrak{h}$, and to $-\operatorname{ad}(a)$ on V. On each subspace $\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}$ of V the only eigenvalue of $-\operatorname{ad}(a)$ is $-\alpha(a)$.

Since each subset $\{h \in \mathfrak{h} \mid \alpha(h) = 0\}$ is Zariski closed in $\mathfrak{h}$, their intersection

$$\bigcap_{\alpha \in \Delta} \{a \in \mathfrak{h} \mid \alpha(a) = 0\}$$

is Zariski closed. Hence its complement in $\mathfrak{h}$ is non-empty. Thus, we can find $a \in \mathfrak{h}$ such that $\alpha(a) \neq 0$ for each $\alpha \in \Delta$. But then $(\mathrm{d}f)\,|_{x=a}$ is an invertible operator on each $\mathfrak{g}^{\mathfrak{h}}_{(\alpha)}$, $\alpha \in \Delta$, so $(\mathrm{d}f)\,|_{x=a}$ is an invertible linear operator on $\mathfrak{g}$.

Then, by Lemma 9.2, the image of the map f contains a non-empty Zariski open subset in $\mathfrak{g}$, which we denote by $\Omega_{\mathfrak{h}}$. Recall that, by definition, the image of f (and thus the subset $\Omega_{\mathfrak{h}}$) consists of points $\sigma(h)$ for some $\sigma \in G$, all $h \in \mathfrak{h}$.

The above arguments hold for any Cartan subalgebra of $\mathfrak{g}$. Taking arbitrary Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g}$, we obtain the corresponding non-empty Zariski open subsets $\Omega_{\mathfrak{h}_1}$ and $\Omega_{\mathfrak{h}_2}$. Let Ω_r be the subset of all regular elements of $\mathfrak{g}$; this is also a non-empty Zariski open subset. The intersection $\Omega_{\mathfrak{h}_1} \cap \Omega_{\mathfrak{h}_2} \cap \Omega_r$ is then non-empty as well. Rephrasing this, there exists a regular element $x \in \mathfrak{g}$, elements $h_1 \in \mathfrak{h}_1, h_2 \in \mathfrak{h}_2$, and automorphisms $\sigma_1, \sigma_2 \in G$ of $\mathfrak{g}$, such that

$$\sigma_1(h_1) = x = \sigma_2(h_2).$$

Since x is a regular element and σ_1 is an automorphism of $\mathfrak{g}$, the element $h_1 = \sigma_1^{-1}(x)$ is regular as well. Hence, by Lemma 9.1, $\mathfrak{h}_1 = \mathfrak{g}_{(0)}^{h_1}$. Similarly, $\mathfrak{h}_2 = \mathfrak{g}_{(0)}^{h_2}$.

The automorphism $\sigma = \sigma_2^{-1}\sigma_1 \in G$ maps h_1 to h_2, and so

$$\sigma(\mathfrak{h}_1) = \sigma\left(\mathfrak{g}_{(0)}^{h_1}\right) = \mathfrak{g}_{(0)}^{\sigma(h_1)} = \mathfrak{g}_{(0)}^{h_2} = \mathfrak{h}_2.$$

This finishes the proof of Chevalley's Theorem. ■

Exercise 9.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. Show that $\mathfrak{h}_1 = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathfrak{h}_2 = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ are Cartan subalgebras, but they are not conjugate by any automorphism of $\mathfrak{g}$.

Bonus Problem. Any Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ is conjugate by its automorphism to one of these two.

Lecture 10

Trace Form and Cartan's Criterion

Definition 10.1. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let π be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V over $\mathbb{F}$. The associated ***trace form*** is a bilinear form on $\mathfrak{g}$ given by the following formula:

$$(a, b)_V = \operatorname{tr}_V \pi(a)\pi(b).$$

Proposition 10.1.

(a) The trace form is symmetric, i.e.

$$(a, b)_V = (b, a)_V.$$

(b) The trace form is ***invariant***, i.e.

$$([a, b], c)_V = (a, [b, c])_V . \tag{1}$$

Proof. (a) follows from the fact that tr AB = tr BA. For (b), we compute

$$\begin{aligned}([a, b], c)_V &= \operatorname{tr}_V \pi([a, b])\pi(c) \\ &= \operatorname{tr}_V (\pi(a)\pi(b)\pi(c) - \pi(b)\pi(a)\pi(c)) \\ &= \operatorname{tr}_V (\pi(a)\pi(b)\pi(c) - \pi(a)\pi(c)\pi(b)) \\ &= \operatorname{tr}_V \pi(a)\pi([b, c]) = (a, [b, c])_V .\end{aligned}$$ □

Exercise 10.1. As above, a bilinear form $(\bullet, \bullet)$ on a Lie algebra $\mathfrak{g}$ is called ***invariant*** if $([a, b], c) = (a, [b, c])$. Check that invariance of $(\bullet, \bullet)$ means that all operators ad a are ***skew-adjoint*** with respect to this form, i.e.

$$((\operatorname{ad} a)b, c) = -(b, (\operatorname{ad} a)c) .$$

Exercise 10.2. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ of characteristic 0, and let $(\bullet, \bullet)$ be an invariant bilinear form on $\mathfrak{g}$. Let D be a derivation of $\mathfrak{g}$, which is a nilpotent operator, skew-adjoint with respect to $(\bullet, \bullet)$. Then

$$\left(e^D a, e^D b\right) = (a, b) \text{ for all } a, b \in \mathfrak{g}.$$

In particular, $(g(a), g(b)) = (a, b)$ for all $g \in G$ from Chevalley's Theorem. This exercise explains why (1) is called the invariance property of the trace form.

Definition 10.2. If $\dim \mathfrak{g} < \infty$, then the trace form of the adjoint representation is called the Killing form:

$$\kappa(a, b) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad} a)(\operatorname{ad} b), \qquad a, b \in \mathfrak{g}.$$

Exercise 10.3.

(a) Show that the trace form of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{F})$ (resp. of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$), associated to the tautological representation of $\mathfrak{g}$ is non-degenerate (resp. if $\operatorname{char} \mathbb{F} \nmid n$).

(b) Show that the Killing form on $\mathfrak{sl}_n(\mathbb{F})$ is non-degenerate, provided that $\operatorname{char} \mathbb{F} \nmid 2n$. Find the radical of the Killing form on $\mathfrak{gl}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$.

Cartan's Lemma. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 (so that $\mathbb{F} \supset \mathbb{Q}$). Let π be a representation of $\mathfrak{g}$ in a finite-dimensional vector space V. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and consider the generalized weight space decomposition of V and the generalized root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (see Theorem 6.2 and 6.3):

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{(\lambda)}, \tag{2}$$

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{(\alpha)}, \qquad \left[\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{(\beta)}\right] \subseteq \mathfrak{g}_{(\alpha+\beta)}, \ \ \mathfrak{g}_{(0)} = \mathfrak{h}, \tag{3}$$

$$\pi\left(\mathfrak{g}_{(\alpha)}\right) V_{(\lambda)} \subseteq V_{(\lambda+\alpha)}. \tag{4}$$

Pick $e \in \mathfrak{g}_{(\alpha)}$, $f \in \mathfrak{g}_{(-\alpha)}$, so that $h := [e, f] \in \mathfrak{g}_{(0)} = \mathfrak{h}$. Suppose that $V_{(\lambda)} \neq 0$. Then

$$\lambda(h) = r\ \alpha(h), \tag{5}$$

where $r \in \mathbb{Q}$ is independent of h.

Proof. Let $U = \bigoplus_{n \in \mathbb{Z}} V_{(\lambda + n\alpha)} \subseteq V$. Then $\dim U < \infty$, and U is $\pi(e)$- ,$\pi(f)$- and $\pi(h)$-invariant, due to (4). Since $\pi(h) = [\pi(e), \pi(f)]$, $\operatorname{tr}_U \pi(h) = 0$, hence we have:

$$0 = \operatorname{tr}_U \pi(h) = \sum_{n \in \mathbb{Z}} \operatorname{tr}_{V_{(\lambda+n\alpha)}} \pi(h), \tag{6}$$

and since $\pi(h)$ on $V_{(\lambda+n\alpha)}$ has in some basis an upper triangular matrix with $(\lambda + n\alpha)(h)$ on the diagonal, (6) can be rewritten as

$$\lambda(h) \sum_{n \in \mathbb{Z}} \dim V_{(\lambda+n\alpha)} = -\alpha(h) \sum_{n \in \mathbb{Z}} n \dim V_{(\lambda+n\alpha)}. \tag{7}$$

Since $V_{(\lambda)} \neq 0$, (5) follows. ∎

Corollary of Cartan's Lemma. Let V be a finite-dimensional vector space over an algebraically closed field of characteristic 0. If $\mathfrak{g} \subseteq \mathfrak{gl}_V$ is a non-zero subalgebra, such that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then $(a, a)_V \neq 0$ for some $a \in \mathfrak{g}$.

Proof. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, due to the generalized root space decomposition (3), we have

$$\mathfrak{h} = \sum_{\alpha \in \mathfrak{h}^*} \left[\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{-\alpha}\right]. \tag{8}$$

We will prove that $(h, h)_V \neq 0$ for some $h \in \mathfrak{h}$. Suppose the contrary. Then we have for $h_\alpha \in \left[\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{-\alpha}\right]$:

$$0 = (h_\alpha, h_\alpha)_V = \operatorname{tr}_V \pi(h_\alpha)^2 = \sum_{\lambda \in \mathfrak{h}^*} \operatorname{tr}_{V_{(\lambda)}} \pi(h_\alpha)^2 = \sum_{\lambda \in \mathfrak{h}^*} \lambda(h_\alpha)^2 \dim V_{(\lambda)}. \tag{9}$$

By Cartan's Lemma we have

$$\lambda(h_\alpha) = r_{\alpha,\lambda} \alpha(h_\alpha) \text{ for some } r_{\alpha,\lambda} \in \mathbb{Q}, \text{ independent of } h_\alpha. \tag{10}$$

Substituting this in (9), we obtain

$$0 = \left(\sum_{\lambda:\ V_{(\lambda)} \neq 0} r_{\alpha,\lambda}^2 \dim V_{(\lambda)} \right) \alpha(h_\alpha)^2. \tag{11}$$

It follows that $\lambda(h_\alpha) = 0$. Indeed, in the contrary case, by (10), $r_{\alpha,\lambda} \neq 0$ and $\alpha(h_\alpha) \neq 0$, which contradicts (11).

Since, by (8), $\mathfrak{h}$ is spanned by all h_α's, it follows that $\lambda = 0$, hence $V = V_{(0)}$.

Since $\pi\left(\mathfrak{g}_{(\alpha)}\right) V = \pi\left(\mathfrak{g}_{(\alpha)}\right) V_{(0)} \subseteq V_{(\alpha)} = 0$ if $\alpha \neq 0$, it follows that $\mathfrak{g}_{(\alpha)} = 0$ for all $\alpha \neq 0$. Hence $\mathfrak{g} = \mathfrak{h}$ is a nilpotent Lie algebra, which contradicts our assumption that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. ∎

Cartan's Criterion. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_V$, where V is a finite-dimensional vector space over an algebraically closed field of characteristic 0. Then the following properties of $\mathfrak{g}$ are equivalent:

(a) $(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}])_V = 0$, i.e. $(a, b)_V = 0$ for all $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}]$,

(b) $(a, a)_V = 0$ for all $a \in [\mathfrak{g}, \mathfrak{g}]$,

(c) $\mathfrak{g}$ is a solvable Lie algebra.

Proof. $(a) \Rightarrow (b)$ is obvious. $(c) \Rightarrow (a)$ follows from Lie's Thorem, since in some basis of V all matrices of elements of $\mathfrak{g}$ are upper triangular, hence of that of $[\mathfrak{g}, \mathfrak{g}]$ are strictly upper triangular, and so $[a, b]_V = 0$ if $a \in \mathfrak{g}, b \in [\mathfrak{g}, \mathfrak{g}]$.

Finally, we prove the nontrivial part that $(b) \Rightarrow (c)$. Suppose that $(a, a)_V = 0$ for all $a \in [\mathfrak{g}, \mathfrak{g}]$, but $\mathfrak{g}$ is not a solvable Lie algebra. Then the derived series of $\mathfrak{g}$ stabilizes at $\mathfrak{g}^{(n)}$:

$$\left[\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}\right] = \mathfrak{g}^{(n)} \neq 0$$

for some n. By the Corollary to Cartan's Lemma, $(a, a)_V \neq 0$ for some $a \in \mathfrak{g}^{(n)}$, which contradicts (b). ■

Corollary of Cartan's Criterion. A finite-dimensional Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic 0 is solvable if and only if $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.

Proof. Consider the adjoint representation

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}.$$

Its kernel is $Z(\mathfrak{g})$, hence $\mathfrak{g}$ is solvable if and only if $\mathrm{ad}\, \mathfrak{g} \subset \mathfrak{gl}_{\mathfrak{g}}$ is a solvable subalgebra. But, by Cartan's criterion applied to the subalgebra $\mathrm{ad}\, \mathfrak{g}$ in $\mathfrak{gl}_{\mathfrak{g}}$, $\mathrm{ad}\, \mathfrak{g}$ is solvable if and only if $\kappa(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$. ■

Exercise 10.4. Consider the following 4-dimensional solvable Lie algebra $D = \mathfrak{heis}_3 + \mathbb{F}d$, where $\mathfrak{heis}_3 = \mathbb{F}p + \mathbb{F}q + \mathbb{F}c$, with brackets

$$[p, q] = c, \qquad [c, p] = [c, q] = 0,$$
$$[d, p] = p, \qquad [d, q] = -q, \qquad [d, c] = 0.$$

Define on D the bilinear form

$$(p, q) = (c, d) = 1, \qquad \text{and the rest are 0.}$$

Show that D is a solvable Lie algebra, $(\bullet, \bullet)$ is a non-degenerate symmetric invariant bilinear form on D, but $(D, [D, D]) \neq 0$, so Cartan's criterion fails for this bilinear form. (Hence it is not a trace form on some finite-dimensional representation of D).

One often can remove the condition that $\mathbb{F}$ is algebraically closed by the following trick. Let $\overline{\mathbb{F}} \supseteq \mathbb{F}$ be the algebraic closure of $\mathbb{F}$. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, let $\overline{\mathfrak{g}} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} \mathfrak{g}$. In other words, choosing a basis $e_1, e_2, \cdots$ of $\mathfrak{g}$ over $\mathbb{F}$, so that $\mathfrak{g} = \bigoplus_i \mathbb{F}e_i$, let $\overline{\mathfrak{g}} = \bigoplus_i \overline{\mathbb{F}}e_i$, which is a Lie algebra over $\overline{\mathbb{F}}$.

Exercise 10.5.

(a) $\mathfrak{g}$ is solvable (resp. nilpotent) if and only if $\overline{\mathfrak{g}}$ is.

(b) Derive Cartan's criterion and its Corollary for char $\mathbb{F} = 0$, but not necessarily $\overline{F} = \mathbb{F}$.

(c) Show that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent if $\mathfrak{g}$ is solvable over any $\mathbb{F}$ of characteristic 0.

(d) $\mathfrak{g}^a_{(0)}$ is a Cartan subalgebra for every regular element $a \in \mathfrak{g}$, for any field $\mathbb{F}$.

(e) Prove Proposition 5.1 for any field $\mathbb{F}$ of characteristic 0.

Lecture 11

The Radical and Semisimple Lie Algebras

Exercise 11.1. Let $\mathfrak{g}$ be a Lie algebra. Show that

(a) If $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ are ideals, then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are ideals; if, moreover, $\mathfrak{a}$ and $\mathfrak{b}$ are solvable Lie algebras, then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are solvable.

(b) If $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal and $\mathfrak{b} \subseteq \mathfrak{g}$ is a subalgebra, then $\mathfrak{a} + \mathfrak{b}$ is a subalgebra.

Definition 11.1. A ***radical*** $R(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ is a solvable ideal in $\mathfrak{g}$ of maximal dimension.

Proposition 11.1. $R(\mathfrak{g})$ contains all solvable ideals of $\mathfrak{g}$. Consequently $R(\mathfrak{g})$ is the sum of all solvable ideals of $\mathfrak{g}$ and therefore it is unique.

Proof. If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$, then $\mathfrak{a} + \mathbb{R}(\mathfrak{g})$ is again a solvable ideal, by Exercise 11.1(a). Since $R(\mathfrak{g})$ has maximal dimension among solvable ideals, we conclude that $\mathfrak{a} + R(\mathfrak{g}) = R(\mathfrak{g})$, hence $\mathfrak{a} \subseteq R(\mathfrak{g})$. Therefore, $R(\mathfrak{g})$ is the sum of all solvable ideals, hence it is unique. ∎

Definition 11.2. A finite-dimensional Lie algebra is called ***semisimple*** if it is non-zero and its radical is zero.

Proposition 11.2. A non-zero finite-dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if either of the following two conditions holds:

(i) any solvable ideal of $\mathfrak{g}$ is zero;

(ii) any abelian ideal of $\mathfrak{g}$ is zero.

Proof. (i) is obviously equivalent to semisimplicity of $\mathfrak{g}$, and (i) $\Rightarrow$ (ii) is obvious as well. Suppose now that $\mathfrak{g}$ contains a non-zero solvable ideal $\mathfrak{r}$. Then for some $k \geqslant 1$ we have the derived series for $\mathfrak{r}$:

$$\mathfrak{r} = \mathfrak{r}^{(0)} \supsetneq \mathfrak{r}^{(1)} \supsetneq \mathfrak{r}^{(2)} \supsetneq \cdots \supsetneq \mathfrak{r}^{(k)} = 0,$$

hence $\mathfrak{r}^{(k-1)}$ is a non-zero abelian ideal of $\mathfrak{g}$. Hence (ii) $\Rightarrow$ (i). ∎

Remark 11.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$, and $R(\mathfrak{g})$ its radical. Then $\mathfrak{s} := \mathfrak{g}/R(\mathfrak{g})$ is a semisimple Lie algebra.

Indeed, suppose the contrary: $\mathfrak{s}$ contains a non-zero solvable ideal $\mathfrak{r}$, then its preimage $\tilde{\mathfrak{r}}$ in $\mathfrak{g}$ contains $R(\mathfrak{g})$ properly, so that $\tilde{\mathfrak{r}}/R(\mathfrak{g}) \approx \mathfrak{r}$, which is solvable. Hence $\tilde{\mathfrak{r}}$ is a larger solvable ideal than $R(\mathfrak{g})$, a contradiction.

So an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ 'reduces' to a solvable Lie algebra $R(\mathfrak{g})$ and a semisimple Lie algebra $\mathfrak{s} = \mathfrak{g}/R(\mathfrak{g})$.

In the case char $\mathbb{F} = 0$ a stronger result holds:

Levi decomposition theorem. If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic 0, then there exists a semisimple subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, complementary to $R(\mathfrak{g})$, i.e.

$$\mathfrak{g} = \mathfrak{s} \oplus R(\mathfrak{g}) \qquad \text{(direct sum as vector spaces).} \tag{1}$$

We shall prove this theorem later in the course after developing a structure theory of semisimple Lie algebras.

The decomposition (1) is a special case of a semidirect product.

Definition 11.3. A decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$, which is a direct sum of subspaces $\mathfrak{h}$ and $\mathfrak{r}$, where $\mathfrak{h}$ is a subalgebra and $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, is called a ***semi-direct product*** of $\mathfrak{h}$ and $\mathfrak{r}$, and it is denoted by

$$\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{r}.$$

The special case when $\mathfrak{h}$ is an ideal too, corresponds to the direct product $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$.

Note that the open end of $\ltimes$ goes on the side of the ideal. When both are ideals, we use $\times$ (or $\oplus$), and we have direct product (or sum) of ideals.

Exercise 11.2. Let $\mathfrak{h}$ and $\mathfrak{r}$ be Lie algebras and let $\gamma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{r})$ be a Lie algebra homomorphism. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ be the direct sum of vector spaces, and extend the bracket on $\mathfrak{h}$ and on $\mathfrak{r}$ to the whole $\mathfrak{g}$ by letting

$$[h, r] = -[r, h] = \gamma(h)(r) \quad \text{for } h \in \mathfrak{h},\ r \in \mathfrak{r}.$$

Show that this provides $\mathfrak{g}$ with a Lie algebra structure, denoted by $\mathfrak{g} = \mathfrak{h} \overset{\gamma}{\ltimes} \mathfrak{r}$. Show that any semidirect product of Lie algebras is obtained in this way. Finally, show that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{r}$ is the direct product of Lie algebras if and only if $\gamma = 0$.

Cartan-Jacobson Theorem. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}_V$, where V is a finite-dimensional vector space over an algebraically closed field of characteristic 0. Suppose that V is irreducible with respect to $\mathfrak{g}$, i.e. any $\mathfrak{g}$-invariant subspace of V is either 0 or V. Then one of the two possibilities holds:

1. $\mathfrak{g}$ is a semisimple Lie algebra,
2. $\mathfrak{g} = (\mathfrak{g} \cap \mathfrak{sl}_V) \oplus \mathbb{F} I_V$ and $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple (hence $R(\mathfrak{g}) = \mathbb{F} I_V$).

Proof. If $\mathfrak{g}$ is not semisimple, then $R(\mathfrak{g})$ is a non-zero solvable ideal of $\mathfrak{g}$. By Lie's Theorem, there exists a linear function λ on $R(\mathfrak{g})$, such that the weight space V_λ is non-zero. By Lie's Lemma, V_λ is $\mathfrak{g}$-invariant. Hence, by irreducibility, $V_\lambda = V$.

Hence $a = \lambda(a) I_V$ for all $a \in R(\mathfrak{g})$, so $R(\mathfrak{g}) = \mathbb{F} I_V$. Hence $(\mathfrak{g} \cap \mathfrak{sl}_V) \cap R(\mathfrak{g}) = 0$, which proves we have case 2, as $\mathfrak{g} \cap \mathfrak{sl}_V$ is semisimple since it is the complement of the radical. ■

Exercise 11.3. Let V be a finite-dimensional vector space over an algebraically closed field of characteristic 0. Show that $\mathfrak{gl}_V$ and $\mathfrak{sl}_V$ are irreducible. Deduce that $\mathfrak{sl}_V$ is a semisimple Lie algebra.

Exercise 11.4. Let V be a finite-dimensional vector space with a symmetric bilinear form $(\bullet, \bullet)$. Let $U \subseteq V$ be a subspace, such that the restriction $(\bullet, \bullet)\ |_{U \times U}$ is non-degenerate. Denote $U^\perp = \{v \in V \mid (v, U) = 0\}$. Then

$$V = U \oplus U^\perp.$$

Proposition 11.3. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and $(\bullet, \bullet)$ a symmetric invariant bilinear form on $\mathfrak{g}$. Then

(a) If $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal, then $\mathfrak{a}^\perp$ is also an ideal.

(b) If, moreover, $(\bullet, \bullet)\ |_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate, then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, a direct sum of Lie algebras.

Proof. (a) $v \in \mathfrak{a}^\perp$ means that $(v, \mathfrak{a}) = 0$. If $b \in \mathfrak{g}$, then $([v, b], \mathfrak{a}) = (v, [b, \mathfrak{a}]) = 0$, since the bilinear form is invariant and $\mathfrak{a}$ is an ideal. Hence $\mathfrak{a}^\perp$ is an ideal of $\mathfrak{g}$. (b) follows from (a) and Exercise 11.4. ∎

Theorem 11.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then the Killing form κ on $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is semisimple. Furthermore, if $\mathfrak{g}$ is semisimple and $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal, then $\kappa\ |_{\mathfrak{a}\times\mathfrak{a}}$ is also non-degenerate and coincides with the Killing form of $\mathfrak{a}$.

Proof. Suppose that the Killing form κ on $\mathfrak{g}$ is non-degenerate. If $\mathfrak{a} \subseteq \mathfrak{g}$ is an abelian ideal, and $x \in \mathfrak{g},\ y \in \mathfrak{a}$, then $(\text{ad } x)(\text{ad } y)z = [x, [y, z]] \in \mathfrak{a}$ for all $z \in \mathfrak{g}$. It follows that for a basis $e_1, \cdots, e_k$ of $\mathfrak{a}$, contained in the basis $e_1, \cdots, e_k, e_{k+1}, \cdots, e_n$ of $\mathfrak{g}$ the matrix of $(\text{ad } x)(\text{ad } y)$ is of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Since the trace of this matrix is 0, we deduce that $\kappa(\mathfrak{g}, \mathfrak{a}) = 0$, and since κ is non-degenerate, we obtain that $\mathfrak{a} = 0$. Hence $\mathfrak{g}$ is semisimple.

Conversely, let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If $\kappa\ |_{\mathfrak{a}\times\mathfrak{a}}$ is degenerate, so that $\mathfrak{b} = \mathfrak{a} \cap \mathfrak{a}^\perp \neq 0$, then $\mathfrak{b}$ is a non-zero ideal of $\mathfrak{g}$, such that $\kappa(b, b) = 0$ for all $b \in \mathfrak{b}$. By Cartan's Criterion (which holds if char $\mathbb{F} = 0$), it follows that $\mathfrak{b}$ is a non-zero solvable ideal of $\mathfrak{g}$, which contradicts semisimplicity of $\mathfrak{g}$. Thus, if $\mathfrak{g}$ is semisimple, the Killing form is non-degenerate, by taking $\mathfrak{a} = \mathfrak{g}$, and also $\kappa\ |_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate.

Hence, by Proposition 11.3(b), $\mathfrak{g}$ is a direct sum of $\mathfrak{a}$ and $\mathfrak{a}^\perp$. Therefore the Killing form of $\mathfrak{a}$ coincides with the Killing form of $\mathfrak{g}$, restricted to $\mathfrak{a}$. ∎

Definition 11.4. A Lie algebra $\mathfrak{g}$ is called ***simple*** if its only ideals are 0 and $\mathfrak{g}$, and $\mathfrak{g}$ is not abelian.

Corollary 11.1 Any semisimple finite-dimensional Lie algebra over a field of characteristic 0 is isomorphic to a direct sum of simple Lie algebras. Conversely, a direct sum of simple Lie algebras is semisimple.

Proof. If $\mathfrak{g}$ is semisimple, but not simple, and if $\mathfrak{a}$ is a proper ideal, then, by Theorem 11.1, the Killing form, restricted to $\mathfrak{a}$ is a non-degenerate, hence $\mathfrak{g}$ is isomorphic to the direct sum of $\mathfrak{a}$ and $\mathfrak{a}^\perp$, both semisimple and having smaller dimension than $\mathfrak{g}$. After finitely many steps we obtain a decomposition of $\mathfrak{g}$ in a direct sum of simple Lie algebras.

The converse statement is obvious. ∎

Exercise 11.5. Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$ of characteristic p. Let $A_p = \mathbb{F}[x]/x^p\mathbb{F}[x]$. Show that

$$\widetilde{\mathfrak{g}} = \mathbb{F}\frac{\mathrm{d}}{\mathrm{d}x} \ltimes (A_p \otimes_{\mathbb{F}} \mathfrak{g})$$

is a semisimple Lie algebra, which is not isomorphic to a direct sum of non-zero simple ideals. Actually the only proper non-zero ideal of $\widetilde{\mathfrak{g}}$ is $A_p \otimes \mathfrak{g}$. (If A is a commutative associative algebra and $\mathfrak{g}$ is a Lie algebra, both over a field $\mathbb{F}$, then $A \otimes_{\mathbb{F}} \mathfrak{g}$ is a Lie algebra with bracket $[f \otimes a, g \otimes b] = fg \otimes [a, b],\ a, b \in \mathfrak{g},\ f, g \in A$).

Theorem 11.2. Let $\mathfrak{g} \subseteq \mathfrak{gl}_V$ be a semisimple subalgebra, where V is a finite-dimensional vector space over a field $\mathbb{F}$ of characteristic 0. Then the trace form $(\bullet, \bullet)_V$ is non-degenerate on V.

Proof. Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be the kernel of $(\bullet, \bullet)_V$. Since, by Corollary 11.1, $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ and $(a, a)_V = 0$ for all $a \in \mathfrak{g}_0$. It follows from Corollary of Cartan's lemma in Lecture 10 that $\mathfrak{g}_0 = 0$. ∎

Lecture 12

Structure Theory of Semisimple Lie Algebras I

In this lecture and a few that follow, we will study the structure of finite-dimensional semisimple Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0 with the aim of classifying them. This will amount to a detailed knowledge of the generalized root space decomposition. The first step is to use the Killing form to understand Cartan subalgebras and their actions under the adjoint representation.

Unless otherwise stated, we will assume throughout, that our base field $\mathbb{F}$ is algebraically closed of characteristic 0.

Exercise 12.1. Show that the Lie algebra $\mathfrak{sl}_n(\mathbb{F})$ is simple for $n \geqslant 2$, iff char $\mathbb{F}$ doesn't divide $2n$, with no other assumptions on $\mathbb{F}$.

Definition 12.1. Let $\mathfrak{g}$ be a Lie algebra over an arbitrary field $\mathbb{F}$. An ***abstract Jordan decomposition*** of an element $a \in \mathfrak{g}$ is a decomposition of the form

$$a = a_s + a_n,$$

where

(a) $\operatorname{ad} a_s$ is a diagonalizable (semisimple) operator on $\mathfrak{g}$,

(b) $\operatorname{ad} a_n$ is a nilpotent operator on $\mathfrak{g}$,

(c) $[a_s, a_n] = 0$.

An element a is called ***semisimple*** (resp. ***nilpotent***) if $\operatorname{ad} a$ is a semisimple (resp. nilpotent) operator on $\mathfrak{g}$.

Exercise 12.2. Abstract Jordan decomposition in a finite-dimensional Lie algebra $\mathfrak{g}$ is unique, when it exists, if and only if its center $Z(\mathfrak{g}) = 0$.

Theorem 12.1. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{F}$. Then

(a) $Z(\mathfrak{g}) = 0$.

(b) All derivations of $\mathfrak{g}$ are inner.

(c) Any $a \in \mathfrak{g}$ admits a unique Jordan decomposition.

Proof. (a) holds since $Z(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$.

(b) By (a) the homomorphism $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der} \mathfrak{g}$ is an embedding $\mathfrak{g} \subseteq \operatorname{Der} \mathfrak{g}$. Consider the trace form

$$(d_1, d_2) = \operatorname{tr}_{\mathfrak{g}} d_1 d_2, \qquad d_1, d_2 \in \operatorname{Der} \mathfrak{g}.$$

Its restriction to $\mathfrak{g}$ is the Killing form, hence non-degenerate since $\mathfrak{g}$ is semisimple, by Theorem 11.1. Hence, by Proposition 11.2 from Lecture 11, we have:

$$\operatorname{Der} \mathfrak{g} = (\operatorname{ad} \mathfrak{g}) \oplus (\operatorname{ad} \mathfrak{g})^{\perp} \quad \text{(direct sum of ideals).}$$

Take $D \in (\operatorname{ad} \mathfrak{g})^{\perp}$. For $a \in \mathfrak{g}$ we have $[D, \operatorname{ad} a] = 0$. Recall from Exercise 2.1 that $[D, \operatorname{ad} a] = \operatorname{ad} D(a)$. Hence $D(a) \in Z(\mathfrak{g})$, and, by (a), $D(a) = 0$ for all $a \in \mathfrak{g}$. Therefore $\operatorname{Der} \mathfrak{g} = \operatorname{ad} \mathfrak{g}$.

