

Basics

$$\begin{aligned} [\hat{x}^n, \hat{p}] &= i\hbar n \hat{x}^{n-1} & [\hat{x}, \hat{p}^n] &= i\hbar n \hat{p}^{n-1} & [\hat{p}, f] &= \frac{\hbar}{i} \frac{\partial f}{\partial x} & \mathbf{J} &= \frac{\hbar}{m} [\Psi^* \nabla \Psi] & [A, BC] &= [A, B]C + B[A, C] & v_g &= \frac{d\omega}{dk} \\ \langle x | p \rangle &= \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} & \int dx \exp [-ax^2 + bx] &= \sqrt{\frac{\pi}{a}} \exp \left[\frac{b^2}{4a} \right] & \varepsilon_{ijk} \varepsilon_{pqk} &= \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} & (\mathbf{a} \times \mathbf{b})_i &= \varepsilon_{ijk} a_j b_k \\ T &= \sum_{ij} |i\rangle T_{ij} \langle j| & p(k) &= \left\| \hat{P}_k |\psi\rangle \right\|^2 & E_i &\leq \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \quad (\langle \psi | \text{ orthogonal to } |0\rangle, \dots, |i-1\rangle) \end{aligned}$$

Uncertainty Relations

$$\Delta H \Delta Q \geq \frac{\hbar}{2} \left| \frac{d \langle \hat{Q} \rangle}{dt} \right| \quad i\hbar \frac{d \langle \hat{Q} \rangle}{dt} = \langle [\hat{Q}, \hat{H}] \rangle \quad \Delta A \Delta B \geq \left| \langle \psi | \frac{1}{2i} [\hat{A}, \hat{B}] | \psi \rangle \right|$$

Operator Exponentials

$$\begin{aligned} e^X e^Y &= \exp \left[X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \right] \\ e^A B e^{-A} &= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots \quad [A, e^B] = [A, B]e^B \text{ when } [[A, B], B] = 0 \end{aligned}$$

Harmonic Oscillator

$$\begin{aligned} \hat{H} &= \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) & L_0 &= \sqrt{\frac{\hbar}{m\omega}} & \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right) & \left[\hat{a}, (\hat{a}^\dagger)^n \right] &= n (\hat{a}^\dagger)^{n-1} & [\hat{a}^n, \hat{a}^\dagger] &= n \hat{a}^{n-1} \\ |n\rangle &= \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |0\rangle & \hat{N} &= \hat{a}^\dagger \hat{a} & \hat{N} |n\rangle &= n |n\rangle & \left[\hat{N}, \hat{a} \right] &= -\hat{a} & \left[\hat{N}, \hat{a}^\dagger \right] &= \hat{a}^\dagger & e^{i\theta \hat{N}} \hat{a} e^{-i\theta \hat{N}} &= \hat{a} e^{-i\theta} \\ \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^\dagger + \hat{a}) & \hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}) & \phi_0(x) &= \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} & \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle & \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\ \Delta x_0 &= \frac{L_0}{\sqrt{2}} & \Delta p_0 &= \frac{\hbar}{\sqrt{2}L_0} & |x_0\rangle_c &= \exp \left[-\frac{x_0^2}{4L_0^2} + \frac{x_0 \hat{a}^\dagger}{\sqrt{2}L_0} \right] |0\rangle & \mathbb{P}(|x_0\rangle_c \rightarrow |n\rangle) &\sim \text{Po} \left(\frac{x_0^2}{2L_0^2} \right) & \hat{a} |\alpha\rangle &= \alpha |\alpha\rangle \\ \hat{D}(\alpha) &= e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-\frac{i}{\hbar}(x_0 \hat{p} - \hat{x} p_0)} & \alpha &= \frac{x_0}{\sqrt{2}L_0} + i \frac{p_0 L_0}{\sqrt{2}\hbar} & |\alpha\rangle &= \hat{D}(\alpha) |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n \geq 0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle & \mathbf{1} &= \int \frac{|\alpha\rangle \langle \alpha|}{\pi} d^2\alpha \\ |\alpha, t\rangle &= e^{-i\omega t(\hat{N} + \frac{1}{2})} |\alpha\rangle = e^{-i\omega t/2} |e^{-i\omega t} \alpha\rangle & \langle \alpha | \beta \rangle &= \exp \left[-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha^* \beta \right] & &= \exp \left[-\frac{1}{2} |\alpha - \beta|^2 + i \text{Im}(\alpha^* \beta) \right] \\ \hat{x}_H(t) &= \hat{x} \cos \omega t + \frac{1}{m\omega} \hat{p} \sin \omega t & \hat{p}_H(t) &= \hat{p} \cos \omega t - m\omega \hat{x} \sin \omega t & \hat{a}(t) &= e^{-i\omega t} \hat{a} \end{aligned}$$

