Basics

$$[\hat{x}^n, \hat{p}] = i\hbar n\hat{x}^{n-1} \qquad [\hat{x}, \hat{p}^n] = i\hbar n\hat{p}^{n-1} \qquad [\hat{p}, f] = \frac{\hbar}{i}\frac{\partial f}{\partial x} \qquad \mathbf{J} = \frac{\hbar}{m}[\Psi^* \nabla \Psi] \qquad [A, BC] = [A, B]C + B[A, C] \qquad v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \qquad \int \mathrm{d}x \exp\left[-ax^2 + bx\right] = \sqrt{\frac{\pi}{a}} \exp\left[\frac{b^2}{4a}\right] \qquad \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \qquad (\mathbf{a} \times \mathbf{b})_i = \varepsilon_{ijk} a_j b_k$$

$$T = \sum_{ij} |i\rangle T_{ij} \langle j| \qquad p(k) = \left\|\hat{P}_k |\psi\rangle\right\|^2 \qquad E_i \le \frac{\langle\psi| \hat{H} |\psi\rangle}{\langle\psi|\psi\rangle} \ (|\psi\rangle \text{ orthogonal to } |0\rangle, \cdots, |i-1\rangle)$$

Uncertainty Relations

$$\Delta H \Delta Q \ge \frac{\hbar}{2}\left|\frac{\mathrm{d}\left\langle\hat{Q}\right\rangle}{\mathrm{d}t}\right| \qquad i\hbar\frac{\mathrm{d}\left\langle\hat{Q}\right\rangle}{\mathrm{d}t} = \left\langle\left[\hat{Q}, \hat{H}\right]\right\rangle \qquad \Delta A \Delta B \ge \left|\langle\psi| \frac{1}{2i}\left[\hat{A}, \hat{B}\right] |\psi\rangle\right|$$

Operator Exponentials

$$e^X e^Y = \exp\left[X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots\right]$$

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \qquad \left[A, e^B\right] = [A, B]e^B \text{ when } [[A, B], B] = 0$$

Harmonic Oscillator

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \qquad L_0 = \sqrt{\frac{\hbar}{m\omega}} \qquad \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \qquad \left[\hat{a}, \left(\hat{a}^\dagger\right)^n\right] = n\left(\hat{a}^\dagger\right)^{n-1} \qquad \left[\hat{a}^n, \hat{a}^\dagger\right] = n\hat{a}^{n-1}$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n |0\rangle \qquad \hat{N} = \hat{a}^\dagger\hat{a} \qquad \hat{N}|n\rangle = n|n\rangle \qquad \left[\hat{N}, \hat{a}\right] = -\hat{a} \qquad \left[\hat{N}, \hat{a}^\dagger\right] = \hat{a}^\dagger \qquad e^{i\theta\hat{N}}\hat{a}e^{-i\theta\hat{N}} = \hat{a}e^{-i\theta}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right) \qquad \phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\Delta x_0 = \frac{L_0}{\sqrt{2}} \qquad \Delta p_0 = \frac{\hbar}{\sqrt{2}L_0} \qquad |x_0\rangle_c = \exp\left[-\frac{x_0^2}{4L_0^2} + \frac{x_0\hat{a}^\dagger}{\sqrt{2}L_0}\right]|0\rangle \qquad \mathbb{P}(|x_0\rangle_c \to |n\rangle) \sim \mathrm{Po}\left(\frac{x_0^2}{2L_0^2}\right) \qquad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} = e^{-\frac{i}{\hbar}(x_0\hat{p} - \hat{x}p_0)} \qquad \alpha = \frac{x_0}{\sqrt{2}L_0} + i\frac{p_0 L_0}{\sqrt{2}\hbar} \qquad |\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n\ge 0}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \qquad \mathbf{1} = \int \frac{|\alpha\rangle\langle\alpha|}{\pi}\mathrm{d}^2\alpha$$