(c): Fix $a \in \mathfrak{g}$, and consider the classical Jordan decomposition

$$\operatorname{ad} a = A_s + A_n,$$

where A_s is diagonalizable on $\mathfrak{g}$, A_s is nilpotent, and $[A_s, A_n] = 0$. Consider the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to ad a:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{F}} \mathfrak{g}^a_{(\lambda)}, \quad \text{where } A_s \,|_{\mathfrak{g}^a_{(\lambda)}} = \lambda I, \; \left[\mathfrak{g}^a_{(\lambda)}, \mathfrak{g}^a_{(\mu)}\right] \subseteq \mathfrak{g}^a_{(\lambda+\mu)}.$$

First, we show that A_s is a derivation of $\mathfrak{g}$. Indeed, for $x \in \mathfrak{g}^a_{(\lambda)}, \; y \in \mathfrak{g}^a_{(\mu)}$ we have:

$$\begin{aligned} A_s([x, y]) &= (\lambda + \mu)[x, y] = [\lambda x, y] + [x, \mu y] \\ &= [A_s x, y] + [x, A_s y] \quad \Longrightarrow \quad A_s \in \operatorname{Der} \mathfrak{g}. \end{aligned}$$

By part (b), $A_s = \operatorname{ad} a_s$ for some $a_s \in \mathfrak{g}$. Letting $a_n = a - a_s$, we have $\operatorname{ad} a_n = A_n$. It remains to check that $[a_s, a_n] = 0$. Since

$$\operatorname{ad}([a_s, a_n]) = [\operatorname{ad} a_s, \operatorname{ad} a_n] = [A_s, A_n] = 0,$$

part (a) gives the result. ■

Theorem 12.2. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{F}$, let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra, and consider the generalized root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_{(\alpha)}, \quad \mathfrak{h} = \mathfrak{g}_{(0)}, \quad \left[\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{(\beta)}\right] \subseteq \mathfrak{g}_{(\alpha+\beta)}.$$

Let κ be the Killing form on $\mathfrak{g}$. Then

(a) $\kappa(\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{(\beta)}) = 0$ if $\alpha + \beta \neq 0$.

(b) $\kappa\,|_{\mathfrak{g}_{(\alpha)} \times \mathfrak{g}_{(-\alpha)}}$ is non-degenerate. In particular, $\kappa\,|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate.

(c) $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$.

(d) $\mathfrak{h}$ consists of semisimple elements, i.e. each ad h for $h \in \mathfrak{h}$ is diagonalizable. Consequently,

$$\mathfrak{g}_{(\alpha)} = \mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a, \text{ for all } h \in \mathfrak{h}\}$$

is a root space.

Proof. (a) If $a \in \mathfrak{g}_{(\alpha)}, b \in \mathfrak{g}_{(\beta)}$, then

$$((\text{ad } a)(\text{ad } b))^N \mathfrak{g}_{(\gamma)} \subseteq \mathfrak{g}_{(\gamma + N(\alpha+\beta))}.$$

Since the RHS is 0 for $N \gg 0$ if $\alpha + \beta \neq 0$, we conclude that the operator (ad a)(ad b) is nilpotent, hence its trace on $\mathfrak{g}$ is 0.

(b) Since the Killing form κ is non-degenerate on $\mathfrak{g}$ by semisimplicity of $\mathfrak{g}$, and $\kappa\left(\mathfrak{g}_{(\alpha)}, \mathfrak{g}_{(\beta)}\right) = 0$ if $\alpha + \beta \neq 0$, by (a), necessarily $\kappa\,|_{\mathfrak{g}_{(\alpha)} \times \mathfrak{g}_{(-\alpha)}}$ is non-degenerate.

(c) Note that $\kappa\,|_{\mathfrak{h} \times \mathfrak{h}}$ is the trace form on $\mathfrak{h}$ under the adjoint representation of it on $\mathfrak{g}$. Since, being nilpotent, $\mathfrak{h}$ is a solvable Lie algebra, by (the easy part of) Cartan's Criterion, we find that

$$0 = \text{tr}_{\mathfrak{g}} \left(\text{ad } \mathfrak{h}, [\text{ad } \mathfrak{h}, \text{ad } \mathfrak{h}]\right) = \kappa\left(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]\right).$$

But by part (b), $\kappa\,|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate. Hence $[\mathfrak{h}, \mathfrak{h}] = 0$, so $\mathfrak{h}$ is an abelian Lie algebra.

(d) Let $h \in \mathfrak{h}$. By Theorem 12.1(c), h has abstract Jordan decomposition $h = h_s + h_n$, where ad h_s is diagonalizable, ad h_n is a nilpotent operator, and $[h_s, h_n] = 0$.

By part (c), $[h, \mathfrak{h}] = 0$. Hence for any $h' \in \mathfrak{h}$ we have: $0 = \text{ad } [h', h] = [\text{ad } h', \text{ad } h]$.

Hence, by Lemma 6.1, we know that $[\text{ad } h', (\text{ad } h)_s] = 0$, yielding

$$0 = [\text{ad } h', (\text{ad } h)_s] = [\text{ad } h', \text{ad } h_s] = \text{ad}\left([h', h_s]\right).$$

Since $Z(\mathfrak{g}) = 0$, it follows that $[h', h_s] = 0$ for all $h' \in \mathfrak{h}$. Since $\mathfrak{h}$ is a Cartan subalgebra, we conclude that $h_s \in \mathfrak{h}$.

It remains to show that $h_n = 0$. Recall that $h_n = h - h_s$, hence $h_n \in \mathfrak{h}$.

Since $\mathfrak{h}$ is a solvable Lie algebra, Lie's Theorem implies that, in some basis of $\mathfrak{g}$, all matrices of elements of ad $\mathfrak{h}$ are upper triangular. As ad h_n is a nilpotent operator, its matrix is strictly upper triangular. Hence $\kappa(\mathfrak{h}, h_n) = 0$. Since, by (b), $\kappa \mid_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, we conclude that $h_n = 0$, implying that $h = h_s$ is a semisimple element. ■

Thus, for a semisimple Lie algebra $\mathfrak{g}$ the generalized root space decomposition is actually a root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha\right), \qquad \mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\},$$

where $\Delta = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$.

Definition 12.2. Elements $\alpha \in \Delta$ are called ***roots*** of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ are called the corresponding ***root spaces***.

Next, we'll be gathering information about roots of $\mathfrak{g}$ and the root spaces $\mathfrak{g}_\alpha$.

We have a canonical linear map of vector spaces

$$\nu : \mathfrak{h} \to \mathfrak{h}^*, \qquad h \mapsto \kappa(h, \bullet).$$

Since the Killing form $\kappa \mid_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate by Theorem 12.2(b), ν is injective, and since $\dim \mathfrak{h} = \dim \mathfrak{h}^* < \infty$, ν is a vector space isomorphism.

Definition 12.3. The Killing form κ on $\mathfrak{h}$ induces a non-degenerate bilinear form on $\mathfrak{h}^*$, using the isomorphism ν:

$$\kappa(\gamma, \gamma') := \kappa\left(\nu^{-1}(\gamma), \nu^{-1}(\gamma')\right), \qquad \text{for } \gamma, \gamma' \in \mathfrak{h}^*.$$

Note that $\kappa(\nu(h), \nu(h')) = \nu(h)(h') = \nu(h')(h)$, for $h, h' \in \mathfrak{h}$.

Theorem 12.3.

(a) If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then

$$[e, f] = \kappa(e, f)\nu^{-1}(\alpha).$$

(b) If $\alpha \in \Delta$, then $\kappa(\alpha, \alpha) \neq 0$.

Proof. (a) We know that $[e, f] \in \mathfrak{h}$. Since, by Theorem 12.2(b), $\kappa \mid_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, it is enough to show that $\kappa\left([e, f] - \kappa(e, f)\nu^{-1}(\alpha), h\right) = 0$ for all $h \in \mathfrak{h}$. We compute:

$$\begin{aligned} \text{LHS} &= \kappa([e, f], h) - \kappa(e, f)\kappa(\nu^{-1}(\alpha), h) \\ &= \kappa(e, [f, h]) - \kappa(e, f)\alpha(h) \qquad \text{by invariance of } \kappa. \end{aligned}$$

But $f \in \mathfrak{g}_{-\alpha}$, hence $[f, h] = \alpha(h)f$, hence

$$\text{RHS} = \alpha(h)\kappa(e, f) - \kappa(e, f)\alpha(h) = 0,$$

proving (a).

(b) Since, by Theorem 12.2(b), $\kappa \mid_{\mathfrak{g}_\alpha\times\mathfrak{g}_{-\alpha}}$ is non-degenerate, there exist $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$, such that $\kappa(e, f) = 1$, hence, by part (a),

$$[e, f] = \nu^{-1}(\alpha).$$

Also $[\nu^{-1}(\alpha), e] = \alpha\left(\nu^{-1}(\alpha)\right)e = \kappa(\alpha, \alpha)e$, and similarly

$$[\nu^{-1}(\alpha), f] = -\alpha(\nu^{-1}(\alpha))f = -\kappa(\alpha, \alpha)f.$$

Suppose the contrary to (c): that $\kappa(\alpha, \alpha) = 0$. Then by the above relations, the Lie algebra

$$\mathfrak{a} = \mathbb{F}e + \mathbb{F}f + \mathbb{F}\nu^{-1}(\alpha)$$

is isomorphic to the nilpotent Lie algebra $\mathfrak{heis}_3$ with center $\mathbb{F}\nu^{-1}(\alpha) = [\mathfrak{a}, \mathfrak{a}]$.

Applying Lie's Theorem to the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$, we can find a basis of $\mathfrak{g}$, such that ad e and ad f have upper triangular matrices, and hence ad $\nu^{-1}(\alpha)$ has strictly upper triangular matrix. Hence $\nu^{-1}(\alpha)$ is an nilpotent element of $\mathfrak{g}$. But $\nu^{-1}(\alpha) \in \mathfrak{h}$ is a semisimple element of $\mathfrak{g}$. So we conclude that $\nu^{-1}(\alpha) = 0$, hence $\alpha = 0$. This contradicts the fact that $\alpha \in \Delta$. ∎

Exercise 12.3.

(a) Show that all derivations of the 2-dimensional non-abelian Lie algebra are inner.

(b) Describe the Lie algebra $\mathrm{Der}(\mathfrak{heis}_3)$. Namely, show that it is isomorphic to the semidirect product of $\mathfrak{sl}_2(\mathbb{F})$ and a 3-dimensional radical, provided that char $\mathbb{F} \neq 2$, while inner derivations form a 2-dimensional abelian ideal.

Bonus Problem. Prove that all derivations of the subalgebra of upper-triangular matrices in $\mathfrak{sl}_n(\mathbb{F})$ are inner.

Exercise 12.4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic 0. Show that $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is non-degenerate. Show that Theorem 12.1 and Theorem 12.2(c) hold (i.e. $\mathbb{F}$ is not necessarily algebraically closed).

Lecture 13

Structure Theory of Semisimple Lie Algebras II

Throughout this and next lecture, $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0.

So far we have proved:

1. The Killing form κ on $\mathfrak{g}$ is non-degenerate.
2. Pick a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$. Then $\mathfrak{h}$ is abelian and consists of semisimple elements of $\mathfrak{g}$, and we have the root space decomposition:
$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}_0 = \mathfrak{h},$$
where
$$\Delta = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$$
is the set of roots; and
$$\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}$$
is the root space, attached to $\alpha \in \Delta$;
$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}.$$
3. κ defines a pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, i.e. the map $\mathfrak{g}_\alpha \to \mathfrak{g}^*_{-\alpha}$, defined by $a \mapsto \kappa(a, \bullet)$ is a vector space isomorphism. In particular, $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$ and $\alpha \in \Delta$ if and only if $-\alpha \in \Delta$.
4. $\kappa\,|_{\mathfrak{h}\times\mathfrak{h}}$ is a non-degenerate bilinear form, hence we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$, defined by $\nu(h)(h') = \kappa(h, h')$ for all $h, h' \in \mathfrak{h}$. The map ν defines a bilinear form on $\mathfrak{h}^*$ by
$$\kappa(\alpha, \beta) = \beta(\nu^{-1}(\alpha)) = \alpha(\nu^{-1}(\beta)) \text{ for } \alpha, \beta \in \mathfrak{h}^*.$$
We proved that $\kappa(\alpha, \alpha) \neq 0$ if $\alpha \in \Delta$.
5. If $\alpha \in \Delta$, $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$, then
$$[e, f] = \kappa(e, f)\nu^{-1}(\alpha) \in \mathfrak{h}.$$

Next, given $\alpha \in \Delta$, pick a non-zero $E \in \mathfrak{g}_\alpha$ and $F \in \mathfrak{g}_{-\alpha}$, such that $\kappa(E, F) = \frac{2}{\kappa(\alpha,\alpha)}$ (which is possible by 3 and the last claim in 4.), and let $H = \frac{2\nu^{-1}(\alpha)}{\kappa(\alpha,\alpha)}$. Then we have:
$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Indeed, we have by 5.:

$$[E, F] = \kappa(E, F)\nu^{-1}(\alpha) = H;$$

next,
$$[H, E] = \frac{2}{\kappa(\alpha, \alpha)}[\nu^{-1}(\alpha), E] = \frac{2}{\kappa(\alpha, \alpha)}\alpha\left(\nu^{-1}(\alpha)\right)E = 2E.$$

The verification of the second equality is similar.

Thus, for each root $\alpha \in \Delta$, we constructed a 3-dimensional subalgebra $\mathfrak{a}_\alpha = \mathbb{F}E + \mathbb{F}H + \mathbb{F}F$ of $\mathfrak{g}$, isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, via the map

$$E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Exercise 13.1. Describe the root space decomposition of $\mathfrak{sl}_n(\mathbb{F})$ and find all $\mathfrak{a}_\alpha$ in it, for $\mathfrak{h}$ = traceless diagonal matrices.

Key $\mathfrak{sl}_2$ Lemma. Let $\mathbb{F}$ be a field of characteristic 0. Let π be a representation of $\mathfrak{sl}_2(\mathbb{F})$ in a vector space V over $\mathbb{F}$ (not necessarily finite-dimensional), and let $v \in V$ be a non-zero vector, such that $\pi(E)v = 0$ and $\pi(H)v = \lambda v$, where $\lambda \in \mathbb{F}$ (such vector is called a singular vector of weight λ). Then

(a) $\pi(H)\pi(F)^n v = (\lambda - 2n)\pi(F)^n v$ for any $n \in \mathbb{Z}_{\geqslant 0}$.

(b) $\pi(E)\pi(F)^n v = n(\lambda - n + 1)\pi(F)^{n-1} v$ for any $n \in \mathbb{Z}_{\geqslant 1}$.

(c) If $\dim V < \infty$, then $\lambda \in \mathbb{Z}_{\geqslant 0}$, the vectors $\pi(F)^j v$ for $0 \leqslant j \leqslant \lambda$ are linearly independent, and $\pi(F)^{\lambda+1} v = 0$. Consequently, in this case the eigenvalues of $\pi(H)$ on the span of the vectors $\pi(F)^j v$, where j$\in \mathbb{Z}_{\geqslant 0}$, are $\lambda, \lambda - 2, \lambda - 4, ..., -\lambda$.

(d) If $\lambda \notin \mathbb{Z}_{\geqslant 0}$, then all $\pi(F)^j v$ for $j \in \mathbb{Z}_{\geqslant 0}$ are linearly independent, hence $\dim V = \infty$.

Proof. (a) We prove this by induction on n. For $n = 0$ it is given to us. Suppose it holds for $n = k - 1$ for some $k > 0$. Then

$$\begin{aligned}
\pi(H)\pi(F)^k v &= \pi(H)\pi(F)\pi(F)^{k-1} v \\
&= [\pi(H), \pi(F)]\,\pi(F)^{k-1} v + \pi(F)\left(\pi(H)\pi(F)^{k-1} v\right) \\
&= -2\pi(F)^k v + \pi(F)(\lambda - 2(k-1))\pi(F)^{k-1} v \\
&= (\lambda - 2k)\pi(F)^k v.
\end{aligned}$$

(b) is proved in a similar way, by induction on $n \in \mathbb{Z}_{\geqslant 1}$, and is left as an exercise.

In order to prove (c) and (d), note that by (a), all vectors $\pi(F)^j v$ for $j \in \mathbb{Z}_{\geqslant 0}$ have distinct eigenvalues with respect to $\pi(H)$. Hence, by linear algebra, those of them which are non-zero, are linearly independent.

Now the first claim of (c) follows from (b). The second claim of (c) follows from (b) as well, since, if $\pi(F)^{\lambda+1}v \neq 0$, (b) implies that all $\pi(F)^n v \neq 0$ for $n > \lambda+1$, which contradicts that V is finite-dimensional .

(d) follows from (b) as well by the same argument. ■

Exercise 13.2. Prove claim (b).

Exercise 13.3. (the F-analogue of the key $\mathfrak{sl}_2$ lemma) Using the notation of the key $\mathfrak{sl}_2$ lemma, if v instead satisfies $\pi(F)v = 0$ and $\pi(H)v = \lambda v$, then we have:

(a) $\pi(H)\pi(E)^n v = (\lambda + 2n)\pi(E)^n v$ for $n \in \mathbb{Z}_{\geqslant 0}$.

(b) $\pi(F)\pi(E)^n v = -n(\lambda + n - 1)\pi(E)^{n-1} v$ for $n \in \mathbb{Z}_{\geqslant 1}$.

(c) If $\dim V < \infty$, then $-\lambda \in \mathbb{Z}_{\geqslant 0}$, the vectors $\pi(E)^j v$ are linearly independent for $0 \leqslant j \leqslant -\lambda$, and $\pi(E)^{-\lambda+1} v = 0$.

(d) If $-\lambda \notin \mathbb{Z}_{\geqslant 0}$, then all $\pi(E)^j v$ for $j \in \mathbb{Z}_{\geqslant 0}$ are linearly independent, hence $\dim V = \infty$.

Hint: One can do it by a direct computation. Alternatively, introduce the automorphism φ of $\mathfrak{sl}_2(\mathbb{F})$ by $\varphi(E) = F, \varphi(F) = E, \varphi(H) = -H$ and its representation $\pi \circ \varphi$ in V.

In order to state the next theorem on the properties of the root space decomposition, we introduce the following notation:

$$A_{\alpha,\beta} = \frac{2\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)} \text{ for } \alpha,\beta \in \Delta, \text{ called } \textbf{\textit{Cartan integers}},$$
$$\widetilde{\Delta} = \Delta \cup \{0\}\,.$$

Theorem 13.1. The root space decomposition of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ and the set of roots $\Delta \subset \mathfrak{h}^* \backslash \{0\}$ satisfy the following properties:

(a) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

(b) (String property) If $\alpha, \beta \in \Delta$, then $\{\beta + n\alpha\}_{n \in \mathbb{Z}} \cap \widetilde{\Delta}$ is a finite connected string

$$\{\beta - p\alpha, \beta - (p-1)\alpha, \cdots, \beta, \beta + \alpha, \cdots, \beta + q\alpha\}$$

where $p, q \in \mathbb{Z}_{\geqslant 0}$ and $p - q = A_{\alpha,\beta} \in \mathbb{Z}$.

(c) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

(d) If $\alpha \in \Delta$, then $n\alpha \in \Delta$ for $n \in \mathbb{F}$ if and only if $n = 1$ or $n = -1$.

Proof. Suppose the contrary to (a): $\dim \mathfrak{g}_\alpha > 1$ for some $\alpha \in \Delta$. Then, by 3., $\dim \mathfrak{g}_{-\alpha} > 1$, and there exists a non-zero vector $v \in \mathfrak{g}_{-\alpha}$, such that $\kappa(E, v) = 0$, where $\mathbb{F}E + \mathbb{F}H + \mathbb{F}F = \mathfrak{a}_\alpha$, $H = \frac{2\nu^{-1}(\alpha)}{\kappa(\alpha,\alpha)}$, $\mathfrak{a}_\alpha \simeq \mathfrak{sl}_2(\mathbb{F})$.

Consider the adjoint representation of $\mathfrak{a}_\alpha$ on $\mathfrak{g}$. We have $(\mathrm{ad}\ E)v = \kappa(E, v)\nu^{-1}(\alpha) = 0$, and

$$(\mathrm{ad}\ H)v = [H, v] = \frac{2[\nu^{-1}(\alpha), v]}{\kappa(\alpha, \alpha)} = -\frac{2\alpha(\nu^{-1}(\alpha))v}{\kappa(\alpha, \alpha)} = -\frac{2\kappa(\alpha, \alpha)v}{\kappa(\alpha, \alpha)} = -2v.$$

Hence, by the key $\mathfrak{sl}_2$ lemma (d), $\dim \mathfrak{g} = \infty$, a contradiction.

(b) Consider the following subspace of $\mathfrak{g}$

$$U_{\alpha,\beta} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}.$$

It is obviously ad $\mathfrak{a}_\alpha$-invariant. The maximal eigenvalue of ad H on $U_{\alpha,\beta}$ is equal to

$$\lambda_{\max} := A_{\alpha,\beta} + 2q, \tag{13.1}$$

and its minimal eigenvalue is equal to

$$\lambda_{\min} := A_{\alpha,\beta} - 2p. \tag{13.2}$$

But by the key $\mathfrak{sl}_2$ lemma (c), the string is connected and $\lambda_{\min} = -\lambda_{\max}$. Hence, adding (13.1) and (13.2), we obtain that $p - q = A_{\alpha,\beta}$. Both p and q are non-negative integers since $\beta \in \Delta$.

(c) Pick the largest integers p and q, such that $\beta - p\alpha, \beta + q\alpha \in \widetilde{\Delta}$, and pick a non-zero vector $v \in \mathfrak{g}_{\beta-p\alpha}$. Then $(\mathrm{ad}\ F)v = 0$ and $(\mathrm{ad}\ H)v = (A_{\alpha,\beta} - 2p)v$.

By Exercise 13.2, $(\mathrm{ad}\ E)^j v \neq 0$ for $0 \leqslant j \leqslant 2p - A_{\alpha\beta} = p + q$. But $q \geqslant 1$ since $\alpha + \beta \in \Delta$, so $(\mathrm{ad}\ E)^{p+1} v \neq 0$. The root of this vector is $\alpha + \beta$ and $(\mathrm{ad}\ E)^p v \in \mathfrak{g}_\beta$, hence $[E, \mathfrak{g}_\beta] \neq 0$. Therefore $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ since $\dim \mathfrak{g}_{\alpha+\beta} = 1$ by (a).

(d) Let $\beta = n\alpha$, $n \neq 0$, be a root. Then, since by (b), $A_{\beta\alpha} \in \mathbb{Z}$, we conclude that $A_{\beta\alpha} = \frac{2\kappa(\alpha, n\alpha)}{\kappa(n\alpha, n\alpha)} = \frac{2n}{n^2} = \frac{2}{n} \in \mathbb{Z}$; also $A_{\alpha\beta} = 2n \in \mathbb{Z}$. Hence the possible values of n are $2, 1, -1, -2, \frac{1}{2}, -\frac{1}{2}$. However $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$ since $\dim \mathfrak{g}_\alpha = 1$, so $2\alpha \notin \Delta$ by (c). Similarly $\frac{1}{2}\alpha \notin \Delta$. Since $\gamma \in \Delta$ if and only if $-\gamma \in \Delta$ by 3., we conclude that $-2\alpha \notin \Delta$ and $-\frac{1}{2}\alpha \notin \Delta$ as well. ■

Exercise 13.4. Find which p and q are possible for $\mathfrak{sl}_n(\mathbb{F})$.

Exercise 13.5. Show that an element $h \in \mathfrak{h}$ is regular if and only if $\alpha(h) \neq 0$ for all $\alpha \in \Delta$.

Exercise 13.6. Show that an elmeent $a \in \mathfrak{g}$ is regular only if it is semisimple.

Remark. By definition, for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$ an element $a \in \mathfrak{g}$ is semisimple if and only if it is contained in a Cartan subalgebra of $\mathfrak{g}$. It is known that this fact holds for any semisimple Lie algebra $\mathfrak{g}$.

Lecture 14

Structure Theory of Semisimple Lie Algebras III

Recall the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right), \quad \mathfrak{h} = \mathfrak{g}_0, \quad [\mathfrak{h}, \mathfrak{h}] = 0, \quad \Delta = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\},$$

$$\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \text{ for all } h \in \mathfrak{h}\}, \quad \dim \mathfrak{g}_\alpha = 1 \text{ if } \alpha \in \Delta, \quad \widetilde{\Delta} = \Delta \cup \{0\}.$$

By Theorem 13.1, the set of roots Δ satisfies the properties:

(i) If $\alpha \in \Delta$, then $n\alpha \in \Delta$ for $n \in \mathbb{F}$ if and only if $n = 1$ or -1.

(ii) (String property) If $\alpha, \beta \in \Delta$, then $\{\beta + n\alpha\}_{n \in \mathbb{Z}} \cap \widetilde{\Delta}$ is a finite connected string

$$\{\beta - p\alpha, \beta - (p-1)\alpha, \cdots, \beta, \beta + \alpha, \cdots, \beta + q\alpha\}$$

where $p, q \in \mathbb{Z}_{\geqslant 0}$ and $p - q = A_{\alpha,\beta} := \frac{2\kappa(\alpha,\beta)}{\kappa(\alpha,\alpha)}$.

The Killing form κ is non-degenerate on $\mathfrak{g}$, as well as its restriction to $\mathfrak{h}$. We have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$, defined by $\nu(h)(h') = \kappa(h, h')$, $h, h' \in \mathfrak{h}$.

Using the root space decomposition of $\mathfrak{g}$, we can rewrite κ on $\mathfrak{h}$ as a sum over roots. Namely, for $h_1, h_2 \in \mathfrak{h}$ we have (since $\dim \mathfrak{g}_\alpha = 1$ if $\alpha \in \Delta$ and $\mathfrak{h}$ is abelian)

$$\kappa(h_1, h_2) = \operatorname{tr}_{\mathfrak{g}} (\text{ad } h_1)(\text{ad } h_2) = \sum_{\alpha \in \Delta} \alpha(h_1)\alpha(h_2). \tag{1}$$

Hence, using ν, we deduce the following formula for κ on $\mathfrak{h}^*$:

$$\kappa(\lambda_1, \lambda_2) = \sum_{\alpha \in \Delta} \alpha\left(\nu^{-1}(\lambda_1)\right) \alpha\left(\nu^{-1}(\lambda_2)\right) = \sum_{\alpha \in \Delta} \kappa(\lambda_1, \alpha)\kappa(\lambda_2, \alpha). \tag{2}$$

Definition 14.1. The $\mathbb{Q}$-span of Δ in $\mathfrak{h}$ is denoted by $\mathfrak{h}^*_{\mathbb{Q}}$.

Theorem 14.1.

(a) Δ spans $\mathfrak{h}^*$ over $\mathbb{F}$.

(b) $\kappa(\alpha, \beta) \in \mathbb{Q}$ for $\alpha, \beta \in \Delta$.

(c) $\kappa \mid_{\mathfrak{h}^*_{\mathbb{Q}} \times \mathfrak{h}^*_{\mathbb{Q}}}$ is a positive definite symmetric bilinear form with values in $\mathbb{Q}$.

Proof. (a) Suppose the contrary: $\mathfrak{h}^*$ is not spanned by Δ over $\mathbb{F}$. Then there exists a non-zero $h \in \mathfrak{h}$, such that $\alpha(h) = 0$ for all $\alpha \in \Delta$.

This implies that $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in \Delta$, and since $[h, \mathfrak{h}] = 0$, we conclude that $h \in Z(\mathfrak{g})$. Hence $h = 0$ since the center of $\mathfrak{g}$ is zero, which is a contradiction.

(b) From equation (2) we obtain for $\lambda \in \mathfrak{h}^*$,

$$\kappa(\lambda, \lambda) = \sum_{\alpha \in \Delta} \kappa(\lambda, \alpha)^2. \tag{3}$$

Recall that $\kappa(\beta, \beta) \neq 0$ for $\beta \in \Delta$, so, by (3), taking $\lambda = \beta$ and multiplying both sides by $(\frac{2}{\kappa(\beta,\beta)})^2$, we obtain:

$$\frac{4}{\kappa(\beta, \beta)} = \sum_{\alpha \in \Delta} \left(\frac{2\kappa(\alpha, \beta)}{\kappa(\beta, \beta)} \right)^2 = \sum_{\alpha \in \Delta} A_{\beta,\alpha}^2.$$

Since, by the string property, $A_{\beta,\alpha} \in \mathbb{Z}$, we conclude that $\kappa(\beta, \beta) \in \mathbb{Q}_{>0}$ for any root β. Since $A_{\alpha,\beta} = \frac{2\kappa(\alpha,\beta)}{\kappa(\beta,\beta)} \in \mathbb{Z}$, we conclude that $\kappa(\alpha, \beta) \in \mathbb{Q}$ for $\alpha, \beta \in \Delta$, proving (b).

(c) It follows from (b) that $\kappa(\lambda, \mu) \in \mathbb{Q}$ for any $\lambda, \mu \in \mathfrak{h}_{\mathbb{Q}}^*$. Since, by (a), Δ spans $\mathfrak{h}^*$, and $\kappa |_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate, we see that $\kappa |_{\mathfrak{h}_{\mathbb{Q}}^* \times \mathfrak{h}_{\mathbb{Q}}^*}$ is non-degenerate as well.

By equation (3), $\kappa(\lambda, \lambda) \geqslant 0$ for all $\lambda \in \mathfrak{h}_{\mathbb{Q}}^*$ since it is a sum of rational squares. This proves part (c) since a non-degenerate symmetric positive semi-definite bilinear form κ on $\mathfrak{h}_{\mathbb{Q}}^*$ is positive definite. ■

The following simple lemma is very important in representation theory, and will be used to prove our next theorem.

Lemma 14.1. Let $\mathfrak{h}$ be a Lie algebra over an infinite field $\mathbb{F}$ and let π be its representation in a vector space V, such that V has a weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \qquad \text{where } V_\lambda = \{v \in V \mid \pi(h)v = \lambda(h)v,\ h \in \mathfrak{h}\}.$$

If $U \subseteq V$ is $\pi(\mathfrak{h})$-invariant subspace, then

$$U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda).$$

Proof. Any $u \in U$ can be written as

$$u = \sum_{i=1}^{n} v_{\lambda_i}, \quad \text{where } v_{\lambda_i} \in V_{\lambda_i} \backslash \{0\},\ \lambda_i \neq \lambda_j. \tag{4}$$

We will prove by induction on n that all $v_{\lambda_i} \in U$. For $n = 1$, $v_{\lambda_1} = u \in U$. If $n > 1$, we apply

$\pi(h)$ to both sides of (4):

$$\pi(h)u = \sum_{i=1}^{n} \lambda_i(h) v_{\lambda_i}, \tag{5}$$

where h is chosen such that $\lambda_i(h) \neq \lambda_j(h)$ for $i \neq j$ (here we use that $\mathbb{F}$ is infinite).

From (4) we obtain:

$$\pi(h)u - \lambda_1(h)u = \sum_{i=2}^{n} (\lambda_i(h) - \lambda_1(h))\, v_{\lambda_i},$$

where each coefficient is not 0.

Hence, by the inductive assumption $v_{\lambda_i} \in U$ for all $i \geqslant 2$, hence also $v_{\lambda_1} \in U$. ∎

Exercise 14.1. Recall that a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^{N} \mathfrak{s}_j$ is a direct sum of simple ideals. Prove that any ideal of $\mathfrak{g}$ is a subsum of this sum and that this decomposition is unique up to permutation of summands.

Next, we examine how a decomposition of $\mathfrak{g}$ in a direct sum of (semisimple) ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, corresponds to a decomposition of the set of roots Δ of $\mathfrak{g}$. Choose Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively; then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta_1 \subset \mathfrak{h}_1^*$ and $\Delta_2 \subset \mathfrak{h}_2^*$ be the sets of roots for $\mathfrak{g}_1$ and $\mathfrak{g}_2$, so that we have root space decompositions

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}_i = \mathfrak{h}_i \oplus \left(\bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha \right) \text{ for } i = 1, 2.$$

Then $\Delta = \Delta_1 \coprod \Delta_2$, and for $\alpha \in \Delta_1$, $\beta \in \Delta_2$ we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, hence $\alpha + \beta \notin \widetilde{\Delta} = \Delta \cup \{0\}$.

Definition 14.2. Let Δ be a finite subset of non-zero vectors in a vector space V. This set is called ***indecomposable*** if it cannot be decomposed into a disjoint union of non-empty subsets Δ_1 and Δ_2, such that

$$\alpha + \beta \notin \widetilde{\Delta} \text{ if } \alpha \in \Delta_1, \beta \in \Delta_2. \tag{5}$$

The following theorem provides a simple way to check if a finite-dimensional Lie algebra is semisimple (resp. simple).

Theorem 14.2. Let $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha \right)$ be a decomposition of a finite-dimensional Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ into a direct sum of subspaces, such that the following properties hold:

(i) $\mathfrak{h}$ is an abelian subalgebra, $\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a, h \in \mathfrak{h}\}$, and $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$,

(ii) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha$, where $h_\alpha \in \mathfrak{h}$ is such that $\alpha(h_\alpha) \neq 0$,

(iii) $\mathfrak{h}^*$ is spanned by Δ.