Spin-1/2

$$\begin{aligned} \boldsymbol{\sigma} &= \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) & \sigma_i \sigma_j &= \delta_{ij} \mathbf{1} + i \varepsilon_{ijk} \sigma_k & [\sigma_i, \sigma_j] &= 2i \varepsilon_{ijk} \sigma_k & \{\sigma_i, \sigma_j\} &= 2\delta_{ij} \mathbf{1} & (\mathbf{n} \cdot \boldsymbol{\sigma})^2 &= \mathbf{1} \\ \hat{\mathbf{S}} &= \frac{\hbar}{2} \boldsymbol{\sigma} & \langle \hat{\mathbf{S}} \rangle_{\mathbf{n}} &= \frac{\hbar}{2} \mathbf{n} & \langle \hat{\mathbf{S}} \cdot \mathbf{n}' \rangle_{\mathbf{n}} &= \frac{\hbar}{2} (\mathbf{n}' \cdot \mathbf{n}) & |\mathbf{n}; +\rangle &= \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} & |\mathbf{n}; -\rangle &= \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix} & \hat{P}_{\mathbf{n}} &= \frac{1}{2} (\mathbf{1} + \mathbf{n} \cdot \boldsymbol{\sigma}) \\ \hat{H} &= h_0 \mathbf{1} + \mathbf{n} \cdot \boldsymbol{\sigma} \Rightarrow \lambda = h_0 \pm n & |\langle \mathbf{n}' | \mathbf{n} \rangle|^2 &= \frac{1 + \mathbf{n} \cdot \mathbf{n}'}{2} = \cos^2 \frac{\gamma}{2} & \left[\mathbf{a} \cdot \hat{\mathbf{S}}, \hat{\mathbf{S}} \right] &= -i\hbar \mathbf{a} \times \hat{\mathbf{S}} & \hat{\boldsymbol{\mu}} &= \gamma \hat{\mathbf{S}} & \boldsymbol{\omega}_L &= -\gamma \mathbf{B} \\ \hat{H}_S &= -\gamma \mathbf{B} \cdot \hat{\mathbf{S}} & \mathbf{B}(t) &= B_0 \mathbf{z} + B_1 (\mathbf{x} \cos \omega t + \mathbf{y} \sin \omega t) \Rightarrow \hat{U} &= \exp \left[\frac{i\omega \hat{S}_z}{\hbar} t \right] \exp \left[i \frac{\gamma \mathbf{B}_R \cdot \hat{\mathbf{S}}}{\hbar} t \right] & \left(\mathbf{B}_R = B_1 \mathbf{x} + B_0 \left(1 - \frac{\omega}{\omega_0} \right) \mathbf{z} \right) \end{aligned}$$