(All these properties hold for a semisimple $\mathfrak{g}$ over an algebraically closed field of characteristic 0.) Then $\mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{h}$ is its Cartan subalgebra. Moreover $\mathfrak{g}$ is simple if and only if the set Δ is indecomposable.

Proof. We need to prove that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{a} = 0$. Note that, since $\mathfrak{a}$ is an ideal, it is ad $\mathfrak{h}$-invariant. Hence, by Lemma 14.1, if $\mathfrak{a} \neq 0$, then either $\mathfrak{g}_\alpha \subseteq \mathfrak{a}$ for some $\alpha \in \Delta$, or $\mathfrak{h} \cap \mathfrak{a} \neq 0$, by the properly that $\dim \mathfrak{g}_\alpha = 1$.

In the first case $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha \subseteq \mathfrak{a}$, and, since $\alpha(h_\alpha) \neq 0$, $[h_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha \subseteq \mathfrak{a}$. So $\mathfrak{a}$ contains a non-abelian subalgebra $\mathbb{F}h_\alpha + \mathfrak{g}_\alpha$, which is impossible since $\mathfrak{a}$ is abelian.

In the second case, $\mathfrak{a}$ contains a non-zero element h from $\mathfrak{h}$. By condition (iii), $\alpha(h) \neq 0$ for some $\alpha \in \Delta$, hence $[h, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha \subseteq \mathfrak{a}$, and $\mathfrak{a}$ again contains a non-abelian subalgebra $\mathbb{F}h + \mathfrak{g}_\alpha$, which is again impossible. So $\mathfrak{a} = 0$, which proves that $\mathfrak{g}$ is semisimple.

It is obvious that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Finally, why $\mathfrak{g}$ is simple if the set $\Delta \subset \mathfrak{h}^*$ is indecomposable? In the contrary case, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are (non-zero) semisimple ideals, and by the discussion preceding Definition 14.2, this implies that the set Δ is not indecomposable. ∎

Exercise 14.2. Prove that if a semisimple Lie algebra $\mathfrak{g}$ with the set of roots Δ is simple, then the set Δ is indecomposable.

The following is an easy method to check if a set Δ is indecomposable.

Exercise 14.3. Show that a finite subset Δ of non-zero vectors in a vector space V is indecomposable if and only if for any $\alpha, \beta \in \Delta$, there exists a sequence $\gamma_1, \cdots, \gamma_s \in \Delta$, such that $\alpha = \gamma_1, \beta = \gamma_s$, and $\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$ for $i = 1, \cdots, s-1$.

Lecture 15

Examples of Classical Semisimple and Simple Lie Algebras

Recall Theorem 14.2, which says that if $\mathfrak{g}$ is a finite-dimensional Lie algebra over an infinite field $\mathbb{F}$, decomposed in a direct sum of subspaces

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta \subseteq \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \right), \tag{1}$$

where $\mathfrak{h}$ is an abelian subalgebra,

$$\begin{gathered} \mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h,a] = \alpha(h)a,\ h \in \mathfrak{h}\}, \quad \dim \mathfrak{g}_\alpha = 1, \\ [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha \in \mathfrak{h}, \quad \alpha(h_\alpha) \neq 0, \quad \text{and} \quad \mathfrak{h}^* \text{ is spanned by } \Delta, \end{gathered} \tag{2}$$

then $\mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{h}$ is its Cartan subalgebra.

Moreover $\mathfrak{g}$ is simple if and only if Δ is an indecomposable set, meaning that for any $\alpha, \beta \in \Delta, \alpha \neq \beta$, these exists a sequence of elements from Δ, $\alpha = \gamma_1, \gamma_2, \cdots, \gamma_s = \beta$, such that $\gamma_i + \gamma_{i+1} \in \Delta \cup \{0\}$ for all $i = 1, \cdots, s-1$.

Example 15.1. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$, $n \geqslant 2$ (special linear Lie algebra). Let D be the subalgebra of all diagonal matrices in $\mathfrak{gl}_n(\mathbb{F})$, and let $\mathfrak{h} = D \cap \mathfrak{sl}_n(\mathbb{F})$ be the subspace of all traceless diagonal matrices. It is a Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{F})$ (by Theorem 8.1). Denote by $\varepsilon_i \in D^*$ the linear function, defined by $\varepsilon_i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_i,\ i = 1, \cdots, n$. They form a basis of D^*. Their restrictions to $\mathfrak{h}$, also denoted by ε_i, are linearly dependent: $\varepsilon_1 + \cdots + \varepsilon_n = 0$. However, the vectors

$$\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$$

form a basis of D^* if char $\mathbb{F} \nmid n$, since

Exercise 15.1. $\begin{vmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = n.$

Hence $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{n-1} - \varepsilon_n\}$ is a basis of $\mathfrak{h}^*$. We have the decomposition (1), where

$$\Delta_{\mathfrak{sl}_n(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j \mid i,j = 1, \cdots, n,\ i \neq j\},\ \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{F}E_{ij}. \tag{3}$$

We have: $[E_{ij}, E_{ji}] = E_{ii} - E_{jj} =: h_{\varepsilon_i - \varepsilon_j}$, and $(\varepsilon_i - \varepsilon_j)(E_{ii} - E_{jj}) = 2 \neq 0$ if char $\mathbb{F} \neq 2$. Hence condition (2) holds if char $\mathbb{F} \neq 2$ and char $\mathbb{F} \nmid n$ (by Exercise 15.1).

Therefore $\mathfrak{sl}_n(\mathbb{F})$ is semisimple if char $\mathbb{F} \nmid 2n$. To check simplicity, consider two roots:

$$\alpha = \varepsilon_i - \varepsilon_j, \quad \beta = \varepsilon_k - \varepsilon_s, \quad \text{where } i \neq j,\ k \neq s,$$

and construct the sequences $\alpha = \gamma_1, \cdots, \gamma_s = \beta$. If $j = k$, the sequence $\alpha = \gamma_1, \gamma_2 = \beta$ works. If $j \neq k$, the sequence $\alpha = \gamma_1, \gamma_2 = \varepsilon_j - \varepsilon_k, \gamma_3 = \beta$ works. Hence $\mathfrak{sl}_n(\mathbb{F})$ is a simple Lie algebra if char $\mathbb{F} \nmid 2n$.

This series of simple Lie algebras is denoted by A_{n-1} (since rank $\mathfrak{sl}_n(\mathbb{F}) = n - 1$), $n \geqslant 2$. This is called the classical series of type A_n, $n \geqslant 1$.

Example 15.2. Let B be a non-degenerate symmetric bilinear form on N-dimensional vector space V over $\mathbb{F}$. Recall the corresponding ***orthogonal Lie algebra***.

$$\mathfrak{o}_{V,B}(\mathbb{F}) = \{a \in \mathfrak{gl}_V(\mathbb{F}) \mid B(au, v) + B(u, av) = 0,\ u, v \in V\} \subseteq \mathfrak{gl}_V(\mathbb{F}), \tag{4}$$

where char $\mathbb{F} \neq 2$. Choosing a basis of V and denoting again by B the matrix of the bilinear form in this basis (which is non-singular symmetric), we proved that we get the subalgebra of $\mathfrak{gl}_N(\mathbb{F})$ (see Exercise 1.4)

$$\mathfrak{o}_{N,B}(\mathbb{F}) = \left\{a \in \mathfrak{gl}_N(\mathbb{F}) \mid a^\top B + Ba = 0\right\}, \tag{5}$$

where $a^\top$ is the transposition of a with respect to the principal diagonal.
If $\mathbb{F}$ is algebraically closed, one can choose a basis of V in which B is any symmetric non-singular $N \times N$ matrix. We choose a basis, such that

$$B = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix},$$

and denote by $\mathfrak{so}_N(\mathbb{F}) = \mathfrak{g}$ the corresponding Lie algebra $\mathfrak{o}_{N,B}(\mathbb{F})$.

Exercise 15.2. Show that

$$\mathfrak{so}_N(\mathbb{F}) = \{a \in \mathfrak{gl}_N(\mathbb{F}) \mid a + a' = 0\},$$

where a' is the transposition of a with respect to the anti-diagonal.

Exercise 15.3.

(a) $\mathfrak{so}_2(\mathbb{F}) = \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{F}\right\}$ is 1-dimensional abelian, hence not semisimple.

(b) Show that $\mathfrak{so}_n(\mathbb{F}) \subseteq \mathfrak{sl}_n(\mathbb{F})$.

We consider separately two cases.

Case 1. $N = 2n + 1$, where n is a positive integer. Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & & \\ & \ddots & & & & & \\ & & a_n & & & & \\ & & & 0 & & & \\ & & & & -a_n & & \\ & & & & & \ddots & \\ & & & & & & -a_1 \end{pmatrix} \quad (= \mathfrak{so}_N(\mathbb{F}) \cap D)\,.$$

This is a Cartan subalgebra by Theorem 8.1, since it contains a diagonal matrix with distinct entries.

Case 2. $N = 2n$, where $n \geqslant 2$ is an integer. Let

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_n & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix} \quad (= \mathfrak{so}_N(\mathbb{F}) \cap D)\,.$$

This is a Cartan subalgebra for the same reason. In both cases, $\dim \mathfrak{h} = n$ and $\varepsilon_1, \cdots, \varepsilon_n$ form a basis of $\mathfrak{h}^*$. Note that

$$\varepsilon_{N+1-j}\,|_{\mathfrak{h}} = -\varepsilon_j\,|_{\mathfrak{h}} \quad \text{and} \quad \varepsilon_{(N+1)/2}\,|_{\mathfrak{h}} = 0 \text{ if } N \text{ is odd.} \tag{6}$$

Next, all the eigenvectors for ad $\mathfrak{h}$ on $\mathfrak{g}$ are the elements

$$E_{i,j} - E_{N+1-j,N+1-i}, \quad i, j \in \{1, \cdots, N\}\,,$$

and the corresponding root is $\varepsilon_i - \varepsilon_j\,|_{\mathfrak{h}}$. Therefore, the set Δ of roots is

$$N = 2n+1: \quad \Delta_{\mathfrak{so}_N(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j), \pm\varepsilon_i \mid i, j \in \{1, \cdots, n\}, i \neq j\}\,; \tag{7}$$

$$N = 2n: \quad \Delta_{\mathfrak{so}_N(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j) \mid i, j \in \{1, \cdots, n\}, i \neq j\}\,. \tag{8}$$

Exercise 15.4.

(a) Using the root space decomposition, prove that $\mathfrak{so}_N(\mathbb{F})$ is semisimple if $N \geqslant 3$.

(b) Show that $\mathfrak{so}_N(\mathbb{F})$ is simple if $N = 3$ or $N \geqslant 5$.

Example 15.3. Consider the special case $\mathfrak{g} = \mathfrak{so}_4(\mathbb{F})$. In this case $\Delta = \{\pm(\varepsilon_1 - \varepsilon_2)\} \coprod \{\pm(\varepsilon_1 + \varepsilon_2)\}$ is the decomposition in a union of two indecomposable subsets $\Delta_1 = \{\pm(\varepsilon_1 - \varepsilon_2)\}$ and $\Delta_2 = \{\pm(\varepsilon_1 + \varepsilon_2)\}$. Then $\mathfrak{g}$ decomposes in a direct sum of two subalgebras, isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ with the sets of roots Δ_1 and Δ_2.

Let $e_1 = E_{12} - E_{34}, f_1 = E_{21} - E_{43}, e_2 = E_{31} - E_{42}, f_2 = E_{13} - E_{24}$. Then

$$[e_1, e_2] = 0, \quad [e_1, f_2] = 0, \quad [e_2, f_1] = 0.$$

Hence $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_i = \mathbb{F}e_i + \mathbb{F}f_i + \mathbb{F}[e_i, f_i]$ $(i = 1, 2)$ are the two ideals of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.

Bonus Problem. Consider the Lorentz Lie algebra $\mathfrak{o}_{4,B}(\mathbb{R})$, where $B = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ (see (5)). Prove that this is a simple 6-dimensional Lie algebra over $\mathbb{R}$. Note that its complexification $\mathfrak{o}_{4,B}(\mathbb{C}) \simeq \mathfrak{so}_4(\mathbb{C})$ is not simple. Thus, though semisimplicity of $\mathfrak{g}$ remains under field extensions, this is false for simplicity.

The series of simple Lie algebras $\mathfrak{so}_{2n+1}(\mathbb{F})$ for $n \geqslant 1$ is called the series of type B_n of classical Lie algebras. The series $\mathfrak{so}_{2n}(\mathbb{F})$ of simple Lie algebras, where $n \geqslant 3$, is called the series of type D_n of classical Lie algebras. In both cases, n is the rank.

Example 15.4. Consider the Lie algebra $\mathfrak{o}_{V,B}$, defined by (4), where V is the N-dimensional vector space over a field $\mathbb{F}$ of characteristic $\neq 2$, and B is skew-symmetric non-singular bilinear form. By linear algebra, $N = 2n$ must be even, and there exists a basis of V in which the bilinear form B has matrix

$$B = \begin{pmatrix} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{pmatrix} \tag{9}$$

We denote $\mathfrak{sp}_{2n}(\mathbb{F})$ the Lie algebra $\mathfrak{o}_{\mathbb{F}^{2n},B}$, where B is the matrix (9). This Lie algebra is called the ***symplectic Lie algebra***.

Exercise 15.5. Repeat the discussion we've done for $\mathfrak{so}_{2n}(\mathbb{F})$ in the case of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{F})$.

First, show that $\mathfrak{sp}_{2n}(\mathbb{F})$ consists of $2n \times 2n$ matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are $n \times n$ matrices over $\mathbb{F}$, and

$$a = -d', \quad b = b', \quad c = c',$$

where $'$ is the transposition with respect to the antidiagonal.

Next, let $\mathfrak{h}$ be the set of all diagonal matrices in $\mathfrak{sp}_{2n}(\mathbb{F})$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & & \\ & & a_n & & & \\ & & & -a_n & & \\ & & & & \ddots & \\ & & & & & -a_1 \end{pmatrix}, \text{ where } a_1, \cdots, a_n \in \mathbb{F} \right\}.$$

Show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{F})$, and find the root space decomposition.

Show that the set of roots is:

$$\Delta_{\mathfrak{sp}_N(\mathbb{F})} = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j), \pm 2\varepsilon_i \mid i, j \in \{1, \cdots, n\}, i \neq j\}.$$

Show that this set is indecomposable.

Prove that $\mathfrak{sp}_{2n}(\mathbb{F})$ is a simple Lie algebra for all $n \geqslant 1$.

These Lie algebras are called type C_n classical simple Lie algebras.

Thus, we have constructed four series of classical simple Lie algebras of rank n:

$$A_n = \mathfrak{sl}_{n+1}(\mathbb{F}) \ (n \geqslant 1), \qquad B_n = \mathfrak{so}_{2n+1}(\mathbb{F}) \ (n \geqslant 1),$$
$$C_n = \mathfrak{sp}_{2n}(\mathbb{F}) \ (n \geqslant 1), \qquad D_n = \mathfrak{so}_{2n}(\mathbb{F}) \ (n \geqslant 3).$$

We conclude the lecture by a discussion on invariant bilinear forms.

Proposition 15.1. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over an algebraically closed field $\mathbb{F}$. Then

(a) Any symmetric invariant bilinear form on $\mathfrak{g}$ is either non-degenerate or identically zero.

(b) Any two non-degenerate symmetric invariant bilinear forms on $\mathfrak{g}$ are proportional: $(a,b)_1 = \lambda(a,b)_2$ for some $\lambda \in \mathbb{F}$, independent of a and b.

Proof. (a) If $(\bullet,\bullet)$ is a symmetric invariant bilinear form on $\mathfrak{g}$, its kernel is an ideal of $\mathfrak{g}$, hence on simple $\mathfrak{g}$ the kernel is either $\mathfrak{g}$ or 0.

(b) Choose a basis of $\mathfrak{g}$, and let B_i be the matrix of $(\bullet,\bullet)_i$ in this basis, $i = 1, 2$. Then

$$\det(B_1 - \lambda B_2) = (\det B_2)(\det(B_2^{-1}B_1 - \lambda I)).$$

If λ_0 is an eigenvalue of the matrix $B_2^{-1}B_1$, then $\det(B_1 - \lambda_0 B_2) = 0$. Hence the bilinear form $(a,b)_1 - \lambda_0(a,b)_2$ is degenerate, and therefore is identically zero by (a). ■

Corollary 15.1. If $\mathfrak{g} \subset \mathfrak{gl}_N(\mathbb{F})$ is a simple Lie algebra and $\mathbb{F}$ is algebraically closed, then $\kappa(a,b) = \lambda \mathrm{tr}_{\mathbb{F}^N}(ab)$ for all $a, b \in \mathfrak{g}$ for some non-zero $\lambda \in \mathbb{F}$, independent of a, b.

Example 15.5. On $\mathfrak{gl}_N(\mathbb{F})$ we have $\mathrm{tr}_{\mathbb{F}^N}(E_{ij}E_{ks}) = \delta_{jk}\delta_{is}$, hence for the induced bilinear form on D^*, where D is the subalgebra of all diagonal matrices, we have

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \qquad i, j = 1, \cdots, N. \tag{10}$$

By Corollary 15.1 and Theorem 14.1(c), for all classical Lie algebras, the Killing form on $\mathfrak{h}^*_{\mathbb{Q}}$ is a multiple of the bilinear form (10) with positive rational coefficient.

Lecture 16

Root Systems

The discussion in previous lectures leads to the following definition.

Definition 16.1. Let V be a finite-dimensional real Euclidean space, i.e. V is a finite-dimensional vector space over $\mathbb{R}$ with symmetric positive definite bilinear form $(\bullet, \bullet)$. Let $\Delta \subset V$ be a subset. Then the pair (V, Δ) is called a ***root system*** if

(i) Δ is finite, $0 \notin \Delta$, Δ spans V over $\mathbb{R}$;

(ii) (string condition) for any $\alpha, \beta \in \Delta$, the set $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \cap (\Delta \cup \{0\})$ is a string

$$\beta - p\alpha, \beta - (p-1)\alpha, \cdots, \beta - \alpha, \beta, \beta + \alpha, \cdots, \beta + q\alpha,$$

where $p, q \in \mathbb{Z}_{\geqslant 0}$ and $p - q = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} (:= A_{\alpha,\beta})$;

(iii) for all $\alpha \in \Delta$, $k \in \mathbb{Z}$, we have $k\alpha \in \Delta$ if and only if $k = 1$ or -1.

Elements of Δ are called ***roots*** and $r = \dim_{\mathbb{R}} V$ is called the ***rank*** of (V, Δ).

Remark 16.1. Multiplying the bilinear form $(\bullet, \bullet)$ by a positive real number, we again get a root system.

Basic Example. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic 0. Choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, let $\Delta \subset \mathfrak{h}^*$ be the set of roots, and $\mathfrak{h}^*_{\mathbb{Q}}$ the $\mathbb{Q}$-span of Δ. Then we know that the Killing form κ on $\mathfrak{h}^*_{\mathbb{Q}}$ is $\mathbb{Q}$-valued and positive definite, by Theorem 14.1(b), (c).

Let $V = \mathbb{R} \otimes_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{Q}}$, i.e. linear combination of roots with real coefficeints, and extend the Killing form from $\mathfrak{h}^*_{\mathbb{Q}}$ to V by bilinearity to a real-valued positve definite symmetric bilinear form on V. Then the pair (V, Δ) is a root system, called the ***root system attached to*** $\mathfrak{g}$.

This construction is independent of the choice of $\mathfrak{h}$, due to Chevalley's Theorem.

Exercise 16.1. Let (V, Δ) be a root system. Then the set Δ is indecomposable if and only if there doesn't exist a decomposition $(V, \Delta) = (V_1, \Delta_1) + (V_2, \Delta_2)$, where $V = V_1 \oplus V_2$, $V_i \neq 0$, $V_1 \perp V_2$, $\Delta_i \subset V_i$, and $\Delta = \Delta_1 \cup \Delta_2$. *(Hint: use the string condition)*

Moreover, the decomposition $\Delta = \coprod_i \Delta_i$ into indecomposable sets corresponds to decomposition of the root system in the orthogonal direct sum of indecomposable root systems.

Definition 16.2. (of Remark 16.1) An isomorphism of indecomposable root systems (V_1, Δ_1) and (V_2, Δ_2) is a vector space isomorphism $\varphi : V_1 \to V_2$, such that $\varphi(\Delta_1) = \Delta_2$ and $(\varphi(\alpha), \varphi(\beta))_2 = c(\alpha, \beta)_1$, for all $\alpha, \beta \in \Delta$, where $c \in \mathbb{R}_{>0}$ is independent of α and β.

Example 16.1. Any root system of rank 1 is isomorphic to $(\mathbb{R}, \Delta = \{1, -1\})$ and $(\alpha, \beta) = \alpha\beta$. This root system is indecomposable, and it is a root system, isomorphic to that, attached $\mathfrak{sl}_2(\mathbb{F}), \mathfrak{so}_3(\mathbb{F})$ and $\mathfrak{sp}_2(\mathbb{F})$.

Proposition 16.1. Let (V, Δ) be an indecomposable root system with the bilinear form $(\bullet, \bullet)$ on V. Then

(a) Any other bilinear form $(\bullet, \bullet)_1$ on V, satisfying the string property, is $(a, b)_1 = c(a, b)$, where $c \in \mathbb{R}_{>0}$ is independent of a and b.

(b) If $(\alpha, \alpha) \in \mathbb{Q}$ for some $\alpha \in \Delta$, then $(\beta, \gamma) \in \mathbb{Q}$ for all $\beta, \gamma \in \Delta$.

Proof. Fix $\alpha \in \Delta$. Since (V, Δ) is indecomposable by Exercise 16.1, for any $\beta \in \Delta$ there exists a sequence of roots $\gamma_1, \cdots, \gamma_s$, such that $\alpha = \gamma_1, \beta = \gamma_s$ and $(\gamma_i, \gamma_{i+1}) \neq 0$ for $i = 1, \cdots, s-1$. Define $c \in \mathbb{R}$ by $(\alpha, \alpha)_1 = c(\alpha, \alpha)$. By the string property, $p - q = \frac{2(\alpha,\gamma_2)}{(\alpha,\alpha)} = \frac{2(\alpha,\gamma_2)_1}{(\alpha,\alpha)_1}$, hence $(\alpha, \gamma_2)_1 = c(\alpha, \gamma_2)$.

Likewise, by the string property $\frac{2(\alpha,\gamma_2)}{(\gamma_2,\gamma_2)} = \frac{2(\alpha,\gamma_2)_1}{(\alpha,\alpha)_1}$, hence $(\gamma_2, \gamma_2)_1 = c(\gamma_2, \gamma_2)$. Continuing this way, we show that $(\gamma_3, \gamma_3)_1 = c(\gamma_3, \gamma_3), \cdots, (\beta, \beta)_1 = c(\beta, \beta)$.

Since $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = \frac{2(\alpha,\beta)_1}{(\alpha,\alpha)_1}$, we conclude that $(\alpha, \beta)_1 = c(\alpha, \beta)$. Since Δ spans V, we conclude that (a) holds. The same argument proves (b). ∎

Now we turn to the construction of roots systems beyond the classical ones.

Definition 16.3. A ***lattice*** in a finite-dimensional real Euclidean space V is a discrete subgroup L with respect to the addition operation in V, which spans V over $\mathbb{R}$. The integer $r = \dim_{\mathbb{R}} V$ is called the ***rank*** of L.

A lattice L is called ***integral*** if $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in L$, and it is called even if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$.

Exercise 16.2. Show that any even lattice is integral.

Example 16.2. For $N \in \mathbb{R}_{>0}$, $\frac{1}{N}\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ with the standard bilinear form

$$(a, b) = \sum_{i=1}^{n} \alpha_i \beta_i, \qquad \text{where } a = (\alpha_1, \cdots, \alpha_n), b = (\beta_1, \cdots, \beta_n).$$

Proposition 16.2. If Δ is a finite subset in a finite-dimensional Euclidean space V, spanning V over $\mathbb{R}$, such that $(a, b) \in \mathbb{Q}$ for all $\alpha, \beta \in \Delta$, then the $\mathbb{Z}$-span of Δ is a lattice in V.

Proof. The only thing to prove is that $\mathbb{Z}\Delta$ is a discrete set. For this choose a basis $\beta_1, \cdots, \beta_r$ of V among the vectors of Δ. Then for any $\alpha \in \Delta$, we have $\alpha = \sum_{i=1}^{r} c_i \beta_i$, where $c_i \in \mathbb{R}$. Hence

$$(\alpha, \beta_j) = \sum_{i=1}^{r} c_i (\beta_i, \beta_j), \quad j = 1, \cdots, r. \tag{1}$$

But $((\beta_i, \beta_j))_{i,j=1}^{r}$ is a Gramm matrix of a linear independent set in a Euclidean space, hence it is non-singular. Hence the c_i can be computed by Cramer's rule, so all $c_i \in \mathbb{Q}$, hence $\mathbb{Z}\Delta \subseteq \mathbb{Q}\{\beta_1, \cdots, \beta_r\}$. But since the set Δ is finite, we conclude that $\mathbb{Z}\Delta \subseteq \frac{1}{N}\mathbb{Z}\{\beta_1, \cdots, \beta_r\}$, where N is a positive integer. But $\frac{1}{N}\mathbb{Z}\{\beta_1, \cdots, \beta_r\}$ is a discrete set, hence $\mathbb{Z}\Delta$ is discrete as well. ■

Bonus Exercise. Show that $\mathbb{Z}\left\{1, \sqrt{2}\right\}$ is not a discrete subset of $\mathbb{R}$.

Definition 16.4. Let (V, Δ) be a root system. Then the $\mathbb{Z}$-span of Δ is called the ***root lattice of this root system***.

By Propositions 16.1(b) and 16.2, the root lattice is indeed a lattice if (V, Δ) is indecomposable (since we can always normalize the bilinear form, such that $(\alpha, \alpha) \in \mathbb{Q}$ for some $\alpha \in \Delta$), hence it holds for any root system.

We list below the four series of indecomposable root systems (V, Δ) and their root lattices L, attached to the simple classical Lie algebras (see Lecture 15).

Type	$\mathfrak{g}$	V	Δ	L
A_r	$\mathfrak{sl}_{r+1}(\mathbb{F})$, $r \geqslant 1$	$\sum_{i=1}^{r+1} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{R}, \sum a_i = 0$	$\varepsilon_i - \varepsilon_j$	$\sum_{i=1}^{r+1} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{Z}, \sum a_i = 0$
B_r	$\mathfrak{so}_{2r+1}(\mathbb{F})$, $r \geqslant 1$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{R}$	$\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j), \pm\varepsilon_i$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{Z}$
C_r	$\mathfrak{sp}_{2r}(\mathbb{F})$, $r \geqslant 1$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{R}$	$\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j), \pm 2\varepsilon_i$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{Z}, \sum a_i \in 2\mathbb{Z}$
D_r	$\mathfrak{so}_{2r}(\mathbb{F})$, $r \geqslant 3$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{R}$	$\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j)i$	$\sum_{i=1}^{r} a_i\varepsilon_i \,\Big\vert\, a_i \in \mathbb{Z}, \sum a_i \in 2\mathbb{Z}$

In the column Δ, $i \neq j$ and $1 \leqslant i, j \leqslant r$.

Explanation of L for A_r. Clearly $\mathbb{Z}\{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ is contained in L. To show the reverse inclusion, write

$$\sum_{i=1}^{r+1} a_i\varepsilon_i = a_1(\varepsilon_1 - \varepsilon_2) + (a_1 + a_2)(\varepsilon_2 - \varepsilon_3) + \cdots + (a_1 + \cdots + a_r)(\varepsilon_r - \varepsilon_{r+1}) + (a_1 + \cdots + a_{r+1})\varepsilon_{r+1}.$$

Since $\sum_{i=1}^{r+1} a_i = 0$, it follows. For B_r the explanation is obvious.

Exercise 16.3. Explain the root lattices in cases C_r and D_r.

Remarks 16.2.

(a) The root lattices of type A_r and D_r are even, hence integral. The root lattice of type B_r is integral, but not even.

(b) The "best" normalization of the bilinear form $(\bullet, \bullet)$ is such that $\max_{\alpha\in\Delta}(\alpha, \alpha) = 2$. For types A_r, B_r and D_r we have this normalization. However, for type C_r, in order to get this normalization, we need to multiply this bilinear form by $1/\sqrt{2}$. Then we get an integral, but not even, lattice.

Now we turn to the construction of "exceptional" root systems. For that we construct some "exceptional" lattices.

Consider the Euclidean space $V^r = \bigoplus_{i=1}^{r} \mathbb{R}\varepsilon_i$, with the bilinear form $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$,

$i, j = 1, \cdots, r$. Consider the following lattices in V^r:

$$\Gamma'_r = \left\{ \sum_{i=1}^r a_i\varepsilon_i \mid \text{either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}$$
$$\supset \Gamma_r = \left\{ \gamma = \sum_{i=1}^r a_i\varepsilon_r \in \Gamma'_r \mid \sum_{i=1}^r a_i \in 2\mathbb{Z} \right\}.$$

Proposition 16.3.

(a) Γ'_r is a lattice, such that $(\alpha, \alpha) \in \mathbb{Z}$ for all $\alpha \in \mathbb{Z}$, if and only if r is divisible by 4.

(b) Γ_r is an even (hence integral) lattice if and only if r is divisible by 8.

Proof. Let $\alpha = \sum_{i=1}^r \alpha_i\varepsilon_i \in \Gamma'_r$. If all $a_i \in \mathbb{Z}$, (α, α) is obviously an integer. If all $a_i \in \frac{1}{2} + \mathbb{Z}$, we write $\alpha = \beta + \rho$, where $\beta = \sum_{i=1}^r b_i\varepsilon_i$ with the $b_i \in \mathbb{Z}$, and $\rho = \frac{1}{2}\sum_{i=1}^r \varepsilon_i$. Then

$$(\alpha, \alpha) = (b, b) + 2(\rho, b) + (\rho, \rho) = \sum_{i=1}^r (b_i^2 + b_i) + \frac{r}{4}. \tag{2}$$

This is an integer if and only if r is divisible by 4, proving (a).

To prove (b), notice that $c^2 \pm c \in 2\mathbb{Z}$ for $c \in \mathbb{Z}$. Hence $(\alpha, \alpha) = \sum_{i=1}^r a_i^2 \equiv \sum_{i=1}^r a_i$ (mod 2), and $(\alpha, \alpha) \in 2\mathbb{Z}$ if $\alpha \in \Gamma_r$ and all $a_i \in \mathbb{Z}$. Finally, $(\alpha, \alpha) \in 2\mathbb{Z}$ for $\alpha = \beta + \rho$ as above if and only if r is divisible by 8, by (2). ∎

Theorem 16.1. Let L be an integral lattice in a finite-dimensional Euclidean space V, let $\Delta = \{\alpha \in L \mid (\alpha, \alpha) = 2\}$ and assume that Δ spans V over $\mathbb{R}$. Then (V, Δ) is a root system.

Proof. Axioms (i) and (iii) of a root system are clear. It remains to prove the string condition. It is clear that the string condition for $\alpha, \beta \in \Delta$ holds if and only if it holds for $-\alpha, \beta$. Hence, without loss of generality we may assume that $(\alpha, \beta) \geqslant 0$. Note that

$$0 \leqslant (\alpha - \beta, \alpha - \beta) = (\alpha, \alpha) - 2(\alpha, \beta) + (\beta, \beta) = 4 - 2(\alpha, \beta)$$

since $(\alpha, \alpha) = (\beta, \beta) = 2$. But $(\alpha, \beta) \in \mathbb{Z}_{\geqslant 0}$ since L is an integral lattice and we assumed that $(\alpha, \beta) \geqslant 0$. Hence the only possibilities for (α, β) are

$$(\alpha, \beta) = 0, 1, \text{ or } 2.$$

If $(\alpha, \beta) = 0$, then $(\alpha \pm \beta, \alpha \pm \beta) = 4$, hence $\alpha \pm \beta \notin \Delta$, and $p = q = 0$, so $p - q = 0 = A_{\alpha,\beta}$ and the string condition holds.

If $(\alpha,\beta)=1$, then $(\beta+j\alpha,\beta+j\alpha)=2+2j+2j^2$, by (3), hence $\beta+j\alpha\in\Delta$ if and only if $j=0$ or -1. Hence $p=1,q=0$ and $p-q=1=A_{\alpha,\beta}$, and the string condition holds.

Finally, if $(\alpha,\beta)=2$, then, by (3), $(\alpha-\beta,\alpha-\beta)=0$, hence $\alpha=\beta$. But then, by axiom (iii), $\beta+j\alpha\in\Delta\cup\{0\}$ if and only if $j=0,-1,-2$. Hence $p=-2,q=0$, and $p-q=-2=A_{\alpha,\beta}$, and the string condition again holds. ■

Exercise 16.4. Show that for $L=\Gamma_r$ with r divisible by 8, the root system, given by Theorem 16.1, is of type D_r if $r>8$.

Exercise 16.5. Let L be a lattice, such that $2(\alpha,\beta)\in\mathbb{Z}$ if $(\alpha,\alpha)=1$ and $(\alpha,\beta)\in\mathbb{Z}$ if $(\alpha,\alpha)=2$ for $\alpha,\beta\in L$. Let

$$\Delta=\{\alpha\in L\mid(\alpha,\alpha)=2\text{ or }1\},$$

and let $V=\mathbb{R}\Delta$. Show that (V,Δ) is a root system.

Lecture 17

Root Systems and Cartan Matrices

Let $V_r = \bigoplus_{i=1}^r \mathbb{R}\varepsilon_i$ be an r-dimensional Euclidean vector space with the symmetric bilinear form defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $i, j = 1, \cdots, r$. Consider in V_8 the lattice $L_{E_8} := \Gamma_8$. Recall that

$$L_{E_8} = \left\{ \sum_{i=1}^{8} a_i \varepsilon_i \middle| \sum_{i=1}^{8} a_i \in 2\mathbb{Z}, \text{ and either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}.$$

As was explained in the last lecture, L_{E_8} is an even lattice, and

$$\Delta_{E_8} = \{\alpha \in L_{E_8} \mid (\alpha, \alpha) = 2\} \subset V_8$$

is a root system. It is immediate to see that

$$\Delta_{E_8} = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\} \cup \left\{ \frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_8) \middle| \text{even number of minus signs} \right\}.$$

Exercise 17.1. Show that $|\Delta_{E_8}| = 240$ and that L_{E_8} is the root lattice for the root system (V_8, Δ_{E_8}).

The root system (V_8, Δ_{E_8}) (of rank 8) is called the ***exceptional root system of type*** E_8. This root system contains two important root subsystems, of type E_7 and E_8. In order to construct them, let (as before) $\rho = \frac{1}{2}\sum_{i=1}^{8} \varepsilon_i$ and $\xi = \varepsilon_7 + \varepsilon_8$ be two roots from Δ_{E_8}. Let

$$V_{E_7} = \{v \in V_8 \mid (\rho, v) = 0\} \supset V_{E_6} = \{v \in V_{E_7} \mid (\xi, v) = 0\},$$

and let $L_{E_i} = V_{E_i} \cap L_{E_8}$, $\Delta_{E_i} = V_{E_i} \cap \Delta_{E_8}$, where $i = 7$ or 6.

Exercise 17.2.

(a) Show that (V_{E_7}, Δ_{E_7}) is a root system of rank 7, for which L_{E_7} is the root lattice. Describe Δ_{E_7} and show that $|\Delta_{E_7}| = 126$.

(b) Show that (V_{E_6}, Δ_{E_6}) is a root system of rank 6, for which L_{E_6} is the root lattice. Describe Δ_{E_6} and show that $|\Delta_{E_6}| = 72$.

The next exceptional root system is of type F_4, which has rank 4. In order to construct it, consider in V_4 the lattice Γ'_4, which we denote by L_{F_4}. Recall that

$$L_{F_4} = \left\{ \sum_{i=1}^{4} a_i \varepsilon_i \middle| \text{ either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}.$$

This lattice satisfies conditions of Exercise 16.5, hence (V_4, Δ_{F_4}) is a root system, where

$\Delta_{F_4} = \{\alpha \in L_{F_4} \mid (\alpha,\alpha) = 2 \text{ or } 1\}$.

It is straightforward to see that

$$\Delta_{F_4} = \{\pm\varepsilon_i \pm \varepsilon_j; \pm\varepsilon_i \mid i,j \in \{1,2,3,4\}, i \neq j\} \cup \left\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\right\}.$$

Exercise 17.3. Show that L_{F_4} is the root lattice for the root system (V_4, Δ_{F_4}), and that $|\Delta_{F_4}| = 48$.

The last exceptional root system is of type G_2 (of rank 2). In order to construct it, recall the root system of type A_2: (V_{A_2}, Δ_{A_2}), where $V_{A_2} = \left\{\sum_{i=1}^3 a_i\varepsilon_i \middle| a_i \in \mathbb{R}, a_1 + a_2 + a_3 = 0\right\}$, $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$; the root lattice is $L_{A_2} = \left\{\sum_{i=1}^3 a_i\varepsilon_i \middle| a_i \in \mathbb{Z}, a_1 + a_2 + a_3 = 0\right\}$, the set of roots $\Delta_{A_2} = \{\alpha \in L_{A_2} \mid (\alpha,\alpha) = 2\} = \{\varepsilon_i - \varepsilon_j \mid i,j \in \{1,2,3\}, i \neq j\}$.

We let $V_{G_2} = V_{A_2}$. Let $L_{G_2} = L_{A_2}$,

$$\Delta_{G_2} = \{\alpha \in L_{G_2} \mid (\alpha,\alpha) = 2 \text{ or } 6\} \, (\supset \Delta_{A_2}).$$

It is straightforward to see that

$$\Delta_{G_2} = \Delta_{A_2} \cup \{\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid i,j,k \in \{1,2,3\} \text{ are distinct}\}.$$

Obviously, L_{G_2} is the root lattice for this root system.

Exercise 17.4. Prove that (V_{G_2}, Δ_{G_2}) is an indecomposable root system with $|\Delta_{G_2}| = 12$.

Bonus Problem. Is it true that for an even lattice L the set $\{\alpha \in L \mid (\alpha,\alpha) = 2 \text{ or } 6\}$ satisfies the string condition?

In order to construct the root space decomposition of a semisimple Lie algebra $\mathfrak{g}$, we needed to "break symmetry": choose a Cartan subalgebra. This has led to the root system (V, Δ) of $\mathfrak{g}$, which is independent, up to isomorphism, of this choice.

In order to study (general) root systems, we need to break symmetry again.

Definition 17.1. Let (V, Δ) be a root system and let $f \in V^*$ be a linear $\mathbb{R}$-valued function on V, such that $f(\alpha) \neq 0$ for all $\alpha \in \Delta$. Then

(i) $\alpha \in \Delta$ is called ***positive*** if $f(\alpha) > 0$ and negative if $f(\alpha) < 0$;

(ii) A positive root is called ***simple*** if it cannot be written as a sum of two positive roots;

(iii) A ***highest*** root θ is that on which f takes maximal value.

Notation 17.1. Δ_+ (resp. Δ_-) is the set of positive (resp. negative) roots; $\Pi \subseteq \Delta_+$ is the set of simple roots. The set Π is called ***indecomposable*** if it cannot be represented as a union of two non-empty subsets, which are orthogonal to each other.

Dynkin Theorem.

(a) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$, then $\alpha - \beta \notin \Delta$ and $(\alpha, \beta) \leqslant 0$.

(b) $\Delta_+ \subseteq \mathbb{Z}_{\geqslant 0}\Pi$.

(c) If $\alpha \in \Delta_+ \backslash \Pi$, then $\alpha - \gamma \in \Delta$ for some $\gamma \in \Pi$; moreover, then $\alpha - \gamma \in \Delta_+$.

(d) Π is a basis of V (over $\mathbb{R}$) and a basis of L over $\mathbb{Z}$.

(e) Δ is indecomposable if and only if Π is indecomposable.

Proof. (a) is proved by contradiction. If $\alpha - \beta = \gamma \in \Delta$, then either

- $\gamma \in \Delta_+$, and so $\alpha = \beta + \gamma$, which contradicts $\alpha \in \Pi$,
- or $\gamma \in \Delta_-$, and so $\beta = \alpha + (-\gamma)$, which contradicts $\beta \in \Pi$.

Finally, $(\alpha, \beta) \leqslant 0$ due to the string condition.

(b) If $\alpha \in \Delta_+$ is simple, there is nothing to prove. Otherwise $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Delta_+$, and then $f(\alpha) = f(\beta) + f(\gamma)$, so both $f(\beta)$ and $f(\gamma)$ are strictly less than $f(\alpha)$. Repeat this process until all summands are simple (which must happen in finitely many steps since Δ is finite), thus yielding α as a sum of simple roots.

(c) If $\alpha - \gamma \notin \Delta$ for all $\gamma \in \Pi$, then the string condition would imply that $\frac{2(\alpha,\gamma)}{(\gamma,\gamma)} = -q \leqslant 0$, hence $(\alpha, \gamma) \leqslant 0$ for all $\gamma \in \Pi$. Then, by (b),

$$(\alpha, \alpha) = \left(\alpha, \sum_{\gamma \in \Pi} k_\gamma \gamma\right) \leqslant 0, \quad \text{where all } k_\gamma \geqslant 0,$$

which would imply that $\alpha = 0$, which is impossible since α is a root. Hence $\alpha - \gamma \in \Delta$ for some $\gamma \in \Pi$. Finally, if $\alpha - \gamma \notin \Delta_+$, we have $\alpha - \gamma = \beta \in \Delta_-$, and then $\gamma = \alpha + (-\beta)$, which would contradict γ being simple.

(d) By (b), $\Delta_+ \subseteq \mathbb{Z}\Pi$, hence, since $\Delta_- = -\Delta_+$, $\Delta = \Delta_+ \cup \Delta_- \subseteq \mathbb{Z}\Pi$. Thus $L = \mathbb{Z}\Pi$, hence V is spanned by Π over $\mathbb{R}$. To prove linear independence of the set Π over $\mathbb{R}$, suppose the contrary: there is a non-trivial linear combination of elements from Π, which is 0. Splitting it into positive and negative parts, and moving the negative part to the other side, we obtain

$$\gamma := \sum_i a_i \alpha_i = \sum_j b_j \alpha_j, \tag{1}$$

where all a_i and b_j are positive integers, the set of i's is non-empty, and $i \neq j$.

However, by (a) we obtain

$$(\gamma, \gamma) = (\sum_i a_i \alpha_i, \sum_j b_j \alpha_j) \leqslant 0,$$

hence $\gamma = 0$. But then we have from (1): $0 = f(\gamma) = \sum_i a_i f(\alpha_i) > 0$, a contradiction.

(e) If (V, Δ) is decomposable, then by (d), so is Π. Conversely, suppose that Π decomposes as $\Pi_1 \coprod \Pi_2$ in two non-empty sets, with $\Pi_1 \perp \Pi_2$, then we will show that $\Delta = (\mathbb{Z}\Pi_1 \cap \Delta) \cup (\mathbb{Z}\Pi_2 \cap \Delta)$.

For this we need to show that any $\alpha \in \Delta_+$ lies in one of the sets $\mathbb{Z}\Pi_i \cap \Delta_+$, $i = 1, 2$. Suppose the contrary, and take the $\alpha \in \Delta_+$ with minimal sum of coefficients in its decomposition as $\sum_{\gamma \in \Pi} k_\gamma \gamma$. By (c), there is $\gamma \in \Pi$, such that $\alpha - \gamma \in \Delta_+$, and by the above minimality, $\alpha - \gamma$ lies in one of the sets $\mathbb{Z}\Pi_i \cap \Delta_+$, so $\alpha = \beta + \gamma$, where $\gamma \in \Pi_1$ and $\beta \in \Pi_2$. But then $\frac{2(\alpha,\beta)}{(\beta,\beta)} = 2 + \frac{2(\gamma,\beta)}{(\beta,\beta)} = 2$. By the string condition, it follows that $\gamma - \beta$ is a root, which contradicts (a). ■

Definition 17.2. Let $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots of Δ, corresponding to the choice of f. This is a basis of V, by Dynkin Theorem (d). Let $A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, which are integers, by the string condition. The matrix $A = (A_{ij})_{i,j=1}^r$ is called the ***Cartan matrix*** of the root system (V, Δ). (We will show later that it is independent of the choice of f, up to a permutation of the set Π.)

Proposition 17.1. The Cartan matrix $A = (A_{ij})$ of the root system (V, Δ) has all entries integers, and satisfies the following properties:

(a) $A_{ii} = 2$ for all i;

(b) If $i \neq j$, then $A_{ij} \leqslant 0$, and $A_{ij} = 0 \Rightarrow A_{ji} = 0$;

(c) All principal minors of A are positive, in particular, $\det A > 0$.

Proof. (a) is immediate, and (b) follows from Dynkin Theorem (a), since $(\alpha, \alpha) > 0$ for $\alpha \neq 0$. In order to prove (c), note that

$$A = \begin{pmatrix} 2/(\alpha_1, \alpha_1) & & 0 \\ & \ddots & \\ & & 2/(\alpha_r, \alpha_r) \end{pmatrix} ((\alpha_i, \alpha_j))_{i,j=1}^{r} .$$

Hence every principal minor of A is equal to the product of a positive number and a principal minor of the Gram matrix of the set Π. The result follows by the Sylvester criterion. ■

Definition 17.3. Let θ be a highest root of Δ, and let $\alpha_0 = -\theta$, $\widetilde{\Pi} = \{\alpha_0, \cdots, \alpha_r\}$. The $(r+1) \times (r+1)$ matrix $\widetilde{A} = (A_{ij})_{i,j=0}^{r}$ is called the ***extended Cartan matrix*** of the root system (V, Δ).

Exercise 17.5. The extended Cartan matrix $\widetilde{A}$ satisfies all properties of Proposition 17.1, except that $\det \widetilde{A} = 0$.

Definition 17.4. An $r \times r$ matrix, satisfying all properties of Proposition 17.1 is called an ***abstract Cartan matrix***.

We will classify abstract Cartan matrices and prove that to each of them corresponds a semi-simple finite-dimensional Lie algebra, in the subsequent lectures.

Lecture 18

Classification of Abstract Cartan Matrices and their Dynkin Diagrams

Recall that an abstract Cartan matrix is an $r \times r$ matrix $A = (A_{ij})$ with integer entries, satisfying the following properties:

(a) $A_{ii} = 2$ for all i;

(b) if $i \neq j$, then $A_{ij} \leqslant 0$, and $A_{ij} < 0 \Rightarrow A_{ji} < 0$;

(c) all principal minors of A are positive, in particular, $\det A > 0$.

In this lecture we classify abstract Cartan matrices A, rather their Dynkin diagrams.

If $r = 1$, then, of course $A = (2)$, and this is the Cartan matrix of type A_1.

There are more possibilities for $r = 2$. In this case $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, where either $a = b = 0$, or a and b are positive integers, such that $ab < 4$ (since $\det A > 0$). Thus, we have 6 possibilities:

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

In the second row the Dynkin diagrams of these abstract Cartan matrices are depicted. For an arbitrary abstract Cartan $r \times r$ matrix its Dynkin diagram $D(A)$ is a graph with r vertices, which are connected by edges as follows: if $i, j \in \{1, \cdots, r\}$, $i \neq j$, the corresponding principal submatrix $\begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{jj} \end{pmatrix}$ is, obviously 2×2 abstract Cartan matrix, listed above, and we connect them as in the above table.

Remark 18.1. A permutation of the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ corresponds to the permutation of the rows and the same permutation of the columns of its Cartan matrix, which corresponds to a permutation of vertices of the Dynkin diagram. Hence the sets of simple roots of root systems are classifed by the Dynkin diagrams.

In particular, there are only 4 Dynkin diagrams with 2 vertices:

(1)

Next we find Cartan matrices and their (connected) Dynkin diagrams of all indecomposable root systems that we constructed so far.

Remarks 18.2.

(a) For a decomposable root system, the Dynkin diagram is a disjoint union of connected Dynkin diagrams, corresponding to indecomposable components by Dynkin Theorem (e).

(b) Any subdiagram of a Dynkin diagram of an abstract Cartan matrix is again the Dynkin diagram of an abstract Cartan matrix.

(c) $D(\widetilde{A})$, the extended Dynkin diagram, is NOT a Dynkin diagram of an abstract Cartan matrix, since $\det \widetilde{A} = 0$.

The list of these connected Dynkin diagrams and their extended Dynkin diagrams is as follows:

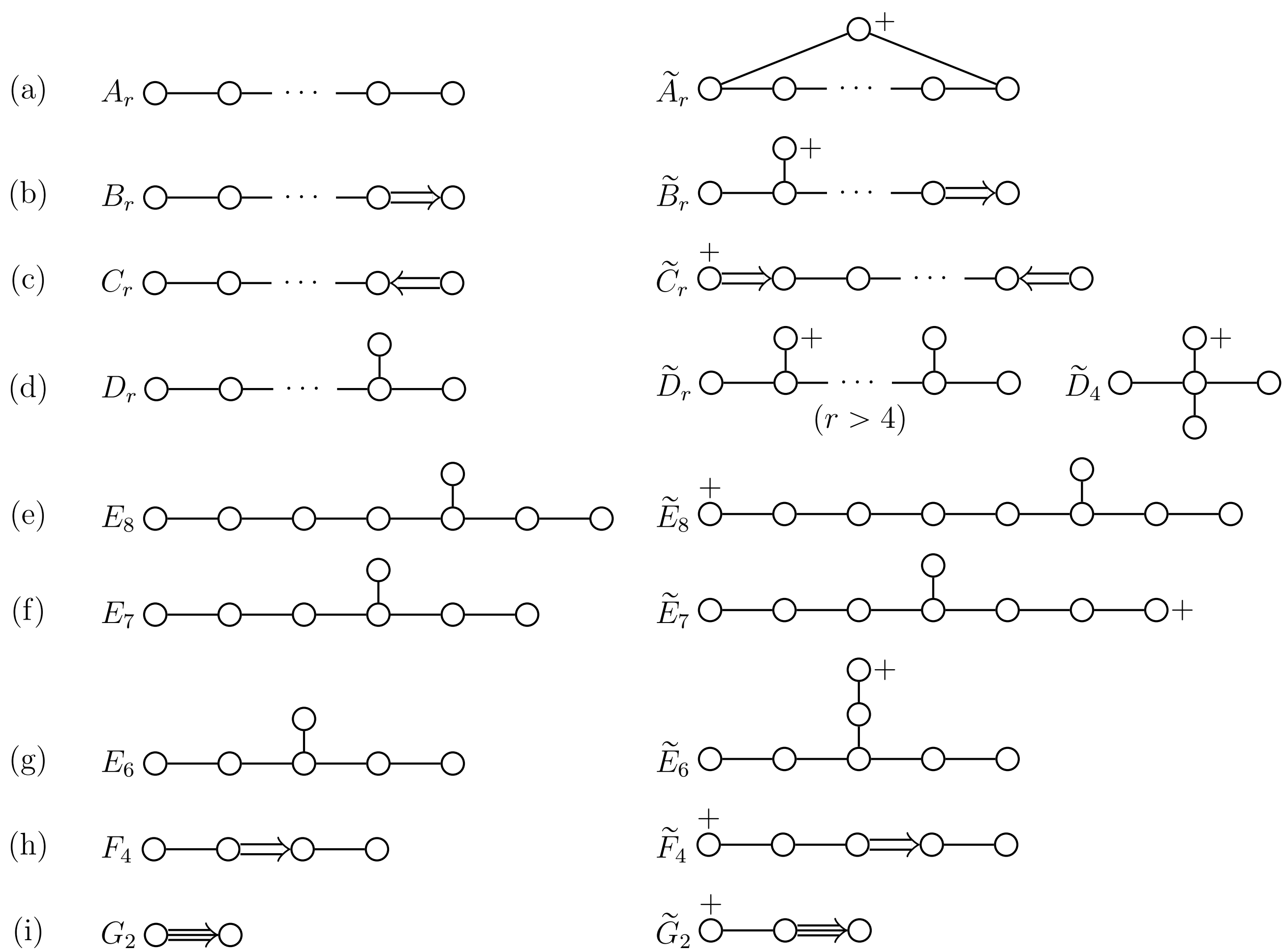


Figure 1: Dynkin diagrams and extended Dynkin diagrams of all the indecomposable root systems constructed so far. Here + corresponds to the vertex added to $D(A)$.

The basic idea is to choose the function $f : V \to \mathbb{R}$, which takes non-zero values on roots, to have integer values on roots, such that $|\{\alpha \in \Delta_+ \mid f(\alpha) = 1\}| = r$. Then Π is the set of these $\alpha's$.

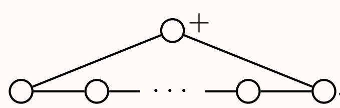
Example 18.1. A_r $(r \geqslant 1)$, $\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i, j \leqslant r+1, i \neq j\}$.

Take $f(\varepsilon_1) = r, f(\varepsilon_2) = r-1, \cdots f(\varepsilon_r) = 1, f(\varepsilon_{r+1}) = 0$. Then $f\,|_\Delta \neq 0$,

$$\begin{aligned} \Delta_+ &= \{\varepsilon_i - \varepsilon_j \mid i < j\}, \\ \Pi &= \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, \cdots, r\}, \\ \theta &= \varepsilon_1 - \varepsilon_{r+1}, \\ A &= \begin{pmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & & \vdots \\ \vdots & & \ddots & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \end{aligned}$$

$D(A) =$ ,

$$\alpha_0 = \varepsilon_{r+1} - \varepsilon_1,$$

$D(\widetilde{A}) =$ .

This is precisely the first row of Figure 1.

Exercise 18.1. Show that $D(A)$ and $D(\widetilde{A})$ are as in the rows 2 and 3 of Figure 1 for B_r and C_r.

We show this below for D_r and E_8.

Example 18.2. D_r $(r \geqslant 3)$, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in \{1, \cdots, r\}, i \neq j\}$.

Take $f(\varepsilon_1) = r-1, f(\varepsilon_2) = r-2, \cdots, f(\varepsilon_r) = 0$. Then $f\,|_\Delta \neq 0$,

$$\begin{aligned} \Delta_+ &= \{\varepsilon_i \pm \varepsilon_j \mid i < j\}, \\ \Pi &= \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \cdots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_{r-1} + \varepsilon_r\}, \\ \theta &= \varepsilon_1 + \varepsilon_2 = -\alpha_0. \end{aligned}$$

So $D(A)$ and $D(\widetilde{A})$ are as in row 4 of Figure 1. Note that $D(D_3) = D(A_3)$.

Example 18.3. E_8, $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid i \neq j\} \cup \left\{\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_8) \mid \text{even number of } -\right\}$.

Let $f(\varepsilon_1) = 23, f(\varepsilon_2) = 6, f(\varepsilon_3) = 5, \cdots, f(\varepsilon_7) = 1, f(\varepsilon_8) = 0$. Then $f\mid_\Delta \neq 0$, and f takes integer values on Δ,

$$\Delta_+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \leqslant 8\} \cup \left\{ \frac{1}{2}(\varepsilon_1 \pm \cdots \pm \varepsilon_8) \mid \text{even number of } - \right\},$$

$$\Pi = \left\{ \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \cdots, \varepsilon_7 - \varepsilon_8, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8) \right\},$$

$$\theta = \varepsilon_1 + \varepsilon_2 = -\alpha_0.$$

So $D(A)$ and $D(\widetilde{A})$ are as in Figure 1.

Exercise 18.2. Show that $D(A)$ and $D(\widetilde{A})$ are as in rows 6 and 7 for E_7 and E_6 in Figure 1.

Exercise 18.3. Show that $D(A)$ and $D(\widetilde{A})$ are as in rows 8 and 9 for F_4 and G_2 in Figure 1.

Now we turn to classification of connected Dynkin diagrams of abstract Cartan matrices. Note that $D(A_1) = D(B_1) = D(C_1)$, $D(B_2) = D(C_2)$, and $D(D_3) = D(A_3)$.

Theorem 18.1. A complete non-redundant list of connected Dynkin diagrams $D(A)$ of abstract Cartan matrices coincides with that of the constructed indecomposable root systems, and is as follows:

$$D(A_r)\ (r \geqslant 2),\ D(B_r)\ (r \geqslant 3),\ D(C_r)\ (r \geqslant 1),\ D(D_r)\ (r \geqslant 4),$$
$$D(E_6),\ D(E_7),\ D(E_8),\ D(F_4),\ \text{and } D(G_2).$$

Proof. We have to prove that there are no other connected Dynkin diagrams. To do this, we find all connected graphs with connections of four types in (1), such that any subgraph corresponds to a principal submatrix with positive determinant (since any subgraph must be a Dynkin diagram).

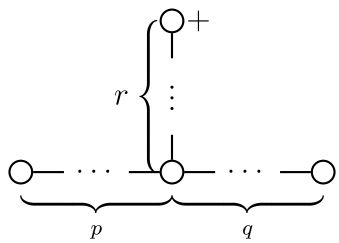
In particular, Dynkin diagrams don't contain any extended Dynkin diagrams, since the corresponding principal submatrix has determinant 0.

Part 1. We first classify all simply-laced connected Dynkin diagrams, i.e. diagrams having only ○ ○ or ○—○ connections (which correspond to symmetric Cartan matrices).

Such a diagram contains no cycles, since otherwise it contains $D(\widetilde{A}_s)$ for some $s \geqslant 2$. It has simple edges, and it may or may not contain branching vertices. If there are no branching vertices, we get $D(A_r)$. If there are 2 branching vertices, the Dynkin diagram contains $D(\widetilde{D}_s)$

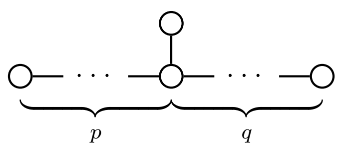
for some $s \geqslant 5$, which is impossible. If there is precisely one branching vertex, the Dynkin diagram has 3 branches, since otherwise it contains $D(\widetilde{D}_4)$ as a subgraph.

Therefore, it remains to consider the case when $D(A)$ is a graph of the form $T_{p,q,r}$, where $p \geqslant q \geqslant r \geqslant 2$:



If $r = 3$, then $T_{p,q,3}$ contains $D(\widetilde{E}_6) = T_{3,3,3}$, which is impossible, so $r = 2$, i.e. $D(A)$ has the form

$T_{p,q,2}$:

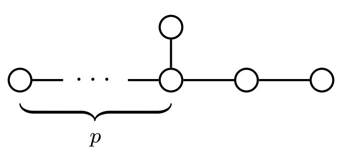


with $p \geqslant q \geqslant 2$.

If $q = 2$, we get $T_{p,q,2} = D_{p+2}$, so it remains to consider the case $q \geqslant 3$.

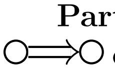
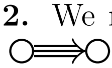
If $q > 3$, then $D(A)$ contains $D(\widetilde{E}_7) = T_{4,4,2}$, which is impossible. So we are left with the case

$T_{p,3,2}$:



with $p \geqslant 3$.

If $p = 3, 4$, or 5, we get E_6, E_7, or E_8 respectively. In the case $p \geqslant 6$, $D(A)$ contains $T_{6,3,2} = D(\widetilde{E}_8)$, which is impossible. This completes the proof in the simply-laced case.

Part 2. We now classify non-simply-laced connected diagrams $D(A)$, i.e. containing  or  as subdiagrams. This can be done by computing many large determinants (which we completely avoided in the simply-laced case). But we would rather argue by the following simple result on determinants:

Exercise 18.4. Let A be an $r \times r$ matrix, $r \geqslant 3$, and let B (resp. C) be the submatrices of A, obtained by removing the first row and column (resp. the first two rows and columns). Then

(a) If the first row (resp. column) of A is $\begin{pmatrix} 2 & -a & 0 & \cdots & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 2 & -b & 0 \cdots & 0 \end{pmatrix}^T$, then

$$\det A = 2 \det B - ab \det C.$$

(b) If

$$A = \begin{pmatrix} c_1 & -a_1 & & & & -b_r \\ -b_1 & c_2 & -a_2 & & 0 & \\ & -b_2 & c_3 & \ddots & & \\ & & \ddots & \ddots & -a_{r-2} & \\ & 0 & & -b_{r-2} & c_{r-1} & -a_{r-1} \\ -a_r & & & & -b_{r-1} & c_r \end{pmatrix},$$

then $\det(A - \varepsilon E_{12}) = \det A - \varepsilon \left(b_1 \det C + \prod_{i=2}^{r} a_i\right)$. In particular, if $b_1 > 0$, $a_i > 0$ for $i = 2, \cdots, r$, $\det C > 0$, and $\varepsilon > 0$, then $\det(A - \varepsilon E_{12}) < \det A$.

Exercise 18.4(b) implies that a non-simply laced Dynkin diagram $D(A)$ has no cycles, since otherwise $\det A < \det \widetilde{A}_r = 0$, a contradiction.

Next, if $D(A)$ without cycles contains the Dynkin diagram of G_2 and is connected, the matrix A contains the following principal submatrix

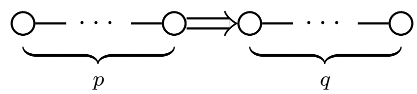
$$M = \begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix} \quad \text{with } a, b > 0,$$

then in Exercise 18.4(a), we have $B = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $C = (2)$, so that $\det M = 2 \det B - ab \det C = 2(1 - ab) \leqslant 0$, a contradiction.

It remains to consider the case when $D(A)$ has only simple or $\geqslant 1$ double connections.

Recall that $\widetilde{C_s}$ with $s \geqslant 2$ cannot be a subdiagram. By Exercise 18.4(a) the subdiagrams with flipped arrow directions also cannot be subdiagrams, since they are obtained from the corresponding matrix of $D(\widetilde{C}_s)$ by replacing some of A, B or C by their transposes, which does not change any of the determinants in the calculation. Hence, by Exercise 18.4(a) each of these variants is impossible.

Therefore $D(A)$ may contain only one double connection. But then it cannot have a branching point, since $D(\widetilde{B}_r)$ contains a double edge and a branching point. So the only remaining case is the diagram



If $p = 1$ (resp. $q = 1$), we get the Dynkin diagram of C_{q+1} (resp. B_{p+1}. If $p = 2 = q$, we get F_4. The diagram $D(\widetilde{F}_4)$ has $p = 3, q = 2$ and the corresponding transpose Cartan matrix has $p = 2, q = 3$, so both cases are impossible. ■

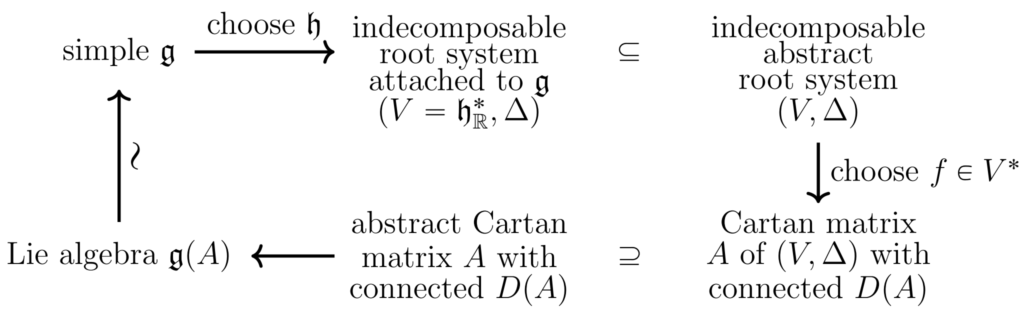
Bonus Problem. Prove that the list of all abstract extended Dynkin diagrams (i.e. connected Dynkin diagrams of the matrices, satisfying all properties, listed in Proposition 17.1, except that the determinant is 0), consists of those in the right column of Figure 1, those obtained from them by reversing the arrows, and the Dynkin diagrams of 2×2 matrices $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$.

Lecture 19

Classification of Simple Finite Dimensional Lie Algebras over an Algebraically Closed Field of Characteristic 0

Since any semisimple Lie algebra decomposes uniquely, up to permutation, in a direct sum of simple ones, the classification of the former reduces to the latter.

Our strategy schematically is as follows:



Semisimple Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0 are classified by the following theorem.

Classification Theorem.

(a) Finite-dimensional semisimple Lie algebras are isomorphic if and only if they have the same Dynkin diagram.

(b) A complete non-redundant list of simple finite-dimensional Lie algebras over $\mathbb{F}$ is the following:

$$\mathfrak{sl}_n(\mathbb{F}) \ (n \geqslant 2), \quad \mathfrak{so}_n(\mathbb{F}) \ (n \geqslant 7), \quad \mathfrak{sp}_{2n}(\mathbb{F}) \ (n \geqslant 2),$$
$$E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2.$$

Corollary 19.1.