Generators

$$\begin{aligned} \hat{T}_x &= e^{-i\hat{p}x/\hbar} & \hat{T}_p &= e^{i\hat{p}\hat{x}/\hbar} & \hat{U}(t, t_0) &= e^{-it\hat{H}/\hbar} & \hat{R}_{\mathbf{n}}(\alpha) &= e^{-i\alpha \hat{S}_{\mathbf{n}}/\hbar} = e^{-i\alpha(\mathbf{n} \cdot \boldsymbol{\sigma})/2} = \mathbf{1} \cos \frac{\alpha}{2} - i(\mathbf{n} \cdot \boldsymbol{\sigma}) \sin \frac{\alpha}{2} \\ \hat{R}_{\mathbf{n}}(\alpha) \hat{S}_{\mathbf{n}'} \hat{R}_{\mathbf{n}}^\dagger(\alpha) &= \hat{S}_{\mathbf{n}''} \quad (\mathbf{n}'' = \mathcal{R}_{\mathbf{n}}(\alpha) \mathbf{n}') \end{aligned}$$

Tensors

$$\begin{aligned} \langle v \otimes w, \tilde{v} \otimes \tilde{w} \rangle &= \langle v, \tilde{v} \rangle \langle w, \tilde{w} \rangle & \hat{A} \otimes \hat{B} &= (A_{ij} B) & (\hat{A} \otimes \hat{B})(\hat{C} \otimes \hat{D}) &= \hat{A} \hat{C} \otimes \hat{B} \hat{D} & \sum_{ij} A_{ij} e_i \otimes f_j \text{ entangled} \Leftrightarrow \det(A) \neq 0. \\ \text{Bell states: } |\Phi^\pm\rangle &= |\Phi_{0,3}\rangle = \frac{|++\rangle \pm |--\rangle}{\sqrt{2}}; |\Psi^+\rangle &= |\Phi_1\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}}, |\Psi^-\rangle &= |\Phi_2\rangle = i \frac{|+-\rangle - |-\rangle}{\sqrt{2}} \end{aligned}$$

Schrödinger and Heisenberg Pictures

$$\hat{H}(t) = i\hbar \frac{\partial \hat{U}}{\partial t} \hat{U}^\dagger = i\hbar \hat{\Lambda}(t) \quad \hat{A}_H(t) \equiv \hat{U}^\dagger \hat{A}_S \hat{U} \quad i\hbar \frac{d\hat{A}_H(t)}{dt} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar \frac{\partial \hat{A}_S(t)}{\partial t}$$

$$[\hat{H}(t), \hat{H}(t')] = 0 \Rightarrow \hat{U}(t, t_0) = \exp \left[-\frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}(t') \right] \& \hat{H}_H(t) = \hat{H}_S(t),$$

$$\hat{U}(t, t_0) = T \exp \left[-\frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}(t') \right] = \sum_k \left(-\frac{i}{\hbar} \right)^k \int_{t_0}^t dt_1 \hat{H}(t_1) \int_{t_0}^{t_1} \dots$$

Angular Momentum

$$[\hat{J}_i, \hat{O}_j] = i\hbar \varepsilon_{ijk} \hat{O}_k \quad [\mathbf{n} \cdot \hat{\mathbf{J}}, \hat{O}] = -i\hbar \mathbf{n} \times \hat{O} \quad [\hat{J}_i, \hat{O}_1 \cdot \hat{O}_2] = 0 \quad (\hat{O} = \hat{\mathbf{r}}, \hat{\mathbf{p}}, \hat{\mathbf{J}}) \quad [\hat{J}_i, \hat{\mathbf{J}}^2] = 0 \quad \hat{R}_{\mathbf{n}}(\theta) = e^{-i\theta \mathbf{n} \cdot \hat{\mathbf{J}}/\hbar}$$

$$\hat{\mathbf{J}}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle \quad \hat{J}_z |j, m\rangle = \hbar m |j, m\rangle \quad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \quad [\hat{J}_z, \hat{J}_\pm] = \pm \hbar \hat{J}_\pm \quad \hat{\mathbf{J}}^2 = \hat{J}_\pm \hat{J}_\mp + \hat{J}_z^2 \mp \hbar \hat{J}_z$$