(a) An arbitrary root system (V, Δ) is isomorphic to a root system, attached to a semisimple finite-dimensional Lie algebra over $\mathbb{F}$.

(b) Any abstract Cartan matrix is that of a finite-dimensional semisimple Lie algebra over $\mathbb{F}$.

Exercise 19.1. Deduce from the Classification Theorem (a) the following isomorphisms:

$$\mathfrak{sl}_2(\mathbb{F}) \simeq \mathfrak{so}_3(\mathbb{F}) \simeq \mathfrak{sp}_2(\mathbb{F}), \qquad \mathfrak{so}_4(\mathbb{F}) \simeq \mathfrak{sl}_2(\mathbb{F}) \oplus \mathfrak{sl}_2(\mathbb{F}),$$
$$\mathfrak{so}_5(\mathbb{F}) \simeq \mathfrak{sp}_4(\mathbb{F}), \qquad \mathfrak{so}_6(\mathbb{F}) \simeq \mathfrak{sl}_4(\mathbb{F}).$$

Beginning of proof of Classification Theorem (a). Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over $\mathbb{F}$. Choose a Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right), \tag{1}$$

where $\Delta \subset \mathfrak{h}^*$ is the set of roots. Choose a linear function on $\mathfrak{h}$, which takes non-zero integer (or rational) values on Δ, and let $\Delta_+ = \{\alpha \in \Delta \mid f(\alpha) > 0\}, \Delta_- = \{\alpha \in \Delta \mid f(\alpha) < 0\}$. Then $\Delta = \Delta_+ \coprod \Delta_-$, where $\Delta_- = -\Delta_+$. Let $\Pi \subseteq \Delta_+$ be the set of simple roots. We will prove later that nothing depends on the choice of f.

Let $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha$, and $\mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha$, which are obviously subalgebras of $\mathfrak{g}$. Then (1) implies the ***triangular decomposition***

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{(direct sum of vector spaces).} \tag{2}$$

Exercise 19.2. Show that for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F}), \mathfrak{so}_n(\mathbb{F})$, or $\mathfrak{sp}_n(\mathbb{F})$, choosing $\mathfrak{h}$ to be all diagonal matrices in $\mathfrak{g}$, then $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) consists of all strictly upper (resp. lower) triangular matrices in $\mathfrak{g}$. (Hence the name triangular decomposition)

Continuation of proof of Classification Theorem (a). Recall that for each $\alpha \in \Delta_+$, we can choose $E_\alpha \in \mathfrak{g}_\alpha$ and $F_\alpha \in \mathfrak{g}_{-\alpha}$, such that $\kappa(E_\alpha, E_{-\alpha}) = \frac{2}{\kappa(\alpha,\alpha)}$, so that, letting $H_\alpha = \frac{2\nu^{-1}(\alpha)}{\kappa(\alpha,\alpha)}$, we obtain a subalgebra $\mathbb{F}E_\alpha \oplus \mathbb{F}H_\alpha \oplus \mathbb{F}F_\alpha$, isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. For the set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$, set

$$E_i = E_{\alpha_i}, \quad H_i = H_{\alpha_i}, \quad F_i = F_{-\alpha_i}, \quad 1 \leqslant i \leqslant r.$$

Then we have:

(a) $[H_i, H_j] = 0$,

(b) $[H_i, E_j] = A_{ij}E_j$,

(c) $[H_i, F_j] = -A_{ij}F_j$,

(d) $[E_i, F_j] = \delta_{ij}H_j$,

where $A = (A_{ij})$ is the Cartan matrix of $\mathfrak{g}$. Indeed, (a) holds since $\mathfrak{h}$ is abelian; (b) and (c) hold since $E_j \in \mathfrak{g}_{\alpha_j}$ and $F_j \in \mathfrak{g}_{-\alpha_j}$. Finally (d) holds for $i = j$ by construction, and for $i \neq j$ by Dynkin Theorem (a) (difference of distinct simple roots is not a root).

Definition 19.1. The relations (a)-(d) are called the ***Chevalley relations***. The elements E_i, F_i, H_i are called ***Chevalley generators***.

Exercise 19.3. Prove that the E_i satisfy the following relations:

$$(\text{ad } E_i)^{1-A_{ij}} E_j = 0 \quad \text{for } i \neq j,$$

and similarly the F_i. These are called ***Serre relations***.

(In the simply-laced case these relatoins mean that $[E_i, E_j] = 0$ (resp. $[[E_i, E_j], E_j] = 0$) if i and j are not connected (resp. connected) in the Dynkin diagram of $\mathfrak{g}$.)

Lemma 19.1. The elements $E_1, \cdots, E_r$ (resp. $F_1, \cdots, F_r$) generate $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). Consequently the Chevalley generators indeed generate the Lie algebra $\mathfrak{g}$.

Proof. We have to prove that for each $\alpha \in \Delta_+$ the root space $\mathfrak{g}_\alpha$ lies in the subalgebra $\mathfrak{n}'_+$ of $\mathfrak{n}_+$, generated by $E_1, \cdots, E_r$, and similarly $\mathfrak{g}_{-\alpha}$ lies in $\mathfrak{n}'_-$, generated by the F_i's.

Given $\alpha \in \Delta_+$, writing $\alpha = \sum_{i=1}^r m_i \alpha_i$, where m_i are non-negative integers, we call $\sum_i m_i$ ($\geqslant 1$) the ***height*** of α, and prove the claim by induction on the height. When height $\alpha = 1$, this root is simple, so $E_\alpha \in \mathfrak{n}'_+$.

For the inductive step, recall that for a non-simple α there exists a simple root α_i, such that $\alpha - \alpha_i \in \Delta_+$ (by Dynkin Theorem (c) in Lecture 17), hence $\mathfrak{g}_\alpha = [\mathfrak{g}_{\alpha-\alpha_i}, E_i]$. By the inductive assumption, $\mathfrak{g}_{\alpha-\alpha_i} \in \mathfrak{n}'_+$, hence $\mathfrak{g}_\alpha \in \mathfrak{n}'_+$. For $\mathfrak{n}_-$ the proof is the same. ■

Definition 19.2. Let A be an abstract Cartan matrix. Denote by $\widetilde{\mathfrak{g}}(A)$ the Lie algebra on generators E_i, F_i and H_i, $1 \leqslant i \leqslant r$, subject to the Chevalley relations (a)-(d). Note that this Lie algebra depends only on A.

The Lie algebra $\widetilde{\mathfrak{g}}(A)$ is infinite-dimensional if $r > 1$ (we will not need this fact), but is closely related to $\mathfrak{g}$, as we have an obvious surjective homomorphism $\varphi : \widetilde{\mathfrak{g}}(A) \to \mathfrak{g}$, sending $E_i \mapsto E_i,\ F_i \mapsto F_i,\ H_i \mapsto H_i$. Surjectivtiy of φ follows from Lemma 19.1.

Lemma 19.2. Let $\widetilde{\mathfrak{n}}_+$ (resp. $\widetilde{\mathfrak{n}}_-$) be the subalgebra of $\widetilde{\mathfrak{g}}(A)$, generated by $E_1, \cdots, E_r$ (resp. $F_1, \cdots, F_r$), and let $\widetilde{\mathfrak{h}}$ be the span of $H_1, \cdots, H_r$. Then

(a) $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}_-$ (direct sum of vector spaces).

(b) If I is an ideal of $\widetilde{\mathfrak{g}}(A)$, then

$$I = (\widetilde{\mathfrak{n}}_+ \cap I) \oplus (\widetilde{\mathfrak{h}} \cap I) \oplus (\widetilde{\mathfrak{n}}_- \cap I).$$

(c) Provided that the Lie algebra $\mathfrak{g}$ is simple, the Lie algebra $\widetilde{\mathfrak{g}}(A)$ contains a unique proper maximal ideal $I(A)$.

Proof. (a) Let $\widetilde{\mathfrak{g}}(A)' = \widetilde{\mathfrak{n}}_+ + \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{n}}_-$, which is a subspace of $\widetilde{\mathfrak{g}}(A)$. It is obvious, by the Jacobi identity, that

$$\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{g}}(A)'\right] \subseteq \widetilde{\mathfrak{g}}(A)'. \tag{3}$$

We will show that

$$[E_i, \widetilde{\mathfrak{g}}(A)'] \subseteq \widetilde{\mathfrak{g}}(A)', \tag{4}$$
$$[F_i, \widetilde{\mathfrak{g}}(A)'] \subseteq \widetilde{\mathfrak{g}}(A)'. \tag{5}$$

Since an arbitrary element of $\widetilde{\mathfrak{g}}(A)$ is an iterated bracket of the E_i, F_i, H_i, by the Jacobi identity, (3), (4) and (5) together would imply that $\widetilde{\mathfrak{g}}(A) \subseteq \widetilde{\mathfrak{g}}(A)'$, hence $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{g}}(A)'$.

Inclusion (5) is proved in the same way as (4). To prove (4), note that, obviously $[E_i, \widetilde{\mathfrak{n}}_+] \subseteq \widetilde{\mathfrak{n}}_+$, and $[E_i, \mathfrak{h}] \subseteq \widetilde{\mathfrak{n}}_+$. Finally, $[E_i, \widetilde{\mathfrak{n}}_-] \subseteq \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{n}}_-$ because $[E_i, F_j] \in \widetilde{\mathfrak{h}}$, and, by the Jacobi identity, for $s \geqslant 2$ we have:

$$[E_i, [F_{i_1}, \cdots, F_{i_s}]] = \delta_{ii_1}\,[H_{i_1}, F_{i_2}, \cdots, F_{i_s}] + \cdots + \delta_{ii_s}\left[F_{i_1}, \cdots, F_{i_{s-1}}, H_{i_s}\right] \subset \widetilde{\mathfrak{n}}_-,$$

by the Chevalley relations (d) and (c). Here $[a_1, ..., a_s]$ stands for an s-fold commutator with brackets taken in arbitrary order.

To show that the sum in $\widetilde{\mathfrak{g}}(A) = \widetilde{\mathfrak{n}}_+ + \widetilde{\mathfrak{h}} + \widetilde{\mathfrak{n}}_-$ is direct, let $h_0 \in \mathfrak{h}$ satisfy $\alpha_i(h_0) = 1$ for all i. Writing $h_0 = \sum_j x_j H_j$, finding such an h_0 is equivalent to solving the linear system of equations

$$\sum_i A_{ij} x_i = 1, \quad j = 1, \cdots, r,$$

which is possible since $\det A \neq 0$.

Then ad h_0 acts with positive (resp. negative) eigenvalues (equal to height α and resp. $-$height α) on $\widetilde{\mathfrak{n}}_+$ (resp. $\widetilde{\mathfrak{n}}_-$) and 0 eigenvalue on $\widetilde{\mathfrak{h}}$, proving that the sum is direct.

To prove (b), we invoke Lemma 19.1, which said for any $\mathfrak{h}_0$-module $V = \bigoplus_{\lambda \in \mathfrak{h}_0^*} V_\lambda$ and any $\mathfrak{h}_0$-invariant subspace $U \subseteq V$, we have $U = \bigoplus_\lambda (U \cap V_\lambda)$. Applying this to the adjoint representation and $\mathfrak{h}_0 = \mathbb{F}h_0$, we see that $\widetilde{\mathfrak{n}}_+$ (resp. $\widetilde{\mathfrak{n}}_-$) is the sum of ad $\mathfrak{h}_0$-eigenspaces with positive (resp. negative) eigenvalues and $\widetilde{\mathfrak{h}}$ has 0 eigenvalue, proving (b).

(d) Since $\mathfrak{g}$ is simple, its Dynkin diagram $D(A)$ is connected. Let I be a proper ideal of $\widetilde{\mathfrak{g}}(A)$. Then $I \cap \widetilde{\mathfrak{h}} = 0$. Indeed, if $a \in I \cap \mathfrak{h}$ is non-zero, then $\alpha_i(a) \neq 0$ for some i, hence $[a, E_i] = \alpha_i(a) E_i \neq 0$, so $E_i \in I$. Hence $H_i \in I$ by Chevalley relation (d). Also, by the Chevalley relations (b) and (c), E_j and F_j are contained in I for all j, such that $A_{ij} \neq 0$. Since $D(A)$ is connected, it follows that all E_j and F_j are contained in I, hence, by Chevalley relation (d), all $H_j \in I$, contradicting properness of I.

Thus we have the decomposition

$$I = (\widetilde{\mathfrak{n}}_+ \cap I) \oplus (\widetilde{\mathfrak{n}}_- \cap I).$$

for any proper ideal, hence for the sum of all proper idealsm, $I(A)$. Hence $I(A)$ is the unique proper maximal ideal. ■

Continuation of proof of the Classification Theorem. By part (c) of Lemma 19.2, the Lie algebra

$$\mathfrak{g}(A) = \widetilde{\mathfrak{g}}(A)/I(A)$$

depends only on the abstract Cartan matrix A, if $D(A)$ is connected. So if $\mathfrak{g}$ is a simple Lie algebra, the kernel of the surjective homomorphism φ is a maximal proper ideal, hence, by Lemma 19.2(a), φ induces an isomorphism $\mathfrak{g}(A) \simeq \mathfrak{g}$. This proves the "if" part of Classification Theorem (a). The "only if" part will follow once we show the independence of A from the choice of f. This will be proved in Lecture 21.

Claim (b) will follow from the classification of Dynkin diagrams of abstract Cartan matrices, once we show that $\dim \mathfrak{g}(A) < \infty$ for $A = E_6, E_7, E_8, F_4, G_2$, This will be proved in Lecture 20. We will prove this by exhibiting an explicit construction of the Lie algebras $\mathfrak{g}(A)$ for these five abstract Cartan matrices A.

Exercise 19.4. Prove that $\dim \widetilde{\mathfrak{g}}(A) = \infty$ if $r > 1$.

Remark 19.1. The Lie algebra $\mathfrak{g}(A)$, for which A satisfies (i), (ii), but not necessarily (iii) is called a ***Kac-Moody algebra***. It is infinite-dimensional if (iii) doesn't hold. Especially important are the Lie algebras $\mathfrak{g}(\widetilde{A})$, where $\widetilde{A}$ is an extended Cartan matrix, called ***affine*** Lie algebras.

Lecture 20

Explicit Construction of Exceptional Simple Lie Algebras, and the Compact Form

In the previous lecture we proved that any simple classical Lie algebra $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{g}(A)$, where A is the Cartan matrix of $\mathfrak{g}$. One of the two things that remained, in order to complete the proof of the Classification Theorem, is to show that the Lie algebra $\mathfrak{g}(A)$ is finite-dimensional for all five exceptional abstract Cartan matrices. We do this by an explicit construction of simple Lie algebras with an abstract Cartan matrix A. Of course, for classical A we recover all clasical (finite-dimensional) simple Lie algebras.

We shall consider two cases: simply laced case, consisting of symmetric abstract Cartan matrices A of type A_r, D_r, E_6, E_7, E_8, for which the set of roots Δ consists of vectors α, such that $(\alpha, \alpha) = 2$ in the even root lattice $L = \mathbb{Z}\Delta$; and the non-simply laced case, of A of type B_r, C_r, F_4, G_2.

Case 1: simply-laced. Let L be an even lattice of rank r such that $\Delta = \{\alpha \in L \mid (\alpha, \alpha) = 2\}$ spans L over $\mathbb{Z}$. Then we know that $(V = \mathbb{R} \otimes_{\mathbb{Z}} L, \Delta)$ is a root system, with a symmetric Cartan matrix $A = ((\alpha_i, \alpha_j))_{i,j=1}^r$.

We expect the corresponding simple Lie algebra to be of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathbb{F} E_\alpha \right), \quad \text{where } \mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} \Delta \quad \text{(direct sum of vector spaces)} \tag{1}$$

with the following brackets, where $h, h' \in \mathfrak{h}$, and $\alpha, \beta \in \Delta$:

(a) $[h, h'] = 0$,

(b) $[h, E_\alpha] = (\alpha, h) E_\alpha = -[E_\alpha, h]$,

(c) $[E_\alpha, E_{-\alpha}] = -\alpha$,

(d) $[E_\alpha, E_\beta] = 0$ if $\alpha + \beta \notin \Delta \cup \{0\}$,

(e) $[E_\alpha, E_\beta] = \varepsilon(\alpha, \beta) E_{\alpha+\beta}$ for some $\varepsilon(\alpha, \beta) \in \mathbb{F}^\times$ if $\alpha + \beta \in \Delta$.

For this algebra $\mathfrak{g}$ to be a Lie algebra we need to construct the non-zero constants $\varepsilon(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$, for which skew-commutativity and Jacobi identity hold. Remarkedly, we can arrange so that $\varepsilon(\alpha, \beta) \in \{\pm 1\}$ for all $\alpha, \beta \in L$.

Once we prove that $\mathfrak{g}$ is a Lie algebra, it will automatically be simple by the criterion from Lecture 14, noting that $\alpha(h)$ means the same as (α, h) if we identify $\mathfrak{h}$ with $\mathfrak{h}^*$, using the bilinear form $(\bullet, \bullet)$ on L. Then $[h, E_\alpha] = \alpha(h) E_\alpha$ for $h \in \mathfrak{h}, \alpha \in \Delta$ and $\alpha(\nu^{-1}(\alpha)) = (\alpha, \alpha) \neq 0$.

Proposition 20.1. There exists a map

$$\varepsilon : L \times L \to \{\pm 1\}$$

with the following two properties:

(a) (bimultiplicativity) $\varepsilon(\alpha, \beta+\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma)$, and $(\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma)$; this implies $\varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1$ and $\varepsilon(-\alpha,\beta) = \varepsilon(\alpha,\beta) = \varepsilon(\alpha,-\beta)$.

(b) $\varepsilon(\alpha,\alpha) = (-1)^{(\alpha,\alpha)/2}$ for all $\alpha \in L$; by expanding $\varepsilon(\alpha+\beta,\alpha+\beta) = \varepsilon(\alpha,\alpha)\varepsilon(\beta,\beta)\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)$, it follows that

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{(\alpha,\beta)}, \quad \text{if } \alpha,\beta \in \Delta. \tag{2}$$

Proof. Choose a set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ for Δ and consider the Dynkin diagram. Direct the edges of the Dynkin diagram arbitrarily and set

(i) $\varepsilon(\alpha_i,\alpha_i) = -1$ for all i;

(ii) $\varepsilon(\alpha_i,\alpha_j) = -\varepsilon(\alpha_j,\alpha_i) = -1$ if $i \neq j$ and there is an edge $\alpha_i \to \alpha_j$ in the Dynkin diagram;

(iii) $\varepsilon(\alpha_i,\alpha_j) = \varepsilon(\alpha_j,\alpha_i) = 1$ if α_i and α_j are not connected in the Dynkin diagram.

This implies that

$$\varepsilon(\alpha_i,\alpha_j)\varepsilon(\alpha_j,\alpha_i) = (-1)^{(\alpha_i,\alpha_j)} \text{ for } i,j = 1,\cdots,r,$$

since $(\alpha_i,\alpha_j) = -1$ (resp. $= 0$) if α_i and α_j are connected in $D(A)$ (resp. not connected).

Then we extend ε from $\Pi \times \Pi \to \{\pm 1\}$ to $\varepsilon : L \times L \to \{\pm 1\}$ by bimultiplicativity, so we get bimultiplicativity of ε for free, and twe just need to check that $\varepsilon(\alpha,\alpha) = (-1)^{(\alpha,\alpha)/2}$ for $\alpha \in \Delta$. Write $\alpha = \sum_{i=1}^r k_i\alpha_i$, where $k_i \in \mathbb{Z}$. Then we have:

$$\begin{aligned}\varepsilon(\alpha,\alpha) &= \prod_{i,j=1}^r \varepsilon(\alpha_i,\alpha_j)^{k_ik_j} = \prod_{i=1}^r \varepsilon(\alpha_i,\alpha_j)^{k_i^2} \prod_{\alpha_i\to\alpha_j} (\varepsilon(\alpha_i,\alpha_j)\varepsilon(\alpha_j,\alpha_i))^{k_ik_j} \\ &= (-1)^{\sum_i k_i^2} \prod_{\alpha_i\to\alpha_j} (-1)^{k_ik_j} = (-1)^{\sum_i k_i^2(\alpha_i,\alpha_i)/2} \prod_{\alpha_i\to\alpha_j} (-1)^{k_ik_j(\alpha_i,\alpha_j)} = (-1)^{(\alpha,\alpha)/2}.\end{aligned}$$

In the second equality we used bimultiplicativity of ε. In the third equality we used that i,j for which α_i and α_j are not connected contribute 1 to the product, and in the final two equalities we used that $(\alpha_i,\alpha_i) = 2$ and $(\alpha_i,\alpha_j) = -1$ if α_i and α_j are connected, and $(\alpha_i,\alpha_j) = 0$ if α_i and α_j are not connected in $D(A)$. ∎

Theorem 20.1. If $\varepsilon : L \times L \to \{\pm 1\}$ satisfies properties (a) and (b) of Proposition 20.1, then skew-commutativity and Jacobi identity holds for the Lie algebra $\mathfrak{g}$, given by (1), with brackets (a) - (e). Consequently $\mathfrak{g}$ is a Lie algebra.

Proof. Note that, since α is a root if and only if $(\alpha, \alpha) = 2$, we have for $\alpha, \beta \in \Delta$:

$$\alpha \pm \beta \in \Delta \quad \text{if and only if} \quad (\alpha, \beta) = \mp 1. \tag{3}$$

Hence, by (2), $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = -1$ if $\alpha, \beta, \alpha + \beta \in \Delta$, proving that $[E_\alpha, E_\beta] = -[E_\beta, E_\alpha]$ in this case (see (e)). In all other cases (a) - (d), the skew-commutativity is obvious.

As we did for skew-commutativity, it is enough to show, in order to prove the Jacobi identity, that

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \tag{4}$$

if x, y, z are either in $\mathfrak{h}$, or one of the E_α.

If one of the elements x, y, z lies in $\mathfrak{h}$, it is immediate to see that (4) holds for any map $\varepsilon : L \times L \to \mathbb{F}^\times$. It remains to check (4) if $x = E_\alpha, y = E_\beta, z = E_\gamma$ for $\alpha, \beta, \gamma \in \Delta$.

If $\alpha + \beta, \alpha + \gamma, \beta + \gamma$ are not in $\Delta \cup \{0\}$, then all brackets $[E_\alpha, E_\beta], [E_\alpha, E_\gamma], [E_\beta, E_\gamma]$ are zero, hence (4) trivially holds.

So we may assume without loss of generality that $\alpha + \beta \in \Delta \cup \{0\}$.

Case 1. $\alpha + \beta = 0$. In this case the LHS of (4) is

$$[-\alpha, E_\gamma] + [[E_{-\alpha}, E_\gamma], E_\alpha] + [[E_\gamma, E_\alpha], E_{-\alpha}]. \tag{5}$$

If $\gamma = \pm\alpha$, then (5) becomes

$$\mp 2E_{\pm\alpha} + [\pm\alpha, E_{\pm\alpha}] + [\pm\alpha, E_{\pm\alpha}] = 0,$$

so in this case Jacobi identity holds for any map ε.

Hence we may assume that $\gamma \pm \alpha \neq 0$. If $\gamma \pm \alpha \notin \Delta$, then $[E_{\pm\alpha}, E_\gamma] = 0$ and $(\alpha, \gamma) = 0$ by (3), so all three terms in (5) vanish.

Otherwise assume that $\gamma + \alpha \in \Delta$, so that $\gamma - \alpha \notin \Delta$ by (3), whence $[E_{-\alpha}, E_\gamma] = 0$. Then the Jacobi identity becomes

$$-(\alpha, \gamma)E_\gamma + \varepsilon(\gamma, \alpha)\varepsilon(\gamma + \alpha, -\alpha)E_\gamma = 0.$$

This is equivalent to

$$-(\alpha, \gamma) + \varepsilon(\gamma, \alpha)\varepsilon(\gamma, -\alpha)\varepsilon(\alpha, -\alpha) = 0,$$

which holds since $(\alpha, \gamma) = -1$ by (3), and $\varepsilon(\gamma, \alpha)\varepsilon(\gamma, -\alpha)\varepsilon(\alpha, -\alpha) = \varepsilon(\gamma, \alpha)\varepsilon(\gamma, \alpha)^{-1}\varepsilon(\alpha, \alpha) = -1$. The case when $\gamma - \alpha \in \Delta$ and $\gamma + \alpha \notin \Delta$ is similar.

Case 2. $\alpha + \beta, \alpha + \gamma, \beta + \gamma \in \Delta$. In this case $(\alpha, \beta) = (\alpha, \gamma) = (\beta, \gamma) = -1$ by (3). This implies that $(\alpha + \beta + \gamma, \alpha + \beta, \gamma) = 2 + 2 + 2 - 6 \cdot 1 = 0$, so $\alpha + \beta + \gamma = 0$. Then $\gamma = -\alpha - \beta \in \Delta$, and (4) becomes equivalent to showing that

$$\begin{aligned} &[[E_\alpha, E_\beta], E_{-\alpha-\beta}] + [[E_\beta, E_{-\alpha-\beta}], E_\alpha] + [[E_{-\alpha-\beta}, E_\alpha], E_\beta] \\ &= \varepsilon(\alpha, \beta)[E_{\alpha+\beta}, E_{-\alpha-\beta}] + \varepsilon(\beta, -\alpha-\beta)[E_{-\alpha}, E_\alpha] + \varepsilon(-\alpha-\beta, \alpha)[E_{-\beta}, E_\beta] \\ &= -\varepsilon(\alpha, \beta)(\alpha + \beta) - \varepsilon(\beta, \alpha)\alpha - \varepsilon(\beta, \alpha)\beta \text{ vanishes.} \end{aligned}$$

But $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha,\beta)} = -1$ by (3), so the Jacobi identity holds in this case too. ■

Remark 20.1. The dimension of $\mathfrak{g}$, given by (1), is equal to $|\Delta| + r$, hence

$$\dim E_6 = 72,\ \dim E_7 = 133,\ \dim E_8 = 248,\ \dim F_4 = 52,\ \dim G_2 = 14.$$

Exercise 20.1. Define on the Lie algebra $\mathfrak{g}$, given by (1), the following bilinear form:

- On $\mathfrak{h} = \mathbb{F} \otimes_{\mathbb{Z}} L$ it is the bilinear form $(\bullet, \bullet)$ on L extended by bilinearity,
- $(\mathfrak{h}, E_\alpha) = 0$ for all $\alpha \in \Delta$,
- $(E_\alpha, E_\beta) = -\delta_{\alpha,-\beta}$ if $\alpha, \beta \in \Delta$.

Show that this is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$.

Exercise 20.2. On the lattice $L = \bigoplus_{i=1}^r \mathbb{Z}\varepsilon_i$ with $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$ consider the bimultiplicative function $\varepsilon : L \times L \to \{\pm 1\}$, defined by

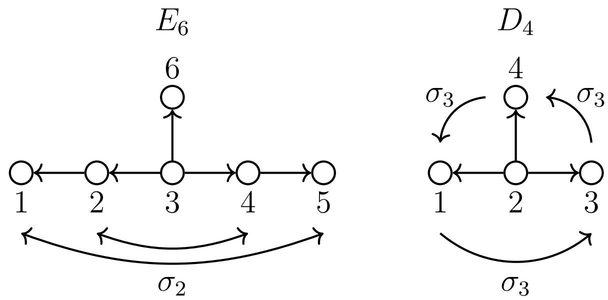
$$\varepsilon(\varepsilon_i, \varepsilon_j) = \begin{cases} -1 & \text{if } i < j, \\ 1 & \text{if } i \geqslant j. \end{cases}$$

Show that the restriction of ε to $L_{A_{r-1}}$ and L_{D_r} satisfies properly (b) of Proposition 20.1.

We have explicitly constructed the simple finite-dimensional Lie algebra of type A_r ($r \geqslant 1$), $D_r (r \geqslant 4)$, E_6, E_7, E_8. The bilinear form, defined by Exercise 20.1, is the unique invariant symmetric non-degenerate bilinear form on these Lie algebras, up to a non-zero constant factor.

Now we turn to the non-simply laced case.

Case 2: non-simply laced. It remains to show that the (simple) Lie algebras $\mathfrak{g}(A)$ for A of type F_4 and G_2 are finite-dimensional . We construct them by constructing embeddings $F_4 \subset E_6$ and $G_2 \subset D_4$ as follows. Consider the Dynkin diagrams of E_6 and D_4 with the following orientations and their automorphism of order 2 and 3 respectively, preserving these orientations:



Since the Dynkin diagram and its orientation is invariant under these automorphisms, we see that $\varepsilon(\sigma(\alpha), \sigma(\beta)) = \varepsilon(\alpha, \beta)$ and $A_{\sigma(i),\sigma(j)} = A_{ij}$ for $\sigma = \sigma_1$ or σ_2, hence the brackets (a) - (d) are preserved by σ.

Also σ preserves all Chevalley relations (a)–(d) from Lecture 19, hence induces an automorphism of the Lie algebra $\widetilde{\mathfrak{g}}(A)$, so that the surjective homomorphism $\varphi : \widetilde{\mathfrak{g}}(A) \to \mathfrak{g}$ is σ-equivariant. Due to its uniqueness, the maximal ideal $I(A)$ of $\widetilde{\mathfrak{g}}(A)$ in Lemma 19.2 is σ-invariant. Hence σ descends to an automorphism of $\mathfrak{g}(A)$, which is again denoted by σ.

Exercise 20.3. Show that for σ_2 the elements $X_1 + X_5, X_2 + X_4, X_3, X_6$, where $X = E, F$ or H, lie in the fixed point set of σ_2 in E_6, and satisfy the Chevalley relation of F_4. Likewise, for σ_3 the elements $X_1 + X_3 + X_4$ and X_2 are fixed by the automorphism σ_3 of D_4, and satisfy the Chevalley relations of G_2.

Denote in both cases the elements of $\mathfrak{g} = \mathbb{E}_6$ or D_4, fixed by $\sigma = \sigma_2$ or σ_3 respectively, by $\mathfrak{g}^\sigma$. By Exercise 20.3, there are surjective homomorphisms from respective subalgebras:

$$\mathfrak{g}(E_6)^{\sigma_2} \xrightarrow{\varphi_2} \mathfrak{g}(F_4) \qquad \text{and} \qquad \mathfrak{g}(D_4)^{\sigma_3} \xrightarrow{\varphi_3} \mathfrak{g}(G_2).$$

It follows that both $\mathfrak{g}(F_4)$ and $\mathfrak{g}(G_2)$ are finite-dimensional (simple) Lie algebras.

Exercise 20.4. Show that $\dim \mathfrak{g}(E_6)^{\sigma_2} = 52$ and $\dim \mathfrak{g}(D_4)^{\sigma_3} = 14$, and therefore, by Remark 20.1, both homomorphisms σ_2 and σ_3 are isomorphisms.

In the last part of this lecture we assume that $\mathbb{F} = \mathbb{C}$, and define the compact form $\mathfrak{g}_{\text{com}} \subset \mathfrak{g}$ where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra over $\mathbb{C}$, and $\mathfrak{g}_{\text{com}}$ is a Lie algebra over $\mathbb{R}$, whose complexification is $\mathfrak{g}$.

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{h}_{\mathbb{R}} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{R}E_\alpha)$, which is a Lie algebra over $\mathbb{R}$, whose complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$

is $\mathfrak{g}$. First, assume that $\mathfrak{g}$ is simply laced. Define an automorphism $\omega_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ by letting it act as -1 on $\mathfrak{h}_{\mathbb{R}}$, and let $\omega_{\mathbb{R}}(E_\alpha) = E_{-\alpha}$.

Exercise 20.5. Check that $\omega_{\mathbb{R}}$ is an automorphism of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ in the simply laced case.

Next, we extend $\omega_{\mathbb{R}}$ from $\mathfrak{g}_{\mathbb{R}}$ to $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$ to be an antilinear automorphism ω, by $\omega(\lambda a) = \overline{\lambda}\omega(a)$ for $\lambda \in \mathbb{C}, a \in \mathfrak{g}_{\mathbb{R}}$.

Definition 20.1. The fixed point set of ω in $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, called the ***compact form*** of $\mathfrak{g}$, denoted $\mathfrak{g}_{\mathrm{com}}$.

Exercise 20.6. If $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then $\omega(A) = -\overline{A}^\top$, and

$$\mathfrak{g}_{\mathrm{com}} = \mathfrak{su}_n = \left\{ A \in \mathfrak{sl}_n(\mathbb{C}) \mid \overline{A}^\top = -A \right\},$$

which is called the special unitary Lie algebra.

Proposition 20.2. The restriction of the invariant bilinear form $(\bullet, \bullet)$ on $\mathfrak{g}$, defined in Exercise 20.1, is negative definite on $\mathfrak{g}_{\mathrm{com}}$.

Proof. We can write

$$\mathfrak{g}_{\mathrm{com}} = \left(\sqrt{-1}\mathfrak{h}_{\mathbb{R}}\right) \oplus \left(\sum_{\alpha \in \Delta_+} \mathbb{R}(E_\alpha + E_{-\alpha}) \right) \oplus \left(\sum_{\alpha \in \Delta_+} \sqrt{-1}\mathbb{R}(E_\alpha - E_{-\alpha}) \right), \tag{6}$$

and these 3 subspaces are orthogonal to each other. Obviously, $\mathfrak{g}$ is the complexification of $\mathfrak{g}_{\mathrm{com}}$. It remains to show that the bilinear form is negative definite on each of these 3 subspaces.

This is true because $(\sqrt{-1}h, \sqrt{-1}h) = -(h,h) < 0$ for $h \in \mathfrak{h}_{\mathbb{R}}$, $(E_\alpha + E_{-\alpha}, E_\beta + E_{-\beta}) = -2\delta_{\alpha,\pm\beta}$, and $\left(\sqrt{-1}(E_\alpha - E_{-\alpha}), \sqrt{-1}(E_\beta - E_{-\beta})\right) = -2\delta_{\alpha,\pm\beta}$ for $\alpha, \beta \in \Delta$.

In the non-simply laced cases the automorphisms σ_2 and σ_3 of E_6 and D_4 commute with ω on the simply laced Lie algebras E_6 and D_4, hence the bilinear form $(\bullet, \bullet)$ remains negative definite on $\mathfrak{g}^\sigma$.

The non-simply laced classical Lie algebras B_r and C_r for $r \geqslant 2$, are fixed point sets of the transposition with respect to the anti-diagonal of the Lie algebras $A_{2r} = \mathfrak{sl}_{2r-1}(\mathbb{C})$ and $A_{2r-1} = \mathfrak{sl}_{2r}(\mathbb{C})$ which commutes with ω as well, hence the propositoin holds for them too. ■

Remark 20.2 $\mathfrak{g}_{\mathrm{com}}$ is the Lie algebra of a maximal compact subgroup of a complex algebraic group whose Lie algebra is $\mathfrak{g}$, hence the name "compact form".

Bonus Problem. Let $\mathfrak{g} \subset \mathfrak{g}'$ be the non-simply laced subalgebra of type B_r, C_r, F_4 or G_2 in the simply laced Lie algebra A_{2r}, A_{2r-1}, E_6 and D_4 respectively, as above. Describe the structure constants of $\mathfrak{g}$ in terms of those of $\mathfrak{g}'$. Show that the root vectors E_α can be chosen in such a way that, for $\alpha, \beta, \alpha + \beta \in \Delta$, one has $[E_\alpha, \mathbb{E}_\beta] = \pm(p+1)E_{\alpha+\beta}$, where p is the maximal non-negative integer such that $\alpha - p\beta \in \Delta$.

Lecture 21

The Weyl Group of a Root System

Let V be a finite-dimensional Euclidean vector space, i.e. a vector space over $\mathbb{R}$ with a positive-definite symmetric bilinear form $(\bullet, \bullet)$. For a non-zero vector $a \in V$, denote by r_a the orthogonal reflection relative to a, i.e. a linear operator on V such that

$$r_a(a) = -a, \quad r_a(v) = v \quad \text{if } (v, a) = 0. \tag{1}$$

Obviously, it is given by the following formula

$$r_a(v) = v - \frac{2(a, v)}{(a, a)} a, \qquad v \in V. \tag{2}$$

Denote by $O_V(\mathbb{R})$ the group of all invertible linear operators on V, preserving the bilinear form $(\bullet, \bullet)$, i.e.

$$O_V(\mathbb{R}) = \{A \in GL_V(\mathbb{R}) \mid (Au, Av) = (u, v),\ u, v \in V\}.$$

It is called the ***orthogonal group***. It is a compact subset of the vector space $\operatorname{End} V$.

Exercise 21.1. Prove the following properties of the reflection r_a:

(a) $r_a \in O_V(\mathbb{R})$,

(b) $r_a = r_{-a}$ and $r_a^2 = I_V$,

(c) $\det r_a = -1$,

(d) $A r_a A^{-1} = r_{A(a)}$ for $A \in O_V(\mathbb{R})$.

Definition 21.1. Let (V, Δ) be a root system. Let W be the subgroup of $O_V(\mathbb{R})$, generated by all r_a, where $a \in \Delta$. This group is called the ***Weyl group*** of the root system (V, Δ), and of the corresponding semisimple Lie algebra $\mathfrak{g}$.

Proposition 21.1. Let (V, Δ) be a root system, and let W be its Weyl group. Then

(a) $w(\Delta) = \Delta$ for all $w \in W$.

(b) W is a finite subgroup of $O_V(\mathbb{R})$.

Proof. For (a) we need to show that

$$r_\alpha(\beta)(= \beta - A_{\alpha,\beta}\alpha) \in \Delta \quad \text{if } \alpha, \beta \in \Delta, \text{ where } A_{\alpha,\beta} = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

First, r_α is non-singular, since $\det r_\alpha = -1$. Second, recall the string property of (V, Δ):

$$\{\beta - k\alpha \mid k \in \mathbb{Z}\} \cap (\Delta \cup \{0\}) = \{\beta - p\alpha, \cdots, \beta, \cdots, \beta + q\alpha\}$$

where $p, q \in \mathbb{Z}_{\geqslant 0}$, $p - q = A_{\alpha,\beta}$. Hence

$$p \geqslant A_{\alpha,\beta}, \qquad q \geqslant -A_{\alpha,\beta}.$$

So, if $(\alpha, \beta) \leqslant 0$ (resp. $\geqslant 0$), we can add α to β at least $-A_{\alpha,\beta}$ times (resp. subtract α from β at least $A_{\alpha,\beta}$ times), which means that $\beta - A_{\alpha,\beta}\alpha \in \Delta \cup \{0\}$. But it cannot be 0 since r_α is a non-singular operator.

(b) If an element w of W fixes all elements of Δ, it musst be I_V since Δ spans V, so the group W embeds in the group of permutations of the finite set Δ. Therefore W is a finite group. ■

Proposition 21.1 shows that the string property of a root system (V, Δ) implies that Δ is W-invariant.

Bonus Problem. Show that the converse is true: if we replace the string property by W-invariance of Δ and that $A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$, then we get an equivalent definition of a root system.

Hint: One has to check this only in the case $\dim V = 2$, i.e. the four rank 2 root systems.

Definition 21.2. Fix $f \in V^*$, which doesn't vanish on Δ, and let, as before, $\Delta_+ = \{\alpha \in \Delta \mid f(\alpha) > 0\}$ be the set of positive roots, and $\Pi = \{\alpha_1, \cdots, \alpha_r\} \subseteq \Delta_+$ be the set of simple roots ($r = \dim V$). Then the reflections $s_i = r_{\alpha_i}$ ($i = 1, \cdots, r$) are called ***simple reflections***.

Theorem 21.1.

(a) $\Delta_+ \setminus \{\alpha_i\}$ is s_i-invariant.

(b) If $\alpha \in \Delta_+ \setminus \Pi$, then there exists i, such that height $s_i(\alpha) <$ height α (height $\sum_i k_i\alpha_i := \sum_i k_i$).

(c) If $\alpha \in \Delta_+ \setminus \Pi$, then there exists a sequence of simple reflections $s_{i_1}, \cdots, s_{i_k}$, such that $s_{i_1} \cdots s_{i_k}(\alpha) \in \Pi$, and also $s_{i_j} \cdots s_{i_k}(\alpha) \in \Delta_+$ for all $1 \leqslant j \leqslant k$.

(d) The group W is generated by simple reflections.

Proof. (a) Recall that each $\alpha \in \Delta_+$ can be written as $\alpha = \sum_j k_j\alpha_j$, where all $k_j \in \mathbb{Z}_{\geqslant 0}$, and

$\alpha_1, \cdots, \alpha_r$ are linearly independent. Since $s_i(\alpha) = \alpha + n\alpha_i \in \Delta$, where $n \in \mathbb{Z}$, we conclude that $s_i(\alpha) \in \Delta_+$ if at least two of the k_i's are positive, i.e. if $\alpha \in \Delta_+ \backslash \Pi$. If $\alpha = \alpha_j,\ j \neq i$, then $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i \in \Delta_+$ since $A_{ij} \leqslant 0$ for $i \neq j$.

(b) Since $s_i(\alpha) = \alpha - \frac{2(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$, we see that if height $s_i(\alpha) \geqslant$ height α for all i, then $(\alpha, \alpha_i) \leqslant 0$ for all i, and then $(\alpha, \alpha) = \sum_i k_i(\alpha, \alpha_i) \leqslant 0$, hence $\alpha = 0$, a contradiction, proving (b).

(c) Just apply (b) finitely many times until we get a simple root.

(d) Denote by W' the subgroup of W, generated by simple reflections. By (c), for any $\alpha \in \Delta_+$ there exists $w \in W'$ such that

$$w(\alpha) = \alpha_j \in \Pi.$$

Hence, by Exercise 21.1(d), $r_\alpha = w^{-1}s_j w$, which lies in W'. So W' contains all r_α with $\alpha \in \Delta_+$. Since $r_\alpha = r_{-\alpha}$, W' contains all reflections, hence $W' = W$. ■

Example 21.1. Recall the root system of type A_r:

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \cdots, r+1\}, i \neq j\} \subset V = \left\{\sum_{i=1}^{r+1} a_i\varepsilon_i \mid \sum_i a_i = 0, a_i \in \mathbb{R}\right\}$$
$$\subset \mathbb{R}^{r+1} = \bigoplus_{i=1}^{r+1} \mathbb{R}\varepsilon_i \qquad \text{with } (\varepsilon_i, \varepsilon_j) = \delta_{ij}.$$

We have

$$r_{\varepsilon_i - \varepsilon_j}(\varepsilon_s) = \varepsilon_s - (\varepsilon_s, \varepsilon_i - \varepsilon_j)(\varepsilon_i - \varepsilon_j) = \begin{cases} \varepsilon_s & \text{if } s \neq i \text{ or } j \\ \varepsilon_j & \text{if } s = i \\ \varepsilon_i & \text{if } s = j \end{cases}.$$

So $r_{\varepsilon_i - \varepsilon_j}$ is the transposition of ε_i and ε_j, and the Weyl group $W_{A_r} = S_{r+1}$, the group of permutations of the set $\{\varepsilon_1, \cdots, \varepsilon_{r+1}\}$.

Exercise 21.2. Compute the Weyl groups for the root system of type B_r, C_r and D_r. In particular, show that for B_r and C_r the Weyl groups are isomorphic, but not isomorphic for D_r.

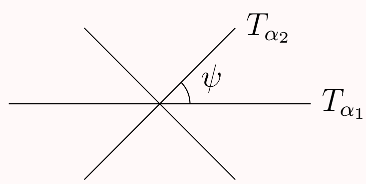
Definition 21.3. Consider the root system (V, Δ), and let T_α be the hyperplane in V, perpendicular to $\alpha \in \Delta_i$; consider V with the usual metric topology and in it the open set $V \backslash \bigcup_{\alpha \in \Delta} T_\alpha$. The connected components of this subset are called ***open chambers***, their closures are called ***closed chambers***. The subset

$$C = \{v \in V \mid (\alpha_i, v) > 0 \text{ for all } \alpha_i \in \Pi\} \tag{3}$$

is called the ***fundamental chamber***, and its closure is called the ***closed fundamental chamber***.

Exercise 21.3. Show that the open fundamental chamber C is an open chamber. (*Hint: if $a, b \in C$, then $ta + (1-t)b \in C$ for $0 \leqslant t \leqslant 1$.*)

Example 21.2. For the root system of type A_2, there are 6 open (closed) chambers, as depicted below:



where α_1 and α_2 are simple roots and $\psi = 60°$. The angle between T_{α_1} and T_{α_2} is the fundamental chamber.

Exercise 21.4. Show that a similar picture holds for all other rank 2 root systems $A_1 + A_1, B_2, G_2$ with $\psi = 90°, 45°, 30°$ respectively. Deduce that $(s_1 s_2)^m = e$, where $m = 2, 3, 4, 6$ for the root systems of type $A_1 + A_1, A_2, B_2, G_2$ respectively, and the number of open chambers is $2m$.

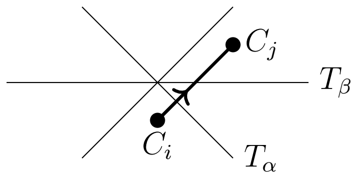
Theorem 21.2.

(a) The Weyl group W permutes all open (hence closed) chambers transitively, i.e. for any two open chambers C_i and C_j there exists $w \in W$, such that $w(C_i) = C_j$.

(b) Let Δ_+ and Δ'_+ be two subsets of positive roots, defined by the linear functions f and $f' \in V^*$ respectively. Then there exists $w \in W$, such that $w(\Delta_+) = \Delta'_+$. In particular, the Cartan matrix of the root system (V, Δ) is independent of the choice of f.

Proof. (a) Choose a segment connecting points in C_i and C_j, which doesn't intersect all $T_\alpha \cap T_\beta$, $\alpha, \beta \in \Delta_+, \alpha \neq \beta$. Let's move along this segment until we hit a hyperplane T_α.

Then replace C_i by $r_\alpha C_i$. After finitely many such steps we hit the chamber C_j



$C_j = r_\beta r_\alpha C_i.$

(b) Any linear function f on V can be written as $f_a(v) = (a, v)$, and it doesn't vanish on all roots if and only if a lies in one of the open chambers. If we move a around this chamber, the set Δ_+ defined by f remains unchanged. If we cross the hyperplane T_α, the chamber C changes to $r_\alpha(C)$. Hence all the subsets of positive roots in Δ are labelled by open chambers, and if $w(C) = C'$, then for the corresponding sets of positive roots Δ_+ and Δ'_+ we get that $w(\Delta_+) = \Delta'_+$. ■

Definition 21.4. Let $s_1, \cdots, s_r$ be the simple reflections in W (they depend on the choice of Δ_+). By Theorem 21.1(d), any w can be written as $w = s_{i_1} \cdots s_{i_t}$. Such a decomposition with minimal possible number of factors t is called a ***reduced*** decomposition, and in this case $t = \ell(w)$ is called the ***length*** of w.

Note that $\det_V w = (-1)^{\ell(w)}$ (since $\det_V s_i = -1$). For example $\ell(e) = 0, \ell(s_i) = 1, \ell(s_i s_j) = 2$ if $i \neq j$, but $= 0$ if $i = j$ since $s_i^2 = e$.

Exchange Lemma. Suppose that $s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-$ where $\alpha_{i_t} \in \Pi$. Then the expression $w = s_{i_1} \cdots s_{i_t}$ is not reduced. More precisely, $w = s_{i_1} \cdots s_{i_{m-1}} s_{i_{m+1}} \cdots s_{i_{t-1}}$ for some $1 \leqslant m \leqslant t-1$.

Proof. Consider the roots

$$\beta_k = s_{i_{k+1}} s_{i_{k+2}} \cdots s_{i_{t-1}}(\alpha_{i_t}) \quad \text{for } 0 \leqslant k \leqslant t-1.$$

Then $\beta_0 \in \Delta_-$ and $\beta_{t-1} = \alpha_{i_t} \in \Delta_+$. Hence there exists a positive integer $m \leqslant t-1$, such that $\beta_{m-1} \in \Delta_-, \beta_m \in \Delta_+$.

But $\beta_{m-1} = s_{i_m}(\beta_m)$, hence, by Theorem 21.1(a), $\beta_m = \alpha_{i_m} \in \Pi$.

Let $\overline{w} = s_{i_{m+1}} \cdots s_{i_{t-1}}$. Since $\overline{w}(\alpha_{i_t}) = \alpha_{i_m}$, by Exercise 21.1(d) it follows that $\overline{w} s_{i_t} \overline{w}^{-1} = s_{i_m}$, or $\overline{w} s_{i_t} = s_{i_m} \overline{w}$. The result follows by multiplying both sides of the last equation by $s_{i_1} \cdots s_{i_m}$ on the left. ■

Corollary 21.1. The Weyl group W acts simply transitively on the set of chambers, i.e. for chambers C_i and C_j there exists a unique $w \in W$, such that $w(C_i) = C_i$.

Proof. By Theorem 21.2(a), W acts transitively on the set of chambers . So we need only to prove that if C is the fundamental chamber, $w \in W$, and $w(C) = C$, then $w = 1$.

In the contrary case, $w(C) = C$ for some $w \neq e$, hence $w(\Delta_+) = \Delta_+$ for Δ_+ corresponding to C. Take a reduced expression $w = s_{i_1} \cdots s_{i_t}$, $t \geqslant 1$. Then $w(\alpha_{i_t}) = s_{i_1} \cdots s_{i_{t-1}}(-\alpha_{i_t}) \in \Delta_+$, hence $s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}) \in \Delta_-$. Therefore, $w = s_{i_1} \cdots s_{i_t}$ is not a reduced expression, a contradiction. ■

Remark 21.1. As we have seen (Exercise 21.4), the generators $s_1, \cdots, s_r$ of the Weyl group W satisfy the relations

$$s_i^2 = e, \quad (s_i s_j)^{m_{ij}} = e \text{ if } i \neq j, \text{ where } m_{ij} \in \mathbb{Z}_{\geqslant 0}, \tag{4}$$

and the possible values of m_{ij} are 2, 3, 4 and 6. One can show that relations (4) generate all relations, i.e. W is a ***Coxeter group***, defined as a group on generators s_i $(i = 1, \cdots, r)$, and relations (4). This property of the Weyl group holds for any Kac-Moody Lie algebra, with the possible values of m_{ij} being 0, 2, 3, 4, and 6. Such a Coxeter group is called crystallographic: in some basis the matrices of the generators s_i have integer entries.

Lecture 22

The Universal Enveloping Algebra of a Lie Algebra and the Casimir Element

Throughout the lecture $\mathfrak{g}$ will denote a Lie algebra, not necessarily finite-dimensional, over any field $\mathbb{F}$.

Definition 22.1. An ***enveloping algebra*** of $\mathfrak{g}$ is a pair (φ, U), where U is a unital associative algebra and $\varphi : \mathfrak{g} \to U_-$ is a Lie algebra homomorphism. (recall that U_- is a Lie algebra structure on U with bracket $[a, b] = ab - ba$).

Example 22.1. If $\varphi : \mathfrak{g} \to \text{End } V$ is a representation of $\mathfrak{g}$, then $(\varphi, \text{End } V)$ is an enveloping algebra of $\mathfrak{g}$.

Definition 22.2. The ***universal enveloping algebra*** of $\mathfrak{g}$ is an enveloping algebra $(\Phi, U(\mathfrak{g}))$ which has the universal mapping property, namely, for any enveloping algebra (φ, U) of $\mathfrak{g}$ there exists a unique associative algebra homomorphism $f : U(\mathfrak{g}) \to U$, such that $\varphi = f \circ \Phi$. This condition can be rephrased as saying that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi} & U \\ {\scriptstyle\Phi}\downarrow & \nearrow {\scriptstyle f} & \\ U(\mathfrak{g}) & & \end{array}$$

Exercise 22.1. Prove that the universal enveloping algebra is unique, up to unique isomorphism, if such an algebra exists.

In order to prove existence, recall that a ***tensor algebra*** $T(V)$ over a vector space V over $\mathbb{F}$ is the following vector space over $\mathbb{F}$:

$$T(V) = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots ,$$

with the concatenation product. This is a unital associative algebra.

Let $J(\mathfrak{g})$ be the two-sided ideal in the algebra $T(\mathfrak{g})$, generated by the set

$$\{a \otimes b - b \otimes a - [a, b] \mid a, b \in \mathfrak{g}\} .$$

Let $U(\mathfrak{g}) = T(\mathfrak{g})/J(\mathfrak{g})$, and define the map $\Phi : \mathfrak{g} \to U(\mathfrak{g})$ by letting

$$\Phi(a) = a \in \mathfrak{g} \subset T(\mathfrak{g}) \mod J(\mathfrak{g}).$$

Exercise 22.2. Show that $(\Phi, U(\mathfrak{g}))$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

The basic result on the universal enveloping algebras is the following Poincare-Birkhoff-Witt (PBW) theorem.

PBW Theorem. Choose a basis of the Lie algebra $\mathfrak{g}$, $\{a_i\}_{i\in I}$, indexed by an ordered set I. Then the monomials

$$1, a_{i_1}a_{i_2}\cdots a_{i_k} \quad \text{with } i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \tag{1}$$

form a basis of $U(\mathfrak{g})$.

Proof. We need to show that the monomials (1) span $U(\mathfrak{g})$, which is the easy part, and that these monomials are linearly independent, which is the hard part.

For both parts introduce the following (partial) ordering on the set of all monomials $A = a_{i_1}\cdots a_{i_k}$: $1 < a_i < a_j$ if $i < j$, and $A < A' = a_{j_1}\cdots a_{j_s}$ if either $k < s$, or $k = s$ but the number of inversions in A is strictly smaller than in A'. Here by an inversion in the monomial A we mean a pair (a_{i_m}, a_{i_n}) such that $i_m, i_n \in I, i_m < i_n$, but $a_{i_m} > a_{i_n}$. Both parts are proved by induction on this ordering.

Easy part. Let the monomial A have an inversion:

$$A = A_1abA_2$$

where A_1 and A_2 are some monomials, and a and b are elements of the basis of $\mathfrak{g}$, such that $a > b$. Then, replacing ab by $ba + [a, b]$, we obtain

$$A = A_1baA_2 + A_1[a, b]A_2. \tag{2}$$

In the first summand the number of inversions is smaller than in A, and the second summand has degree smaller than the degree of A. In both cases we may apply induction on our ordering to write both summands as a linear combination of monomials from (1).

Hard part. Let B be a vector space over $\mathbb{F}$ with basis

$$1, b_{i_1}\cdots b_{i_k} \quad \text{with } i_1 \leqslant \cdots \leqslant i_k \text{ in } I. \tag{3}$$

Define the linear map $\sigma : T(\mathfrak{g}) \to B$ by (we shall skip the $\otimes$ signs):

(i) $\sigma(1) = 1, \sigma(a_{i_1}\cdots a_{i_k}) = b_{i_1}\cdots b_{i_k}$ if $i_1 \leqslant \cdots \leqslant i_k$,

(ii) if $i_{s+1} < i_s$, let inductively for $A = A_1 a_{i_s} a_{i_{s+1}} A_2$ (cf. (2)):

$$\sigma(A) = \sigma(A_1 a_{i_{s+1}} a_{i_s} A_2) + \sigma(A_1 [a_{i_s}, a_{i_{s+1}}] A_2).$$

By induction on our ordering, we can use (ii) to reduce $\sigma(a_{i_1} \cdots a_{i_k})$ to a linear combination of σ, applied to the ordered monomials (1) and use (i). The difficulty is to show that σ is well-defined, i.e. independent of the order, in which the inversions are resolved.

Case 1. We swap the order of two inversions which don't overlap. Namely, consider the monomial

$$A = A_1 ab A_2 cd A_3,$$

where A_1, A_2, A_3 are some monomials and a, b, c, d are basis elements of $\mathfrak{g}$, such that $a > b$ and $c > d$. Then

$$\sigma(A) = A_1 ba A_2 cd A_3 + A_1 [a, b] A_2 cd A_3 \tag{4}$$

or

$$\sigma(A) = A_1 ab A_2 dc A_3 + A_1 ab A_2 [c, d] A_3. \tag{5}$$

By the inductive assumption, the map σ is well-defined on each summand of (4) and (5) so it suffices to show that there exists further reduction of (4) and (5) that gives the same result. Indeed, if we replace ab by $ba + [a, b]$ in (4), and cd by $dc + [c, d]$ in (5), we obtain the same result.

Note that in this case we didn't use axioms of a Lie algebra at all.

Case 2. We swap the order of two inversions which overlap. Namely, consider the monomial

$$A = A_1 cba A_2,$$

where a, b, c are elements of the basis of $\mathfrak{g}$, such that $c > b > a$. We have two reductions:

$$\sigma(A) = A_1 bca A_2 + A_1 [c, b] a A_2, \tag{6}$$

or

$$\sigma(A) = A_1 cab A_2 + A_1 c [b, a] A_2. \tag{7}$$

Hence, by the inductive assumption, the map σ is well-defined on each summand of (6) and (7), so it suffices to show that there exist further reductions of (6) and (7) that give the same result.

The further reductions of (6) and (7) ignoring the irrelevant factors A_1 and A_2, are as follows, respectively:

$$bca + [c,b]a \to bac + b[c,a] + [c,b]a \to abc + [b,a]c + b[c,a] + [c,b]a \tag{8}$$
$$cab + c[b,a] \to acb + [c,a]b + c[b,a] \to abc + a[c,b] + [c,a]b + c[b,a] \tag{9}$$

Consider the summand $[b,a]c$ in the last reduction in (8). The element $[b,a]$ is a linear combination of some elements of the basis $\{a_i\}_{i\in I}$. If such an element, say d, is less than c, we keep dc unchanged in $[b,a]c$; otherwise we replace dc by $cd + [d,c]$, so that $[b,a]c$ gets replaced by $cd + [d,c]$. We do the same with the term $c[b,a]$ in (9).

As a result the difference of the reductions of this term in (8) and the corresponding term $c[b,a]$ in (9) becomes $[[b,a],c]$.

Indeed, if a non-zero multiple of a basis element in $[b,a]$, which we denote by $[b,a]_i$, is smaller than c, we keep $[b,a]_ic$ unchanged; otherwise we replace it by $[[b,a]_i,c] + c[b,a]_i$ in (8). We reduce similarly the term $c[b,a]$ in (9): if $[b,a]_i < c$, we replace $c[b,a]_i$ by $[c,[b,a]_i] + [b,a]_ic$, and if $[b,a]_i > c$, we keep $c[b,a]_i$ unchanged.

The difference between these two reductions is $-[c,[b,a]_i]$ in the first case and $[[b,a]_i,c]$ in the second, which is the same due to skew-commutativity in $\mathfrak{g}$.

We perform similar reductions for the remaining two pairs:

- $b[c,a]$ in (8) and $[c,a]b$ in (9), and
- $[c,b]a$ in (8) and $a[c,b]$ in (9).

As a result, we obtain that the two reductions differ by

$$[[b,a],c] + [b,[c,a]] + [[c,b],a],$$

which is 0 by the Jacobi identity in $\mathfrak{g}$ since $[b,[c,a]] = [[a,c],b]$ by the skew-commutativity in $\mathfrak{g}$.

Next, we need to show that $J(\mathfrak{g})$ lies in $\ker\sigma$. But $J(\mathfrak{g})$ is an ideal of $T(\mathfrak{g})$, generated by the elements

$$ab - ba - [a,b], \quad \text{where } a < b,\ a,b \in \mathfrak{g}, \tag{10}$$

since $ba - ab - [b.a] =$ (10) with the $-$ sign.

Applying σ to (10), we obtain $ab - ([b,a] + ab) - [a,b] = 0$ due to skew-commutativity in $\mathfrak{g}$.

Hence σ induces a linear map: $U(\mathfrak{g}) \to B$, which is surjective, and since monomials (3) form a basis of B, monomials (1) form a basis of $U(\mathfrak{g})$. ■

Now, that we have an understanding of $U(\mathfrak{g})$, we define a very important element of this algebra. For that we need

Exercise 22.3. Let V be a d-dimensional vector space over $\mathbb{F}$, and let $\{a_1, \cdots, a_d\}$ be a basis of V. Let $(\bullet, \bullet)$ be a non-degenerate bilinear form of V. Then there exists a unique ***dual basis*** $\{b_1, \cdots, b_d\}$ of V, so that $(a_i, b_j) = \delta_{ij}$ for $i, j = 1, \cdots, d$.

Definition 22.3. Let $\mathfrak{g}$ be a d-dimensional Lie algebra over $\mathbb{F}$, equipped with a non-degenerate invariant symmetric bilinear form $(\bullet, \bullet)$. Choose a basis $\{a_1, \cdots, a_d\}$ of $\mathfrak{g}$, and let $\{b_1, \cdots, b_d\}$ be the dual basis. Then the ***Casimir element*** $\Omega \in U(\mathfrak{g})$ is defined as $\Omega = \sum_{i=1}^{d} a_i b_i$.

Exercise 22.4. Show that Ω is independent of the choice of the basis $\{a_1, \cdots, a_d\}$ of $\mathfrak{g}$. (Hint: Write a change of basis as $(a'_1 \cdots a'_d) = (a_1 \cdots a_d)A$, where A is a non-singular $d \times d$ matrix over $\mathbb{F}$.)

Lemma 22.1. Let $\mathfrak{g}$ be a d-dimensional Lie algebra over $\mathbb{F}$ with a non-degenerate symmetric invariant bilinear form $(\bullet, \bullet)$. Let $\{a_1, \cdots, a_d\}$ be a basis of $\mathfrak{g}$, and $\{b_1, \cdots, b_d\}$ the dual basis. For $x \in \mathfrak{g}$ write

$$[x, a_i] = \sum_{k=1}^{d} \alpha_{ik} a_k, \quad [x, b_j] = \sum_{k=1}^{d} \beta_{jk} b_k.$$

Then $\alpha_{ij} = -\beta_{ji}$.

Proof. We have:

$$([x, a_i], b_j) = \Big(\sum_k \alpha_{ik} a_k, b_j\Big) = \sum_{k=1}^{d} \alpha_{ik}(a_k, b_j) = \sum_{k=1}^{d} \alpha_{ik}\delta_{kj} = \alpha_{ij}, \tag{11}$$

and similarly

$$([x, b_j], a_i) = \beta_{ji}, \tag{12}$$

Using invariance of the bilinear form $(\bullet, \bullet)$, we obtain from (11) and (12):

$$\alpha_{ij} = ([x, a_i], b_j) = (x, [a_i, b_j]) = -(x, [b_j, a_i]) = -([x, b_j], a_i) = -\beta_{ji}.$$ ■

Exercise 22.5. Using Lemma 22.1 show that the Casimir element Ω is a central element of the algebra $U(\mathfrak{g})$, i.e. $a\Omega = \Omega a$ for any $a \in U(\mathfrak{g})$.

(Hint: use the fact that in an associative algebra the operator ad a, defined by $(\text{ad } a)b = ab - ba$, is a derivation of this algebra.)

Lecture 23

Vanishing of First Cohomology, and the Weyl, Levi and Maltsev Theorems

First, note that, by the universality property of $U(\mathfrak{g})$, any representation π of the Lie algebra $\mathfrak{g}$ in a vector space V extends uniquely to an associative algebra homomorphism $U(\mathfrak{g}) \to \operatorname{End} V$ (cf. Example 22.1)

We will use the more efficient language of $\mathfrak{g}$-***modules*** instead of representations π of $\mathfrak{g}$ in V:

$$\begin{aligned} \pi(a)v = a \cdot v \text{ (or } av) \quad &\text{for } a \in \mathfrak{g}, v \in V, \text{ so that} \\ [a, b]v = abv - bav \quad &\text{for } a, b \in \mathfrak{g}, v \in V. \end{aligned}$$

Definition 23.1. A ***1-cocycle*** of a $\mathfrak{g}$-module V is a linear map $f : \mathfrak{g} \to V$, such that

$$f([a, b]) = a \cdot f(b) - b \cdot f(a) \quad \text{for } a, b \in \mathfrak{g}. \tag{1}$$

Example 23.1. For any $v \in V$ we have a trivial 1-cocyle $f_v(a) = a \cdot v$ for all $a \in \mathfrak{g}$. This is a 1-cocycle since the LHS of (1) is equal to $[a, b] \cdot v$, and its RHS is $abv - bav$, which are equal by the definition of a $\mathfrak{g}$-module.