$$\hat{J}_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} |j, m \pm 1\rangle \quad \hat{L}_z = -i\hbar \frac{\partial}{\partial \phi} \quad \mathcal{L}^2 = -\hbar^2 \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right)$$

$$d\Omega = \sin \theta \, d\theta \, d\phi \quad \int d\Omega = \int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\phi \quad \langle \Omega | \Omega' \rangle = \delta(\Omega - \Omega') = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

$$|\Omega\rangle = |\theta\phi\rangle \quad Y_{\ell m}(\theta, \phi) = \langle \Omega | \ell, m \rangle \quad \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} |\ell, m\rangle \langle \ell, m| = \mathbf{1} \quad \hat{\mathbf{L}}^2 = \hat{\mathbf{r}}^2 \hat{\mathbf{p}}^2 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})^2 + i\hbar \hat{\mathbf{r}} \cdot \hat{\mathbf{p}} \quad \mathbf{r} \cdot \mathbf{p} = \frac{\hbar}{i} r \frac{\partial}{\partial r}$$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \dots \oplus |j_1 - j_2|$$

Central Potentials

$$[\hat{\mathbf{L}}, \hat{H}] = 0 \quad \psi_{E\ell m}(r, \theta, \phi) = \frac{u_{E\ell}(r)}{r} Y_{\ell m}(\theta, \phi) \quad \frac{d^2 u}{dr^2} - \frac{\ell(\ell+1)}{r^2} u - \frac{2m}{\hbar^2} V(r) u = -\frac{2m}{\hbar^2} E u \quad \frac{\mathbf{p}^2}{2m} = \frac{1}{2mr^2} \mathcal{L}^2 - \hbar^2 \frac{1}{r} \frac{\partial^2}{\partial r^2} r$$

$$\hat{H} = -\frac{\hbar^2}{2mr} \frac{\partial^2}{\partial r^2} r + \frac{1}{2mr^2} \mathcal{L}^2 + V(r) \quad \int_0^\infty dr |u_{E\ell}(r)|^2 = 1 \quad V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$$

Density Matrices

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \quad \langle \hat{Q} \rangle = \text{Tr}(\hat{Q}\rho) \quad i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \rho] \quad \tilde{\rho} = \sum_i (|i\rangle \langle i|)\rho(|i\rangle \langle i|) \quad \rho_A = \text{Tr}_B \rho_{AB}$$

$$\xi(\rho) = \text{Tr}(\rho^2) \quad \bar{\rho} = \frac{\mathbf{1}}{N} \quad \rho = \frac{1}{2} (\mathbf{1} + \mathbf{a} \cdot \boldsymbol{\sigma}) \quad (s = \frac{1}{2}, |\mathbf{a}| \leq 1) \quad \frac{d\xi}{dt} = 0 \quad \text{Tr}_A(A \otimes B) = \text{Tr}(A)B$$

$$\rho(t) = \hat{U} \rho \hat{U}^\dagger \quad |\psi_{AB}\rangle = \sum_{k=1}^r \sqrt{p_k} |k_A\rangle \otimes |k_B\rangle \quad \left(\rho_A = \sum_{k=1}^r p_k |k_A\rangle \langle k_A|, \quad \rho_B = \sum_{k=1}^r p_k |k_B\rangle \langle k_B| \right)$$

$$\rho_{\text{th}} = \frac{1}{Z} \exp \left[-\frac{\hat{H}}{k_B T} \right] \quad \frac{d\rho}{dt} = \frac{1}{i\hbar} [\hat{H}, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho \} \right)$$