Definition 23.2. Denote by $Z^1(\mathfrak{g}, V)$ the vector space of all 1-cocycles of the $\mathfrak{g}$-module V, and by $B^1(\mathfrak{g}, V)$ the subspace of trivial ones. The ***first cohomology of*** $\mathfrak{g}$ ***with coefficients in*** V is defined as the factor space

$$H^1(\mathfrak{g}, V) = Z^1(\mathfrak{g}, V)/B^1(\mathfrak{g}, V).$$

So $H^1(\mathfrak{g}, V) = 0$ means that all 1-cocycles are trivial.

The key result of this lecture is the

First cohomology vanishing theorem. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field F of characteristic 0, and let V be a finite-dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, V) = 0$.

The proof of this theorem is based on the following technical lemma, which holds over an arbitrary field.

Lemma 23.1. Let $\mathfrak{g}$ be a d-dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $(\bullet, \bullet)$. Let $\{a_i\}$ be a basis of $\mathfrak{g}$ and $\{b_i\}$ the dual basis.

Let $f \in Z^1(\mathfrak{g}, V)$ for a $\mathfrak{g}$-module V. Then for any $a \in \mathfrak{g}$ we have

$$a \sum_{j=1}^{d} a_j f(b_j) = \Omega(f(a)),$$

where $\Omega \in U(\mathfrak{g})$ is the Casimir element of $U(\mathfrak{g})$.

Exercise 23.1. Prove this lemma, using Lemma 22.1 on dual bases, and that $\sum_i a_i b_i = \sum_i b_i a_i$ (which follows from Exercise 22.4).

Corollary 23.1(cf. Exercise 22.5)**.** Ω commutes with the action of $\mathfrak{g}$ in any $\mathfrak{g}$-module V.

Proof. Take $f = f_v$, $v \in V$, then Lemma 23.1 becomes $a\Omega(v) = \Omega a(v)$. ∎

Exercise 23.2. For any $\mathfrak{g}$-modules V_1 and V_2, we have:

$$H^1(\mathfrak{g}, V_1 \oplus V_2) = H^1(\mathfrak{g}, V_1) \oplus H^1(\mathfrak{g}, V_2).$$

Proof of the 1st cohomology vanishing theorem. First, we may assume that the $\mathfrak{g}$-module V is faithful. Indeed, if its kernel $\mathfrak{g}_0$ is non-zero, it is a semisimple ideal of $\mathfrak{g}$ since $\mathfrak{g}$ is semisimple. In particular, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$. If $f \in Z^1(\mathfrak{g}, V)$, then

$$f([a, b]) = af(b) - bf(a) = 0 \text{ if } a, b \in \mathfrak{g}_0.$$

Hence $f(\mathfrak{g}_0) = f([\mathfrak{g}_0, \mathfrak{g}_0]) = 0$. This means that f induces a well-defined 1-cocycle on $\mathfrak{g}/\mathfrak{g}_0$ with values in V. Replacing $\mathfrak{g}$ by $\mathfrak{g}/\mathfrak{g}_0$, we may assume that $\mathfrak{g} \subseteq \mathfrak{gl}_V$. By Theorem 11.2, the trace form $(\bullet, \bullet)_V$ is non-degenerate on $\mathfrak{g}$, hence we may consider the corresponding Casimir element $\Omega = \sum_i a_i b_i$ for $\mathfrak{g}$, where a_i and b_i are dual bases of $\mathfrak{g}$ with respect to $(\bullet, \bullet)_V$.

We prove the theorem by induction on $\dim V$. With respect to the operator Ω we have the decomposition

$$V = V_{(0)} \oplus V' \quad \text{(direct sum of vector spaces)}, \tag{2}$$

where $V_{(0)}$ is the generalized 0-th eigenspace of Ω and $V' = \Omega^N(V)$ for sufficiently large N. (If $\mathbb{F}$ is algebraically closed, we can take $V' = \bigoplus_{\lambda \neq 0} V_{(\lambda)}$.)

By Corollary 23.1, $\mathfrak{g}$ commutes with Ω on V, hence $V_{(0)}$ and V' are submodules of the

$\mathfrak{g}$-module V. By Exercise 23.2, we have

$$H^1(\mathfrak{g}, V) = H^1(\mathfrak{g}, V_{(0)}) \oplus H^1(\mathfrak{g}, V').$$

Hence, by the inductive assumption, $H^1(\mathfrak{g}, V) = 0$ if both $V_{(0)}$ are V' are non-zero.

Thus, the proof reduces to two cases:

Case 1. Ω is invertible on V.

Case 2. Ω is a nilpotent operator on V.

In Case 1, we use Lemma 23.1, which says that for $m = \sum_{j=1}^{\dim \mathfrak{g}} a_j f(b_j)$, where $f \in Z^1(\mathfrak{g}, V)$, we have:

$$a(m) = \Omega(f(a)),$$

hence $\Omega^{-1}(am) = a\Omega^{-1}(m)$ by Corollary 23.1, and therefore $f(a) = \Omega^{-1}(am) = a(\Omega^{-1}(m))$. Hence $f = f_{\Omega^{-1}(m)}$ is a trivial 1-cocycle.

In Case 2 we have

$$\mathrm{tr}_V \Omega = \sum_{i=1}^{\dim \mathfrak{g}} \mathrm{tr}_V a_i b_i = \sum_{i=1}^{\dim \mathfrak{g}} (a_i, b_i)_V = \dim \mathfrak{g}.$$

But since Ω is a nilpotent operator, it follows that its trace is 0, hence $\mathfrak{g} = 0$. But then, of course, $H^1(\mathfrak{g}, V) = 0$. ∎

Bonus Problem. Does the converse hold, that if $H^1(\mathfrak{g}, V) = 0$ for a finite-dimensional $\mathfrak{g}$ and any finite-dimensional $\mathfrak{g}$-module V, then $\mathfrak{g}$ is a semisimple Lie algebra?

Next, we use the first cohomology vanishing theorem to prove the following three fundamental theorems, which hold over any field $\mathbb{F}$ of characteristic 0.

Weyl's Complete Reducibility Theorem. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra and let V be a finite-dimensional $\mathfrak{g}$-module. Then for any submodule $U \subseteq V$, there exists a complementary submodule U' so that $V = U \oplus U'$.

Levi's Theorem. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $R(\mathfrak{g})$ be its radical. Then there exists a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, complementary to $R(\mathfrak{g})$, so that $\mathfrak{g} = \mathfrak{s} \ltimes R(\mathfrak{g})$. (Note that $\mathfrak{s}$ is automatically semisimple)

Maltsev's Theorem. In the notation of Levi's theorem, any semisimple subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}$ is conjugate to $\mathfrak{s}$, that is there exists an automorphism σ of $\mathfrak{g}$, such that $\sigma(\mathfrak{s}_1) \subseteq \mathfrak{s}$.

The proofs of all three theorems use the first cohomology vanishing theorem, and the notion of a projector P of a vector space V onto a subspace U. Recall that a linear operator P on V is called a ***projector*** onto U if it has the following two properties:

(i) $P(V) \subseteq U$,

(ii) $P(u) = u$ for $u \in U$.

Note that $P^2 = P$ and

$$V = (\text{Ker } P) \oplus U, \tag{3}$$

since $v = (v - P(v)) + P(v)$ and $P(v - P(v)) = 0$ by $P^2 = P$.

Observe also that, given a projector $P : V \twoheadrightarrow U$, any other projector $P_1 : V \twoheadrightarrow U$ is obtained as $P_1 = P + A$ where A is an operator satisfying

$$A(V) \subseteq U \quad \text{and} \quad A(U) = 0. \tag{4}$$

Proof of Weyl's theorem. Consider End V as a (finite-dimensional) $\mathfrak{g}$-module with $a \in \mathfrak{g}$ acting as $a \cdot A = aA - Aa$. Pick an arbitrary projector $P_0 : V \twoheadrightarrow U$ and consider the following (trivial) 1-cocycle f for the $\mathfrak{g}$-module End V:

$$f(a) = a \cdot P_0 = aP_0 - P_0 a, \quad a \in \mathfrak{g}. \tag{5}$$

Let $M \subseteq \text{End } V$ be the subspace, consisting of all A such that $A(V) \subseteq U, A(U) = 0$.

Exercise 23.3.

(a) M is a submodule of the $\mathfrak{g}$-module End V.

(b) The 1-cocycle f defined by (5) is actually a 1-cocycle with values in M.

Thus, f defines a class $[f] \in H^1(\mathfrak{g}, M)$. Since $H^1(\mathfrak{g}, M) = 0$, f is a trivial cocycle, hence there exists $A \in M$, such that $f(a) = a \cdot A$ for all $a \in \mathfrak{g}$. This means that

$$aP_0 - P_0 a (= f(a)) = a \cdot A = aA - Aa, \quad \text{for all } a \in \mathfrak{g}.$$

Rearranging terms, we see that

$$a(P_0 - A) = (P_0 - A)a, \quad \text{for all } a \in \mathfrak{g}.$$

Thus $P := P_0 - A$ is a projector of V onto U which commutes with each operator from $\mathfrak{g}$.

Hence $\ker P$ is $\mathfrak{g}$-invariant, and so

$$V = U \oplus \ker P$$

is the desired decomposition of V in a direct sum of $\mathfrak{g}$-submodules. ∎

Proof of Levi's theorem. We proceed by induction on $\dim \mathfrak{g}$. First, we reduce to the case when the radical $R(\mathfrak{g})$ is abelian. If not then $\widetilde{\mathfrak{g}} = \mathfrak{g}/[R(\mathfrak{g}), R(\mathfrak{g})]$ has smaller dimension than $\mathfrak{g}$, and so, by the inductive assumption, $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{s}} \ltimes R(\widetilde{\mathfrak{g}})$, where $\widetilde{\mathfrak{s}}$ is a complementary subalgebra. Hence $\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g})$, where $\mathfrak{g}_1$ is the preimage of $\widetilde{\mathfrak{s}}$ in $\mathfrak{g}$. Obviously, $\dim \mathfrak{g}_1 < \dim \mathfrak{g}$, and applying the inductive assumption to $\mathfrak{g}_1$, we obtain $\mathfrak{g}_1 = \mathfrak{s} \ltimes R(\mathfrak{g}_1)$. Hence $\mathfrak{g} = \mathfrak{s} \ltimes (R(\mathfrak{g}_1) + R(\mathfrak{g}))$.

Exercise 23.4. Prove that $R(\mathfrak{g}_1) + R(\mathfrak{g})$ is a solvable ideal of $\mathfrak{g}$ (hence actually $R(\mathfrak{g}_1) \subseteq R(\mathfrak{g})$).

It remains to prove the theorem in the case $R(\mathfrak{g})$ is abelian.

Consider the following $\mathfrak{g}$-module structure on End $\mathfrak{g}$:

$$a \cdot m = (\text{ad } a)m - m(\text{ad } a) \quad \text{for } a \in \mathfrak{g}, m \in \text{End } \mathfrak{g}.$$

Further, consider the following $\mathfrak{g}$-submodule of End $\mathfrak{g}$:

$$\widetilde{M} = \{m \in \text{End } \mathfrak{g} \mid m(\mathfrak{g}) \subseteq R(\mathfrak{g});\ m(R(\mathfrak{g})) = 0\}.$$

It is trivial to see that $\widetilde{M}$ is a submodule. (In that, it is a particular case of the submodule M, constructed in the proof of Weyl's theorem with $V = \mathfrak{g}, U = R(\mathfrak{g})$, and $\mathfrak{g}$ acting by the adjoint action.)

Consider also the submodule of the $\mathfrak{g}$-module $\widetilde{M}$:

$$\widetilde{R} = \{\text{ad } a \mid a \in R(\mathfrak{g})\}.$$

It is a submodule since $R(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$. Hence $M = \widetilde{M}/\widetilde{R}$ is a $\mathfrak{g}$-module.

We now see that $R(\mathfrak{g})$ acts trivially on M. Indeed, if $r \in R(\mathfrak{g})$ and $m \in \widetilde{M}$, then for any $b \in \mathfrak{g}$ we have

$$(r \cdot m)(b) = (\text{ad } r)m(b) - m(\text{ad } r)(b) = [r, m(b)] - m[r, b].$$

Since $m(b) \in R(\mathfrak{g})$, which is abelian, we see that $[r, m(b)] = 0$. Since $R(\mathfrak{g})$ is an ideal and $m \cdot R(\mathfrak{g}) = 0$, we see that $m \cdot [r, b] = 0$. Hence $r \cdot m = 0$, as was claimed, and M is actually a module over the semisimple Lie algebra $\mathfrak{s} = \mathfrak{g}/R(\mathfrak{g})$.

Now let $P_0 : \mathfrak{g} \twoheadrightarrow R(\mathfrak{g})$ be an arbitrary projector, and consider the following 1-cocycle

$f : \mathfrak{s} \to M$,

$$f(a) = (\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a}),$$

where $\tilde{a}$ is any preimage of $a \in \mathfrak{s}$ under the canonical map $\mathfrak{g} \to \mathfrak{g}/R(\mathfrak{g}) = \mathfrak{s}$.

Exercise 23.5. Check that $f : \mathfrak{s} \to M$ is a well-defined 1-cocycle.

Thus f defines a cohomology class $[f] \in H^1(\mathfrak{s}, M) = 0$, hence we can find $m \in M$, such that

$$a \cdot m = (\text{ad } \tilde{a})P_0 - P_0(\text{ad } \tilde{a}), \qquad a \in \mathfrak{s}.$$

This just means that

$$(\text{ad } \tilde{a})(P_0 - \tilde{m}) - (P_0 - \tilde{m})(\text{ad } \tilde{a}) = \text{ad } r_a, \tag{6}$$

where $\tilde{m} \in \widetilde{M}$ is a lift of m under the canonical map $\widetilde{M} \to M$, and $r_a \in R(\mathfrak{g})$ is an element, depending on $a \in \mathfrak{s}$.

Now, consider the projector $P = P_0 - \tilde{m} : \mathfrak{g} \twoheadrightarrow R(\mathfrak{g})$.

Case 1. All the $r_a = 0$ in (6). Then all ad $\tilde{a}$ commute with the projector P, and $\mathfrak{g} = (\ker P) \oplus R(\mathfrak{g})$ is a direct sum of ideals, and, in particular, $\ker P$ is a complementary subalgebra to $\mathfrak{g}$.

Case 2. $r_a \neq 0$ for some $a \in \mathfrak{g}$. Let $\mathfrak{g}_1 = \{a \in \mathfrak{g} \mid [P, \text{ad } a] = 0\}$. Then $\mathfrak{g}_1$ is a proper subalgebra of $\mathfrak{g}$. Since $R(\mathfrak{g})$ is an abelian ideal of $\mathfrak{g}$, we have

$$((\text{ad } r)P - P(\text{ad } r))(b) = [r, P(b)] - P([r, b]) = 0 - [r, b] = -(\text{ad } r)(b)$$

for any $r \in R(\mathfrak{g})$ and $b \in \mathfrak{g}$. Thus we may write

$$(\text{ad } a)P - P(\text{ad } a) = \text{ad } r_a = P(\text{ad } r_a) - (\text{ad } r_a)P,$$

or $P(\text{ad}(a + r_a)) = (\text{ad}(a + r_a))P$.

That is, any element a of $\mathfrak{g}$ differs from an element of $\mathfrak{g}_1$ by an element of $R(\mathfrak{g})$, i.e.

$$\mathfrak{g} = \mathfrak{g}_1 + R(\mathfrak{g}) = \mathfrak{s} + (R(\mathfrak{g}_1) + R(\mathfrak{g})),$$

where we have the decomposition $\mathfrak{g}_1 = \mathfrak{s} + R(\mathfrak{g}_1)$ by the inductive assumption. But by Exercise 23.4, the ideal $R(\mathfrak{g}_1) + R(\mathfrak{g})$ is solvable, hence coincides with $R(\mathfrak{g})$, since $R(\mathfrak{g})$ is the maximal solvable ideal. Hence $\mathfrak{g} = \mathfrak{s} + R(\mathfrak{g})$. ∎

Proof of Maltsev's theorem. By Levi's theorem, $\mathfrak{g} = \mathfrak{s} + R(\mathfrak{g})$. Suppose that $\mathfrak{s}_1$ is a semisimple subalgebra of $\mathfrak{g}$. We need to construct an automorphism σ of $\mathfrak{g}$, such that $\sigma(\mathfrak{s}_1) \subseteq \mathfrak{s}$.

First, consider the case when $R(\mathfrak{g})$ is abelian. Let $P_\mathfrak{s}$ and P_R be the projections of $\mathfrak{g}$ onto $\mathfrak{s}$ and $R(\mathfrak{g})$ respectively with respect to the decomposition $\mathfrak{g} = \mathfrak{s} \oplus R(\mathfrak{g})$ (direct sum of vector spaces). Since $R(\mathfrak{g})$ is a ideal, it is easy to see that P_s is a homomorphism of Lie algebras. Also note that $P_s + P_R = I_\mathfrak{g}$, the identity map. Consider the map $f : \mathfrak{s}_1 \twoheadrightarrow R(\mathfrak{g})$, defined by

$$f(a) = P_R(a). \tag{7}$$

Note that $R(\mathfrak{g})$ is an $\mathfrak{s}_1$-module by the adjoint action, and we claim that f is a 1-cocycle of this $\mathfrak{s}_1$-module, i.e. $f \in Z^1(\mathfrak{s}_1, R(\mathfrak{g}))$.

Exercise 23.6. Prove that claim.

Thus f defines a class $[f] \in H^1(\mathfrak{s}_1, R(\mathfrak{g}))$, which is trivial by the vanishing of the first cohomology of the module $R(\mathfrak{g})$ over the semisimple Lie algebra $\mathfrak{s}_1$. Hence there exists an element $r \in R(\mathfrak{g})$, such that $P_R(a) = a \cdot r = [a, r]$ for all $a \in \mathfrak{s}_1$. Hence

$$P_R = -\text{ad } r \qquad \text{on } \mathfrak{s}_1. \tag{8}$$

Now consider the automorphism $\sigma = e^{\text{ad } r}$. Since $R(\mathfrak{g})$ is an abelian ideal, $(\text{ad } r)^2 = 0$, so that $\sigma = I_\mathfrak{g} + \text{ad } r$. Hence, if $a \in \mathfrak{s}_1$, we have by (8):

$$\sigma(a) = (I_\mathfrak{g} + \text{ad } r)(a) = (I_\mathfrak{g} - P_R)(a) = P_\mathfrak{s}(a) \in \mathfrak{s}.$$

Hence $\sigma(\mathfrak{s}_1) \subseteq \mathfrak{s}$, as desired. This finishes the case when $R(\mathfrak{g})$ is abelian.

The general case uses this one by the following exercise.

Exercise 23.7. Let $N(\mathfrak{g})$ be the subalgebra $R(\mathfrak{g})$, consisting of nilpotent elements

(a) Using Proposition 5.2, show that

$$\mathfrak{s}_1 (= [\mathfrak{s}_1, \mathfrak{s}_1]) \subseteq \mathfrak{s} + N(\mathfrak{g}).$$

(b) Prove by induction on k that there exists $r_k \in N(\mathfrak{g})$, such that $e^{\text{ad } r_k}(\mathfrak{s}_1) \subseteq \mathfrak{s} + N(\mathfrak{g})^{(k)}$.

Since $N(\mathfrak{g})^{(k)} = 0$ for sufficiently large k, this completes the proof of Maltsev's theorem.

Lecture 24

Height Weight Modules over a Semisimple Lie Algebra

In the remainder of the course $\mathbb{F}$ is an algebraically closed field of characteristic 0, and $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra over $\mathbb{F}$.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and consider the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right),$$

where $\Delta \subset \mathfrak{h}^*$ is the set of roots, and

$$\mathfrak{g}_\alpha = \{a \in \mathfrak{g} \mid [h, a] = \alpha(h)a \text{ or all } h \in \mathfrak{h}\}$$

is the root space attached to $\alpha \in \Delta$. Recall that $\dim \mathfrak{g}_\alpha = 1$ and choose a non-zero vector $E_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in \Delta$.

Choose a subset of positive roots $\Delta_+ \subset \Delta$, and let $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\pm\alpha}$. Then we have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad \text{(direct sum of vector spaces)}, \tag{1}$$

where $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are subalgebras of $\mathfrak{g}$ with bases $\{E_\alpha\}_{\alpha \in \Delta_+ (\text{resp. } -\Delta_+)}$. Let $\Delta_+ = \{\beta_1, \cdots, \beta_N\}$ be the set of all positive roots, then $-\Delta_+ = \{-\beta_1, \cdots, -\beta_N\}$, and let $\{H_1, \cdots, H_r\}$ be a basis of $\mathfrak{h}$. Then the set

$$\{E_{-\alpha}\}_{\alpha \in \Delta_+} \cup \{H_i\}_{i=1}^r \cup \{E_\alpha\}_{\alpha \in \Delta_+} \tag{2}$$

is an ordered basis of $\mathfrak{g}$. Hence, by the PBW theorem, the monomials

$$E_{-\beta_1}^{m_1} \cdots E_{-\beta_N}^{m_N} H_1^{s_1} \cdots H_r^{s_r} E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} \quad \text{with } m_i, k_i, s_j \in \mathbb{Z}_{\geqslant 0}$$

form a basis of the universal enveloping algebra $U(\mathfrak{g})$, while the monomials

$$E_{-\beta_1}^{m_1} \cdots E_{-\beta_N}^{m_N} \quad (\text{resp. } E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}) \tag{3}$$

form a basis of $U(\mathfrak{n}_-)$ (resp. $U(\mathfrak{n}_+)$). It follows that

$$U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \quad \text{(tensor product of vector spaces)} \tag{4}$$

The subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}_+$ is called a ***Borel*** subalgebra of $\mathfrak{g}$.

Exercise 24.1. Show that $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are maximal nilpotent subalgebras of $\mathfrak{g}$ and $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$, and that $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$.

Since, by Weyl's complete reducibility theorem, any finite-dimensional $\mathfrak{g}$-module decomposes into a direct sum of irreducible $\mathfrak{g}$-modules, it suffices to study finite-dimensional irreducible $\mathfrak{g}$-modules in order to understand all the finite-dimensional ones.

Proposition 24.1. Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. Then there exists $\Lambda \in \mathfrak{h}^*$ and a non-zero vector $v_\Lambda \in V$ with the following three properties:

(i) $hv_\Lambda = \Lambda(h)v_\Lambda$ for $h \in \mathfrak{h}$,

(ii) $\mathfrak{n}_+ v_\Lambda = 0$,

(iii) $U(\mathfrak{g})v_\Lambda = V$.

It follows from (4) and (i) that (iii) is equivalent to

(iii′) $U(\mathfrak{n}_-)v_\Lambda = V$.

Proof. By Lie's theorem, the solvable Lie algebra $\mathfrak{b}$ has a non-zero vector $v \in V$, such that

$$\mathfrak{b}v = \Lambda(\mathfrak{b})v \qquad \text{for some } \Lambda \in \mathfrak{b}^*.$$

But $\Lambda([\mathfrak{b}, \mathfrak{b}]) = 0$ since $\mathbb{F}v$ is a 1-dimensional $\mathfrak{b}$-module, hence, by Exercise 24.1, $\Lambda(\mathfrak{n}_+) = 0$. Hence (i) and (ii) hold for $v_\Lambda = v$. Property (iii) follows from the irreducibility of the $\mathfrak{g}$-module V since $U(\mathfrak{g})v$ is a non-zero submodule of V.

Definition 24.1. A $\mathfrak{g}$-module V (not necessarily finite-dimensional), satisfying the three properties (i)-(iii) from Proposition 24.1, is called a ***highest weight*** module with ***highest weight*** Λ, and v_Λ is called a highest weight vector.

Definition 24.2. For an arbitrary $\mathfrak{g}$-module V, for $\lambda \in \mathfrak{h}^*$, we denote by $V_\lambda = \{v \in V \mid hv = \lambda(h)v, \text{ all } h \in \mathfrak{h}\}$ the weight space of $\mathfrak{h}$ in V, attached to λ. A non-zero vector $v \in V_\lambda$ is called ***singular*** of weight λ if $\mathfrak{n}_+ v = 0$; if such a vector exists, then λ is called a ***singular weight*** of the $\mathfrak{g}$-module V.

Example 24.1. Any $\Lambda \in \mathfrak{h}^*$ is a singular weight of a highest weight $\mathfrak{g}$-module with highest weight Λ.

Notation. Given $\Lambda \in \mathfrak{h}^*$, let $D(\Lambda) = \left\{\Lambda - \sum_{\alpha \in \Delta_+} k_\alpha \alpha \mid k_\alpha \in \mathbb{Z}_{\geqslant 0}\right\}$.

Theorem on Highest Weight Modules. Let V be a highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$. Then

(a) $V = \bigoplus_{\lambda \in D(\Lambda)} V_\lambda$.

(b) $V_\Lambda = \mathbb{F} v_\Lambda$ and $\dim V_\lambda < \infty$.

(c) V is an irreducible $\mathfrak{g}$-module if and only if Λ is the only singular weight of V.

(d) V contains a unique proper maximal submodule.

(e) If v is a singular vector with weight λ, then

$$\Omega(v) = (\lambda + 2\rho, \lambda)v,$$

where $(\bullet, \bullet)$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, Ω is the corresponding Casimir element, and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha.$$

(f) $\Omega\mid_V = (\Lambda + 2\rho, \Lambda) I_V$.

(g) If λ is a singular weight of V, then

$$(\lambda + \rho, \lambda + \rho) = (\Lambda + \rho, \Lambda + \rho).$$

(h) If $\Lambda \in \mathfrak{h}^*_{\mathbb{Q}}$, then the number of singular weights of the $\mathfrak{g}$-module V is finite.

Proof. By property (iii′) of a highest weight module,

$$V = U(\mathfrak{n}_-)v_\Lambda = \sum \mathbb{F} E_{-\beta_1}^{m_1} \cdots E_{-\beta_N}^{m_N} v_\Lambda.$$

But the weight of $E_{-\beta_1}^{m_1} \cdots E_{-\beta_N}^{m_N} v_\Lambda$ is $\Lambda - \sum_{i=1}^N m_i \beta_i \in D(\Lambda)$, proving (a) and (b).

In order to prove (c) and (d), note that any $\mathfrak{g}$-submodule U of V we have, by Lemma 14.1 and (a),

$$U = \bigoplus_{\lambda \in D(\Lambda)} (U \cap V_\lambda). \tag{5}$$

Choose $\lambda = \Lambda - \sum_{\alpha \in \Delta_+} k_\alpha \alpha \in D(\Lambda)$ with minimal $\sum_{\alpha \in \Delta_+} k_\alpha$ height α, such that $U \cap V_\lambda \neq 0$, and let v be a non-zero vector there. Then $E_\alpha v = 0$ for all $\alpha \in \Delta_+$. Hence v is a singular vector, and therefore

$$U(\mathfrak{g})v = U(\mathfrak{n}_-)v,$$

which is a proper submodule, unless $\lambda = \Lambda$, proving (c).

The sum of all proper submodules of V is again a proper submodule because it doesn't contain v_Λ by (5). Then, this sum is a unique (proper) maximal submodule, proving (d).

In order to prove (e), consider the basis (2) of $\mathfrak{g}$, where we choose E_{β_j} and $E_{-\beta_j}$, such that $(E_{\beta_j}, E_{-\beta_j}) = 1$. Then the basis of $\mathfrak{g}$, dual to (2), is

$$\{E_{-\alpha}\}_{\alpha\in\Delta_+} \cup \left\{H^i\right\}_{i=1}^r \cup \{E_\alpha\}_{\alpha\in\Delta_+},$$

where $\{H^i\}$ is the basis of $\mathfrak{h}$, dual to $\{H_i\}$.

Then, by the definition, we obtain the following formula for the Casimir element:

$$\Omega = \sum_{i=1}^r H_i H^i + \sum_{j=1}^N E_{\beta_j} E_{-\beta_j} + \sum_{j=1}^N E_{-\beta_j} E_{\beta_j}.$$

Using that $[E_\alpha, E_{-\alpha}] = \nu^{-1}(\alpha)$ if $(E_\alpha, E_{-\alpha}) = 1$, we can rewrite this as follows

$$\Omega = 2\sum_{j=1}^N E_{-\beta_j} E_{\beta_j} + \sum_{i=1}^r H_i H^i + \sum_{j=1}^N \nu^{-1}(\beta_j).$$

Since the last sum is $2\nu^{-1}(\rho)$, applying this to a singular vector v_λ, we obtain

$$\Omega v_\lambda = \left(\sum_{i=1}^r \lambda(H_i)\lambda(H^i)v_\lambda + 2(\rho,\lambda)\right) v_\lambda.$$

Since $\sum_{i=1}^r \lambda(H_i)\lambda(H^i) = (\lambda,\lambda)$, (e) is proved.

Here we used

Exercise 24.2. Let V be a finite-dimensional vector space with a non-degenerate symmetric bilinear form $(\bullet,\bullet)$. Then it induces a bilinear form $(\bullet,\bullet)$ on V^*, and for dual bases $\{a_i\}$ and $\{b_i\}$ of V we have

$$(\lambda,\lambda) = \sum_{i=1}^{\dim V} \lambda(a_i)\lambda(b_i) \qquad \text{for any } \lambda \in V^*.$$

End of the proof of the Theorem. Claim (f) follows from (e) since Ω commutes with $U(\mathfrak{g})$ on

V, hence

$$\Omega\left(E_{-\beta_1}^{m_1}\cdots E_{-\beta_N}^{m_N}v_\Lambda\right) = E_{-\beta_1}^{m_1}\cdots E_{-\beta_N}^{m_N}\Omega\left(v_\Lambda\right),$$

and we can use (e).

Claim (g) follows from (f) and (e). Finally, by (g), the set of singular weights lies in a compact set in $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\otimes_{\mathbb{Q}}\mathfrak{h}_{\mathbb{Q}}^*$, and since it also lies in a discrete set $D(\Lambda)$, their intersection is finite. ■

Exercise 24.3. Let V be a $\mathfrak{g}$-module, and $v \in V$ be such that $U(\mathfrak{g})v = V$ (then v is called a ***cyclic*** vector of the $\mathfrak{g}$-module V). Let $\operatorname{Ann} v = \{a \in U(\mathfrak{g}) \mid av = 0\}$. This is a left ideal of the associative algebra $U(\mathfrak{g})$, and V is isomorphic to $U(\mathfrak{g})/\operatorname{Ann} v$ (with $\mathfrak{g}$ acting by multiplication on the left), as $\mathfrak{g}$-modules. Show that if V' is another $\mathfrak{g}$-module with cyclic vector v', and $\operatorname{Ann} v' \supseteq \operatorname{Ann} v$, then the $\mathfrak{g}$-module V' is isomorphic to a factor module of V.

Definition 24.3. Given $\Lambda \in \mathfrak{h}^*$, a ***Verma module*** $M(\Lambda)$ is a highest weight $\mathfrak{g}$-module with highest weight Λ, such that any other highest weight module with highest weight Λ is a factor module of $M(\Lambda)$.

Proposition 24.2.

(a) For any $\Lambda \in \mathfrak{h}^*$, there exists a Verma module $M(\Lambda)$, unique up to isomorphism.

(b) $M(\Lambda)$ has a unique proper maximal submodule $J(\Lambda)$, so that the factor-module $L(\Lambda) = M(\Lambda)/J(\Lambda)$ is irreducible.

(c) The $\mathfrak{g}$-modules $M(\Lambda)$ and $M(\Lambda')$ (resp. $L(\Lambda)$ and $L(\Lambda')$) are isomorphic if and only if $\Lambda = \Lambda'$.

(d) The vectors $E_{-\beta_1}^{m_1}\cdots E_{-\beta_N}^{m_N}v_\Lambda$ with $m_1, \cdots, m_N \in \mathbb{Z}_{\geqslant 0}$ form a basis of $M(\Lambda)$.

Proof. (a) It is clear from the definition and Exercise 24.2 that

$$M(\Lambda) = U(\mathfrak{g})/U(\mathfrak{g})(\mathfrak{n}_+ + \{h - \Lambda(h) \text{ with } h \in \mathfrak{h}\})$$

with $\mathfrak{g}$ acting by multiplication on the left.

(b) follows from the Theorem on highest weight modules (d), and (c) follows from (b).

In order to prove (d), note that by (3) and the PBW theorem we have

$$U(\mathfrak{n}_-) \cap U(\mathfrak{g})(\mathfrak{n}_+ + \{h - \Lambda(h) \text{ with } h \in \mathfrak{h}\}) = 0.$$

Hence the map $U(\mathfrak{n}_-) \to M(\Lambda)$, defined by $u \mapsto u \cdot v_\Lambda$ is injective, but by (iii′) it is surjective. Hence $M(\Lambda)$ is a free module of rank 1 over $U(\mathfrak{n}_-)$, and (d) follows from the PBW theorem for $\mathfrak{n}_-$. ■

Exercise 24.4. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}F + \mathbb{F}H + \mathbb{F}E$, so that $\mathfrak{h} = \mathbb{F}H, \mathfrak{n}_- = \mathbb{F}F, \mathfrak{n}_+ = \mathbb{F}E$. Then $\Lambda \in \mathfrak{h}^*$ is identified with $\Lambda(H) \in \mathbb{F}$, and $M(\Lambda) = \operatorname{span}\{F^n v_\Lambda \mid n \in \mathbb{Z}_{\geqslant 0}\}$, on which $\mathfrak{sl}_2(\mathbb{F})$ acts according to the key $\mathfrak{sl}_2$ lemma. This module is irreducible, i.e. isomorphic to $L(\Lambda)$ if and only if $\Lambda(H) \notin \mathbb{Z}_{\geqslant 0}$, by this lemma, and, if $\Lambda(H) \in \mathbb{Z}_{\geqslant 0}$, then $J(\Lambda) = \operatorname{span}\{F^n v_\Lambda \mid n > \Lambda(H)\}$, so that $L(\Lambda) = M(\Lambda)/J(\Lambda) = \operatorname{span}\{F^n v_\Lambda \mid 0 \leqslant n \leqslant \Lambda(H)\}$ is $(\Lambda(H) + 1)$-dimensional.

Lecture 25

Dimensions and Characters of Finite-Dimensional Irreducible Modules over Semisimple Lie Algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, choose a Cartan subalgebra $\mathfrak{h}$, let $\Delta \subset \mathfrak{h}^*$ be the set of roots, and

$$\Pi = \{\alpha_1, \cdots, \alpha_r\} \subseteq \Delta_+ \subset \Delta$$

be the subsets of simple and positive roots. We have the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ be the Borel subalgebra, recall that $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}_+$. Let, as before,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \in \mathfrak{h}^*. \tag{1}$$

Fix a non-degenerate invariant symmetric bilinear form $(\bullet, \bullet)$ on $\mathfrak{g}$, and let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the corresponding vector space isomorphism; it induces a non-degenerate symmetric bilinear form $(\bullet, \bullet)$ on $\mathfrak{h}^*$, and $(\alpha, \alpha) \neq 0$ for all $\alpha \in \Delta$.

Let $\{E_i, F_i, H_i\}_{i=1}^r$ be the Chevalley generators of $\mathfrak{g}$, where $H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}, E_i \in \mathfrak{g}_{\alpha_i}, F_i \in \mathfrak{g}_{-\alpha_i}$, so that E_i, H_i, F_i form the standard basis of the subalgebra $\mathfrak{a}_i$, isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Define the set of ***dominant weights***

$$P_+ = \{\lambda \in \mathfrak{h}^* \mid \lambda(H_i) \in \mathbb{Z}_{\geqslant 0} \text{ for all } i = 1, \cdots, r\}.$$

The two main theorems of finite-dimensional representation theory of $\mathfrak{g}$ are the following.

Cartan's Theorem. The set of $\mathfrak{g}$-modules $\{L(\Lambda)\}_{\Lambda \in P_+}$ is, up to isomorphism, a complete non-redundant list of all irreducible finite-dimensional $\mathfrak{g}$-modules. (Recall that $L(\Lambda)$ is the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$.)

Weyl's Dimension Formula. If $\Lambda \in P_+$, then

$$\dim L(\Lambda) = \prod_{\alpha \in \Delta_+} \frac{(\Lambda + \rho, \alpha)}{(\rho, \alpha)}.$$

Proof of Cartan's Theorem. By Proposition 24.1, any irreducible finite-dimensional $\mathfrak{g}$-module is isomorphic to one of the $L(\Lambda), \Lambda \in \mathfrak{h}^*$. Note that $v_\Lambda \in L(\Lambda)$ satisfies $E_i v_\Lambda = 0$, $H_i v_\Lambda = \Lambda(H_i) v_\Lambda$. Hence, by the key $\mathfrak{sl}_2$ lemma, $\Lambda(H_i) \in \mathbb{Z}_{\geqslant 0}$ for all $i = 1, \cdots, r$, if $L(\Lambda)$ is finite-dimensional, and so $\Lambda \in P_+$. Conversely, if $\Lambda \in P_+$, then, by Weyl's dimensional formula, which will be proved shortly, $\dim L(\Lambda) < \infty$ (we shall see in a moment that $(\rho, \alpha) \neq 0$ for

$\alpha \in \Delta$). That $\{L(\Lambda)\}_{\Lambda \in P_+}$ is a non-redundant list follows from Proposition 24.2(c). ■

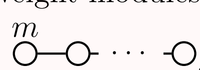
The $\mathfrak{g}$-modules $L(\Lambda)$ with $\Lambda \in P_+$ are depicted by labeled Dynkin diagrams, by writing $\Lambda = \sum_{i=1}^{r} m_i \Lambda_i$, where $\Lambda_i \in \mathfrak{h}^*$ are the ***fundamental weights***, defined by $\Lambda_i(H_j) = \delta_{ij}$, and $m_i \in \mathbb{Z}_{\geqslant 0}$, by putting m_i against the i-th node of the Dynkin diagram of $\mathfrak{g}$ if $m_i \neq 0$.

Example 25.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$, and choose its triangular decomposition such that $\mathfrak{h}$ consists of traceless diagonal matrices, and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) of strictly upper (resp. lower) triangular matrices. Then $H_i = E_{i,i} - E_{i+1,i+1}$ for $i = 1, \cdots, n-1$. Let V_m be the space of all homogeneous polynomials in $x_1, \cdots, x_n$ of degree m, and define a $\mathfrak{g}$-module structure on V_m by letting

$$(a_{ij}) \cdot P(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i \frac{\partial P(x_1, \cdots, x_n)}{\partial x_j}.$$

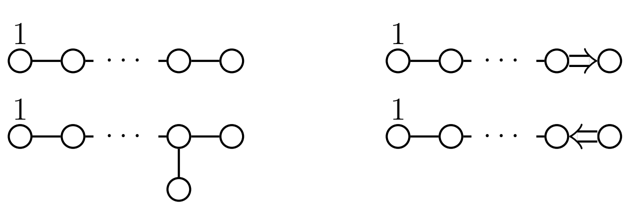
It is easy to see that this is a $\mathfrak{g}$-module. If $P(x_1, \cdots, x_n)$ is annihilated by $\mathfrak{n}_+$, i.e. by all $x_i \frac{\partial}{\partial x_j}$ with $i < j$, then $\frac{\partial P}{\partial x_j} = 0$ for all $j \geqslant 2$, hence P is a constant multiple of x_1^m. In other words, $\mathbb{F}x_1^m$ are all singular vectors of the $\mathfrak{g}$-module V_m. Also, since, up to a non-zero constant factor,

$$\left(x_n \frac{\partial}{\partial x_1}\right)^{k_n} \cdots \left(x_2 \frac{\partial}{\partial x_1}\right)^{k_1} x_1^m = x_1^{m-k_2-\cdots-k_n} x_2^{k_1} \cdots x_n^{k_n},$$

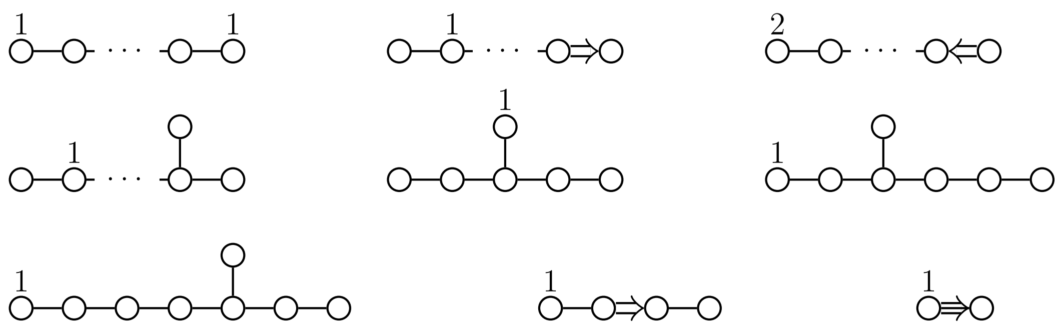
we see that $U(\mathfrak{n}_-)x_1^m = V_m$. Thus V_m is a highest weight module over $\mathfrak{g}$ with the highest weight vector x_1^m, $\mathbb{F}x_1^m$ being the only singular vectors. Since $H_i = x_i \frac{\partial}{\partial x_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}}$, we see that the weight of x_1^m is $m\Lambda_1$. Hence, by the Theorem on highest weight modules, the $\mathfrak{g}$-module V_m is isomorphic to $L(m\Lambda_1)$, and its Dynkin diagram is 

.

Exercise 25.1. Prove that for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$ with $n \geqslant 2$, $\mathfrak{so}_n(\mathbb{F})$ with $n \geqslant 7$ and $\mathfrak{sp}_n(\mathbb{F})$ with $n \geqslant 4$, the tautological module is irreducible, and is depicted by the following labeled Dynkin diagrams:



Exercise 25.2. Show that for the adjoint representation of simple Lie algebras of type A_r $(r \geqslant 2), B_r$ $(r \geqslant 3), C_r$ $(r \geqslant 1), D_r$ $(r \geqslant 4), E_6, E_7, E_8, F_4, G_2$, the labeled Dynkin diagrams are respectively



Now we turn to the proof of Weyl's dimension formula. First we take care of vector ρ.

Lemma 25.1. Recall ρ, given by formula (1). Then $\rho(H_i) = 1$ for all $i = 1, \cdots, r$.

Proof. Consider the simple reflection $s_i = r_{\alpha_i}$, and recall that the set $\Delta_i = \Delta_+ \setminus \{\alpha_i\}$ is s_i-invariant. Hence $\rho = \frac{1}{2}\alpha_i + \rho_1$, where $\rho_1 = \frac{1}{2}\sum_{\alpha \in \Delta \setminus \{\alpha_i\}} \alpha$ is fixed by s_i. So $s_i(\rho) = -\frac{1}{2}\alpha_i + \rho_1 = \rho - \alpha_i$. But $s_i(\lambda) = \lambda - \lambda(H_i)\alpha_i$, hence $\rho(H_i) = 1$. □

Corollary 25.2. $(\rho, \alpha) \neq 0$ for all $\alpha \in \Delta_+$.

Proof. We can take the normalization of the bilinear form $(\bullet, \bullet)$ such that $(\alpha, \alpha) > 0$ for all $\alpha \in \Delta$. Since $H_i = \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}$, we see that

$$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i), \qquad u = 1, \cdots, r. \tag{2}$$

Hence for $\alpha = \sum_i k_i \alpha_i \in \Delta_+, (\rho, \alpha) = \frac{1}{2}\sum_i k_i(\alpha_i, \alpha_i) > 0$. □

Exercise 25.3. For a root system (V, Δ) with Cartan matrix A, let $\Delta^\vee = \left\{\alpha^\vee = \frac{2\nu(\alpha)}{(\alpha,\alpha)} \mid \alpha \in \Delta\right\}$, where $\nu : V \to V^*$ is the vector space isomorphism induced by the bilinear form on V. Show that

(a) $(V^*, \Delta^\vee)$ is a root system (called the ***dual root system***) and its Cartan matrix is $A^\top$.

(b) Show that if $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ is a set of simple roots for (V, Δ), then $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_r^\vee\}$ is a set of simple roots for the dual root system and $\rho(\alpha_i^\vee) = 1$ for all $i = 1, \cdots, r$. Consequently,

$$\rho(\alpha^\vee) = \text{height } \alpha^\vee \qquad \text{for } \alpha^\vee \in \Delta_+^\vee.$$

(c) The Weyl dimension formula can be rewritten as a ratio of two positive integers:

$$\dim L(\Lambda) = \frac{\prod_{\alpha^\vee \in \Delta_+^\vee} (\Lambda(\alpha^\vee) + \text{height } \alpha^\vee)}{\prod_{\alpha^\vee \in \Delta_+^\vee} \text{height } \alpha^\vee}. \tag{3}$$

Example 25.2. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. By Cartan's theorem, all irreducible finite-dimensional $\mathfrak{g}$-modules are, up to isomorphism, $V_m = L(m\Lambda_1)$ which has dimension $m+1$. This is consistent with (3), whose numerator is $m+1$ and denominator is 1.

Exercise 25.4. Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{F})$ and $\Lambda = m_1\Lambda_1 + m_2\Lambda_2$ (whose Dynkin diagram is $\overset{m_1}{\circ}$—$\overset{m_2}{\circ}$ where $m_1, m_2 \in \mathbb{Z}_{\geqslant 0}$). Then $\Lambda(\alpha_i^\vee) = m_i$, $i = 1, 2$, the set of positive coroots is $\alpha_1^\vee, \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee$, and we have

$$(\Lambda + \rho)(\alpha_i^\vee) = m_i + 1, \quad i = 1, 2, \quad (\Lambda + \rho)(\alpha_1^\vee + \alpha_2^\vee) = m_1 + m_2 + 2,$$

so that the Weyl dimension formula gives

$$\dim L(\Lambda) = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}.$$

(However, to construct explicitly the $\mathfrak{sl}_3(\mathbb{F})$-modules $L(\Lambda)$ with $m_i > 0$ and $m_1 + m_2 > 2$ is a rather hard problem; recall that $\overset{1}{\circ}$—$\overset{1}{\circ}$ is the adjoint module).

Exercise 25.5. Given a finite-dimensional $\mathfrak{g}$-module V, show that V^* is a $\mathfrak{g}$-module as well, letting

$$(a \cdot f)(v) = -f(a \cdot v).$$

Show that for $\Lambda = \sum_{i=1}^n m_i\Lambda_i$, where $m_i \in \mathbb{Z}_{\geqslant 0}$, the A_n-module $L(\Lambda)^*$ is isomorphic to $L(\Lambda^*)$, where $* : P_+ \to P_+$ is defined by

$$*(\Lambda_i) = \Lambda_{n-i+1}, \quad i = 1, \cdots, n.$$

We shall deduce the Weyl dimension formula from the Weyl character formula, introduced below. For this and for the proof of the Weyl character formula we shall need the following framework.

Definition 25.1. Recall that for an abelian group V its ***group ring*** $\mathbb{Z}[V]$ is defined as the unital commutative associative ring with basis $\{e^a \mid a \in V\}$ over $\mathbb{Z}$ and product $e^a e^b = e^{a+b}$ for $a, b \in V$. We shall consider $\mathbb{Z}[V]$ with $V = \mathfrak{h}^*_{\mathbb{Q}}$, and its "completion" $\mathbb{Z}[[V]]$, consisting of all sums $\sum_{a \in V} m_a e^a$ where $m_a \in \mathbb{Z}$. This completion is a module over the ring $\mathbb{Z}[V]$, but it is not a ring.

Definition 25.2. A $\mathfrak{g}$-module M is called a ***weight module*** if $M = \bigoplus_{\lambda \in V} M_\lambda$ as $\mathfrak{h}$-module, where $\dim M_\lambda < \infty$ for all λ (recall that M_λ denotes the weight space, attached to the weight λ). The ***character*** of a weight module M is

$$\text{ch } M = \bigoplus_{\lambda \in V} (\dim M_\lambda) e^\lambda.$$

(This is an element of $\mathbb{Z}[[V]]$, and it lies in $\mathbb{Z}[V]$ if and only if $\dim V < \infty$.)

Introduce the ***Weyl denominator***

$$R = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) \in \mathbb{Z}[V]. \tag{4}$$

Weyl's character formula. For $\Lambda \in P_+ (\subset V)$ we have

$$e^\rho R \text{ ch } L(\Lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda + \rho)}. \tag{5}$$

Corollary 25.3 (Weyl's denominator identity) $e^\rho R = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}$. (6)

Proof. Since $L(0)$ is the trivial 1-dimensional $\mathfrak{g}$-module, its character ch $L(0) = 1$. Hence (6) follows from (5) with $\Lambda = 0$. □

For the dual Lie algebra $\mathfrak{g}(A^\top)$ we obtain a similar formula

$$e^{\rho^\vee} R^\vee = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho^\vee)}, \tag{7}$$

where $(\rho^\vee, \alpha_i) = 1$ and $R^\vee = \prod_{\alpha \in \Delta_+^\vee} (1 - e^{-\alpha})$.

We first deduce Weyl's dimension formula from Weyl's character formula (5), and then prove (5) next time.

Let $P^\vee_{++} = \{\lambda \in V \mid (\lambda, \alpha_i) \in \mathbb{Z}_{\geqslant 1}\}$. Given $\mu \in P^\vee_{++}$, define a ring homomorphism

$$F_\mu : \mathbb{Z}[[e^{-\alpha_1}, \cdots, e^{-\alpha_r}]] \to \mathbb{Z}[[q]], \qquad F_\mu(e^{-\alpha_i}) = q^{(\mu,\alpha_i)}.$$

Formula (5) can be rewritten as

$$Re^{-\Lambda} \text{ ch } L(\Lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)-(\Lambda+\rho)}. \tag{8}$$

Its LHS lies in $\mathbb{Z}[[e^{-\alpha_1}, \cdots, e^{-\alpha_r}]$. It follows from Lemma 26.1 from the next lecture that this holds for the RHS as well. Recall $\rho^\vee \in V$, defined by $(\rho^\vee, \alpha_i) = 1$ for $i = 1, \cdots, r$, and apply $F_{\rho^\vee}$ to both sides of (8). We obtain in the LHS:

$$\prod_{\alpha \in \Delta_+} (1 - q^{(\rho^\vee,\alpha)}) \sum_\lambda (\dim L(\Lambda)_\lambda) q^{(\Lambda-\lambda,\rho^\vee)}. \tag{9}$$

Next in the RHS we have:

$$\sum_{w \in W} (-1)^{\ell(w)} q^{-(\Lambda+\rho,\rho^\vee)+(w(\Lambda+\rho),\rho^\vee)},$$

and using the W-invariance of the bilinear form $(\bullet, \bullet)$, we get:

$$q^{-(\Lambda+\rho,\rho^\vee)} \sum_{w \in W} (-1)^{\ell(w)} q^{(w(\Lambda+\rho),\rho^\vee)} = q^{-(\Lambda+\rho,\rho^\vee)} F_{\Lambda+\rho} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho^\vee)-\rho},$$

which, by (7), is equal to $\prod_{\alpha \in \Delta^\vee_+} (1 - q^{(\Lambda+\rho,\alpha)})$. From this and (9) we obtain

$$\prod_{\alpha \in \Delta^\vee_+} (1 - q^{(\rho^\vee,\alpha)}) \sum_\lambda (\dim L(\Lambda)_\lambda) q^{(\Lambda-\lambda,\rho^\vee)} = \prod_{\alpha \in \Delta_+} (1 - q^{(\Lambda+\rho,\alpha^\vee)}).$$

Hence $\sum_\lambda \dim L(\Lambda)_\lambda q^{(\Lambda-\lambda,\rho^\vee)} = \prod_{\alpha \in \Delta_+} \frac{1-q^{(\Lambda+\rho,\alpha^\vee)}}{1-q^{(\rho,\alpha^\vee)}}$.

As $q \to 1$, by L'Hôpital's rule, the limit of each factor in the RHS is $\frac{(\Lambda+\rho,\alpha^\vee)}{(\rho,\alpha^\vee)}$, hence the limit of the LHS exists and is equal to $\dim L(\Lambda)$. This proves the Weyl dimension formula in the form (3).

Example 25.3. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$. By Example 25.2, all finite-dimensional irreducible $\mathfrak{g}$-modules are $L(m\Lambda_1)$, whose highest weight is $m\Lambda_1 = \frac{m}{2}\alpha$ where α is the positive root of $\mathfrak{g}$ and $m \in \mathbb{Z}_{\geqslant 0}$. Its weight space decomposition is

$$L(m\Lambda_1) = \bigoplus_{j=0}^{m} L(m\Lambda_1)_{\left(\frac{m}{2}-j\right)\alpha},$$

where $L(m\Lambda_1)_{\left(\frac{m}{2}-j\right)\alpha} = \mathbb{F}F^j v_{m\Lambda_1}$ are 1-dimensional. Hence

$$\text{ch } L(m\Lambda_1) = e^{\frac{1}{2}m\alpha} + e^{\frac{1}{2}(m-2)\alpha} + +e^{\frac{1}{2}(m-4)\alpha} + \cdots + +e^{-\frac{1}{2}\alpha}, \tag{10}$$

which also can be written as

$$\left(e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}\right) \text{ ch } L(m\Lambda_1) = e^{\frac{1}{2}(m+1)\alpha} - e^{-\frac{1}{2}(m+1)\alpha}. \tag{11}$$

Formuala (11) coincides with the Weyl character formula for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, since $\rho = \frac{1}{2}\alpha$.

Lecture 26

Proof of the Weyl Character Formula

As in the previous lecture, let $V = \mathfrak{h}^*_{\mathbb{Q}}$. We view V as an abelian group with respect to the addition operation and consider its group ring $\mathbb{Z}[V]$. Consider also the set $\mathbb{Z}[[V]]$ of all linear combinations $\sum_{\lambda \in V} a_\lambda e^\lambda$, where $a_\lambda \in \mathbb{Z}$. It is an abelian group, but not a ring. We have the inclusion $\mathbb{Z}[V] \subset Z[[V]]$ as abelian groups, with the action of the Weyl group W, given by $w \sum_\lambda a_\lambda e^\lambda = \sum_\lambda a_\lambda e^{w(\lambda)}$.

Also the abelian group $\mathbb{Z}[[V]]$ is a module over the ring $\mathbb{Z}[V]$, with a compatible action of W:

$$w(pf) = w(p)w(f) \qquad \text{for } p \in \mathbb{Z}[V], f \in \mathbb{Z}[[V]].$$

Of course, the character of any weight $\mathfrak{g}$-module lies in $\mathbb{Z}[[V]]$.

Lemma 26.1. If $\Lambda \in P_+$, then $w(\text{ch } L(\Lambda)) = \text{ch } L(\Lambda)$.

Proof. Consider the subalgebra $\mathfrak{a}_i = \mathbb{F}E_i + \mathbb{F}H_i + \mathbb{F}F_i + T_{\alpha_i}$ of $\mathfrak{g}$, where $T_{\alpha_i} = \{h \in \mathfrak{h} \mid \alpha_i(h) = 0\}$, so that $\mathfrak{h} = T_{\alpha_i} \oplus \mathbb{F}H_i$.

By the key $\mathfrak{sl}_2$-lemma, $E_i\left(F_i^{\Lambda(H_i)+1} v_\Lambda\right) = 0$, also $E_j\left(F_i^{\Lambda(H_i)+1} v_\Lambda\right) = 0$ for $j \neq i$, since $[E_j, F_i] = 0$ in this case. So, $F_i^{\Lambda(H_i)+1} v_\Lambda$ is a singular vector of $L(\Lambda)$, and since $L(\Lambda)$ is an irreducible $\mathfrak{g}$-module, it has no singular weights different from Λ, and therefore $F_i^{\Lambda(H_i)+1} v_\Lambda = 0$. But $L(\Lambda) = U(\mathfrak{g}) v_\Lambda$ and ad F_i is locally nilpotent on $\mathfrak{g}$, hence on $U(\mathfrak{g})$, hence F_i is locally nilpotent on $L(\Lambda)$. The same obviously holds for the E_i. We use here Lemma 6.2 with $\alpha = \lambda = 0$.

(Recall that an operator A on a vector space V is called ***locally nilpotent***, if for any $v \in V$, $A^N v = 0$ for $N \gg 0$.)

It follows that any weight vector $v \in L(\Lambda)$ lies in an $\mathfrak{a}_i$-invariant finite-dimensional subspace. Hence, by the Weyl complete reducibility theorem, applied to $\mathfrak{sl}_2(\mathbb{F})$, $L(\Lambda)$ decomposes in a direct sum of finite-dimensional $\mathfrak{sl}_2(\mathbb{F})$-modules, which are also $\mathfrak{h}$-modules, hence $\mathfrak{a}_i$-modules. But the character of such a module is of the form (see Example 25.3)

$$e^{m\alpha_i/2} + e^{(m-2)\alpha_i/2} + e^{(m-4)\alpha_i/2} + \cdots + e^{-m\alpha_i/2}, \qquad m \in \mathbb{Z}_{\geqslant 0},$$

which is r_{α_i}-invariant. Hence ch $L(\Lambda)$ is r_{α_i}-invariant for all i, and therefore W-invariant. ■

Lemma 26.2. R ch $M(\Lambda) = e^\Lambda$, where $R = \prod_{\alpha \in \Delta_+}(1 - e^{-\alpha})$, and $\Lambda \in V$.

Proof. By Proposition 24.2(d), we have

$$\begin{aligned}\operatorname{ch} M(\Lambda) &= \sum_{(m_1,\cdots,m_N)\in\mathbb{Z}^N_{\geqslant 0}} e^{\Lambda - m_1\beta_1-\cdots-m_N\beta_N}\\ &= e^{\Lambda} \sum_{(m_1,\cdots,m_N)\in\mathbb{Z}^N_{\geqslant 0}} e^{-m_1\beta_1-\cdots-m_N\beta_N}\\ &= e^{\Lambda}\prod_{\alpha\in\Delta_+}\left(1-e^{-\alpha}\right)^{-1}\end{aligned}$$

by the geometric progression: $\sum_{n\geqslant 0} e^{-n\alpha} = (1-e^{-\alpha})^{-1}$. ■

Lemma 26.3. $w(e^{\rho}R) = (-1)^{\ell(w)}e^{\rho}R$ for $w \in W$, i.e. $e^{\rho}R$ is W-anti-invariant.

Proof. Since the group W is generated by the simple reflections $s_i = r_{\alpha_i}$, it suffices to check that $s_i(e^{\rho}R) = -e^{\rho}R$. For that we rewrite R as

$$R = (1-e^{-\alpha_i})R_1, \qquad \text{where } R_1 = \prod_{\alpha\in\Delta_+\setminus\{\alpha_i\}}(1-e^{-\alpha}).$$

Since $\Delta_+\setminus\{\alpha_i\}$ is s_i-invariant, R_1 is s_i-invariant, hence $s_i(R) = (1-e^{\alpha_i})R_1$, and $s_i(e^{\rho}R) = e^{\rho-\alpha_i}(1-e^{\alpha_i})R_1 = e^{\rho}(e^{-\alpha_i}-1)R_1 = -e^{\rho}R$. ■

Lemma 26.4. Let $\Lambda \in V$ and let M be a highest weight module over $\mathfrak{g}$ with highest weight Λ. Recall $D(\Lambda) = \{\Lambda - \sum_{i=1}^{r} k_i\alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$ and let $B(\Lambda) = \{\lambda \in D(\Lambda) \mid (\lambda+\rho, \lambda+\rho) = (\Lambda+\rho, \Lambda+\rho)\}$. Then $\operatorname{ch} V = \sum_{\lambda\in B(\Lambda)} a_\lambda \operatorname{ch} L(\lambda)$, where $a_\Lambda = 1$ and $a_\lambda \in \mathbb{Z}_{\geqslant 0}$.

Proof. It is by induction on $\sum_{\lambda\in B(\Lambda)} \dim V_\lambda$, which is finite since $\dim V_\lambda < \infty$ and $B(\Lambda)$ is a finite set by Theorem on highest weight modules (h) from Lecture 24.

If $\sum_{\lambda\in B(\Lambda)} \dim V_\lambda = 1$, then Λ is the only singular weight of V, hence, by Theorem on highest weight modules (c) from Lecture 24, $V = L(\Lambda)$, so $\operatorname{ch} V = \operatorname{ch} L(\Lambda)$. If there is another singular weight $\lambda \neq \Lambda$, then $\lambda \in B(\Lambda)$ by Theorem on highest weight modules (g), and let v_λ be a corresponding singular vector, let $U = U(\mathfrak{g})v_\lambda$, and consider the following exact sequence of $\mathfrak{g}$-modules

$$0 \to U \to V \to V/U \to 0.$$

Then $\operatorname{ch} V = \operatorname{ch} U + \operatorname{ch} V/U$, and we can apply the inductive assumption to each of the two terms on the right. ■

Lemma 26.5. In the assumptions of Lemma 26.4, and $V = L(\Lambda)$, we have

$$\text{ch } L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda \text{ ch } M(\lambda), \text{ where } b_\Lambda = 1,\, b_\lambda \in \mathbb{Z}. \tag{1}$$

Proof. By Lemma 26.4, we have for any $\mu \in B(\Lambda)$,

$$\text{ch } M(\mu) = \sum_{\lambda \in B(\Lambda)} a_{\lambda,\mu} \text{ch } L(\lambda), \qquad a_{\lambda,\mu} \in \mathbb{Z}_{\geqslant 0},\, a_{\mu,\mu} = 1.$$

Let $B(\Lambda) = \{\lambda_1, \cdots, \lambda_m\}$, and order this set in such a way that $\lambda_i - \lambda_j \notin \{\sum_{i=1}^r k_i \alpha_i \mid k_i \in \mathbb{Z}_{\geqslant 0}\}$ if $i > j$. We get a system of linear equations

$$\text{ch } M(\lambda_j) = \sum_{i=1}^m a_{ij} \text{ ch } L(\lambda_i), \qquad j = 1, \cdots, m,$$

where $a_{ij} \in \mathbb{Z}$, $a_{ii} = 1$ and $a_{ij} = 0$ for $i > j$. So the matrix $(a_{ij})_{i,j=1}^m$ of this system is upper triangular with 1's on the diagonal and integers over the diagonal. Hence its inverse, which expresses the ch $L(\lambda_i)$'s in terms of the ch $M(\lambda_i)$'s is an upper triangular matrix with 1's on the diagonal and integers over the diagonal. In particular, this proves (1). ■

End of Proof of Weyl's Character Formula. Multiply both sides of (1) with $\Lambda \in P_+$, which lie in $\mathbb{Z}[[V]]$, by $e^\rho R \in \mathbb{Z}[V]$, and use Lemma 26.2 to obtain

$$e^\rho R \text{ ch } L(\Lambda) = \sum_{\lambda \in B(\Lambda)} b_\lambda e^{\lambda+\rho}, \qquad b_\Lambda = 1,\, b_\lambda \in \mathbb{Z}. \tag{2}$$

By Lemma 26.1, ch $L(\Lambda)$ is W-invariant, hence, by Lemma 26.3, the LHS of (2) is W-anti-invariant, so the RHS is as well. Hence, using simple transitivity of the action of W on open Weyl chambers and that $\Lambda + \rho$ lies in the open fundamental chamber, we can rewrite (2) as follows

$$e^\rho R \text{ ch } L(\Lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\Lambda+\rho)} + \sum_{\lambda \in B(\Lambda)\setminus\{\Lambda\}} b_\lambda \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}, \qquad \lambda + \rho \in P_+. \tag{3}$$

It remains to show that the second sum in the RHS of (3) is zero. For that it suffices to show that the set

$$\{\lambda \in B(\Lambda) \mid \lambda \neq \Lambda, \lambda + \rho \in P_+\}$$

is empty. In the contrary case, for λ from this set we have $\lambda = \Lambda - \sum_i k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\geqslant 0}$

and not all of them are 0, and also $(\lambda+\rho,\lambda+\rho)=(\Lambda+\rho,\Lambda+\rho)$. Hence we have

$$\begin{aligned}0&=(\Lambda+\rho,\Lambda+\rho)-(\lambda+\rho,\lambda+\rho)\\&=(\Lambda-\lambda,\lambda+\Lambda+2\rho)\\&=\left(\sum_i k_i\alpha_i,\Lambda\right)+\left(\sum_i k_i\alpha_i,\lambda+\rho\right)+\left(\sum_i k_i\alpha_i,\rho\right).\end{aligned}\tag{4}$$

Since $(\Lambda,\alpha_i)=\frac{2\Lambda(H_i)}{(\alpha_i,\alpha_i)}\geqslant 0$, and similarly $(\lambda+\rho,\alpha_i)\geqslant 0$, and $(\rho,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)>0$, where we use the Killing form, the RHS of (4) is positive, which is a contradiction. ■