Notation:	$\begin{matrix} J & J & \dots \\ M & M & \dots \end{matrix}$
$m_1 \quad m_2$	Coefficients
$\begin{matrix} 1/2 \times 1/2 \\ +1/2+1/2 \\ 1 \\ +1/2-1/2 \\ -1/2+1/2 \\ -1/2-1/2 \end{matrix}$	$\begin{matrix} 5/2 & 5/2 & 3/2 \\ +2+1/2 & 1 & -3/2+3/2 \\ +2-1/2 & 1/5 & 4/5 \\ +1+1/2 & 4/5-1/5 & 1/2+1/2 \\ 0+1/2 & 3/5 & 2/5 \\ 0+1/2 & 2/5 & 3/2 \end{matrix}$
$\begin{matrix} 1 \times 1/2 \\ +3/2 & 3/2 & 1/2 \\ +1+1/2 & 1 & +1/2+1/2 \\ +1-1/2 & 1/3 & 2/3 \\ 0+1/2 & 2/3-1/3 & -1/2-1/2 \\ 0-1/2 & 2/3 & 1/3 \\ -1+1/2 & 1/3-2/3 & -3/2 \end{matrix}$	$\begin{matrix} 0-1/2 & 2/5 & 2/5 & 5/2 & 3/2 \\ -1+1/2 & 2/5 & -3/5 & 3/2 & -3/2 \\ -1+1/2 & 2/5 & 3/5 & 5/2 & 3/2 \\ -1+1/2 & 2/5 & -3/5 & 5/2 & -3/2 \\ -1+1/2 & 2/5 & 3/5 & -3/2 & -3/2 \end{matrix}$
$\begin{matrix} 2 \times 1 \\ +3 & 3 & 2 \\ +2+1 & 1 & +2 \\ +2 & 0 & 1/3 & 2/3 \\ +1+1 & 2/3-1/3 & +1 & +1 \\ +2-1 & 1/15 & 1/3 & 3/5 \\ +1 & 0 & 8/15 & 1/6-3/10 \\ 0+1 & 2/5 & -1/2 & 1/10 \end{matrix}$	$\begin{matrix} 2 & 2 & 1 & 1 & +1 \\ +3/2+1/2 & 1 & +1 & +1 & +1 \\ +3/2-1/2 & 1/4 & 3/4 & -1/4 & 0 \\ +1/2+1/2 & 3/4 & -1/4 & 2 & 0 \\ +1/2-1/2 & 1/2 & 1/2 & -1 & 1 \end{matrix}$
$\begin{matrix} 1 \times 1 \\ +2 & 2 & 1 \\ +1+1 & 1 & +1 \\ +1 & 0 & 1/2 & 1/2 \\ 0+1 & 1/2-1/2 & 0 & 0 \\ +1-1 & 1/6 & 1/2 & 1/3 \\ 0 & 0 & 2/3 & -1/3 \\ -1+1 & 1/6-1/2 & 1/3 & -1 \end{matrix}$	$\begin{matrix} 3/2 \times 1 \\ +5/2 & 5/2 & 3/2 \\ +3/2+1 & 1 & +3/2+3/2 \\ +3/2-1 & 0 & 2/5 & 3/5 \\ +1/2+1 & 3/5 & -2/5 & 1/2 \\ +3/2-1 & 1/10 & 2/5 & 1/2 \\ +1/2+1 & 3/10 & -8/15 & 1/6 \\ +1/2-1 & 1/6 & -1/2 & -1/2 \\ -1/2+1 & 1/2 & 1/2 & -1 \end{matrix}$
$\begin{matrix} 3/2 \times 1/2 \\ +5/2 & 5/2 & 3/2 \\ +2+1/2 & 1 & -3/2+3/2 \\ +2-1/2 & 1/5 & 4/5 \\ +1+1/2 & 4/5-1/5 & 1/2+1/2 \\ 0+1/2 & 3/5 & 2/5 \\ 0+1/2 & 2/5 & 3/2 \end{matrix}$	$\begin{matrix} 2 & 1 & 1 & 1 & 1 \\ +3/2-1/2 & 1 & +1 & +1 & +1 \\ +3/2-1/2 & 1/4 & 3/4 & -1/4 & 0 \\ +1/2+1/2 & 3/4 & -1/4 & 2 & 0 \\ +1/2-1/2 & 1/2 & 1/2 & -1 & 1 \end{matrix}$
$\begin{matrix} 1/2 \times 1/2 \\ +1/2+1/2 \\ 1 \\ +1/2-1/2 \\ -1/2+1/2 \\ -1/2-1/2 \end{matrix}$	$\begin{matrix} 0-1/2 & 2/5 & 2/5 & 5/2 & 3/2 \\ -1+1/2 & 2/5 & -3/5 & 3/2 & -3/2 \\ -1+1/2 & 2/5 & 3/5 & 5/2 & 3/2 \\ -1+1/2 & 2/5 & -3/5 & 5/2 & -3/2 \\ -1+1/2 & 2/5 & 3/5 & -3/2 & -3/2 \end{matrix}$
$\begin{matrix} 1 \times 1/2 \\ +3/2 & 3/2 & 1/2 \\ +1+1/2 & 1 & +1/2+1/2 \\ +1-1/2 & 1/3 & 2/3 \\ 0+1/2 & 2/3-1/3 & -1/2-1/2 \\ 0-1/2 & 2/3 & 1/3 \\ -1+1 & 1/3-2/3 & -3/2 \end{matrix}$	$\begin{matrix} 3/2 \times 1/2 \\ +5/2 & 5/2 & 3/2 \\ +3/2+1 & 1 & +3/2+3/2 \\ +3/2-1 & 0 & 2/5 & 3/5 \\ +1/2+1 & 3/5 & -2/5 & 1/2 \\ +3/2-1 & 1/10 & 2/5 & 1/2 \\ +1/2+1 & 3/10 & -8/15 & 1/6 \\ +1/2-1 & 1/6 & -1/2 & -1/2 \\ -1/2+1 & 1/2 & 1/2 & -1 \end{matrix}$
$\begin{matrix} 2 \times 1 \\ +3 & 3 & 2 \\ +2+1 & 1 & +2 \\ +2 & 0 & 1/3 & 2/3 \\ +1+1 & 2/3-1/3 & +1 & +1 \\ +2-1 & 1/15 & 1/3 & 3/5 \\ +1 & 0 & 8/15 & 1/6-3/10 \\ 0+1 & 2/5 & -1/2 & 1/10 \end{matrix}$	$\begin{matrix} 3/2 \times 1 \\ +5/2 & 5/2 & 3/2 \\ +3/2+1 & 1 & +3/2+3/2 \\ +3/2-1 & 0 & 2/5 & 3/5 \\ +1/2+1 & 3/5 & -2/5 & 1/2 \\ +3/2-1 & 1/10 & 2/5 & 1/2 \\ +1/2+1 & 3/10 & -8/15 & 1/6 \\ +1/2-1 & 1/6 & -1/2 & -1/2 \\ -1/2+1 & 1/2 & 1/2 & -1 \end{matrix}$
$\begin{matrix} 1 \times 1 \\ +2 & 2 & 1 \\ +1+1 & 1 & +1 \\ +1 & 0 & 1/2 & 1/2 \\ 0+1 & 1/2-1/2 & 0 & 0 \\ +1-1 & 1/6 & 1/2 & 1/3 \\ 0 & 0 & 2/3 & -1/3 \\ -1+1 & 1/6-1/2 & 1/3 & -1 \end{matrix}$	$\begin{matrix} 3/2 \times 1/2 \\ +5/2 & 5/2 & 3/2 \\ +2+1/2 & 1 & -3/2+3/2 \\ +2-1/2 & 1/5 & 4/5 \\ +1+1/2 & 4/5-1/5 & 1/2+1/2 \\ 0+1/2 & 3/5 & 2/5 \\ 0+1/2 & 2/5 & 3/2 \end{matrix}$