$$|\alpha, t\rangle = e^{-i\omega t\left(\hat{N} + \frac{1}{2}\right)}|\alpha\rangle = e^{-i\omega t/2}\left|e^{-i\omega t}\alpha\right\rangle \qquad \langle\alpha|\beta\rangle = \exp\left[-\frac{|\alpha|^2 + |\beta|^2}{2} + \alpha^*\beta\right] = \exp\left[-\frac{1}{2}|\alpha - \beta|^2 + i\ \mathrm{Im}(\alpha^*\beta)\right]$$

$$\hat{x}_H(t) = \hat{x}\cos\omega t + \frac{1}{m\omega}\hat{p}\sin\omega t \qquad \hat{p}_H(t) = \hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t \qquad \hat{a}(t) = e^{-i\omega t}\hat{a}$$

Spin-1/2

$$\boldsymbol{\sigma} = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) \qquad \sigma_i\sigma_j = \delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k \qquad [\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1} \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})^2 = \mathbf{1}$$

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\boldsymbol{\sigma} \qquad \left\langle\hat{\mathbf{S}}\right\rangle_{\mathbf{n}} = \frac{\hbar}{2}\mathbf{n} \qquad \left\langle\hat{\mathbf{S}}\cdot\mathbf{n}'\right\rangle_{\mathbf{n}} = \frac{\hbar}{2}(\mathbf{n}'\cdot\mathbf{n}) \qquad |\mathbf{n}; +\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \qquad |\mathbf{n}; -\rangle = \begin{pmatrix}-e^{-i\phi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix} \qquad \hat{P}_{\mathbf{n}} = \frac{1}{2}(\mathbf{1} + \mathbf{n}\cdot\boldsymbol{\sigma})$$

$$\hat{H} = h_0\mathbf{1} + \mathbf{n}\cdot\boldsymbol{\sigma} \Rightarrow \lambda = h_0 \pm n \qquad |\langle\mathbf{n}'|\mathbf{n}\rangle|^2 = \frac{1 + \mathbf{n}\cdot\mathbf{n}'}{2} = \cos^2\frac{\gamma}{2} \qquad \left[\mathbf{a}\cdot\hat{\mathbf{S}}, \hat{\mathbf{S}}\right] = -i\hbar\mathbf{a}\times\hat{\mathbf{S}} \qquad \hat{\boldsymbol{\mu}} = \gamma\hat{\mathbf{S}} \qquad \boldsymbol{\omega}_L = -\gamma\mathbf{B}$$

$$\hat{H}_S = -\gamma\mathbf{B}\cdot\hat{\mathbf{S}} \qquad \mathbf{B}(t) = B_0\mathbf{z} + B_1(\mathbf{x}\cos\omega t + \mathbf{y}\sin\omega t) \Rightarrow \hat{U} = \exp\left[\frac{i\omega\hat{S}_z}{\hbar}t\right]\exp\left[i\frac{\gamma\mathbf{B}_R\cdot\hat{\mathbf{S}}}{\hbar}t\right] \qquad \left(\mathbf{B}_R = B_1\mathbf{x} + B_0\left(1 - \frac{\omega}{\omega_0}\right)\mathbf{z}\right)$$

Generators

$$\hat{T}_x = e^{-i\hat{p}x/\hbar} \qquad \hat{\tilde{T}}_p = e^{ip\hat{x}/\hbar} \qquad \hat{U}(t, t_0) = e^{-it\hat{H}/\hbar} \qquad \hat{R}_{\mathbf{n}}(\alpha) = e^{-i\alpha\hat{S}_{\mathbf{n}}/\hbar} = e^{-i\alpha(\mathbf{n}\cdot\boldsymbol{\sigma})/2} = \mathbf{1}\cos\frac{\alpha}{2} - i(\mathbf{n}\cdot\boldsymbol{\sigma})\sin\frac{\alpha}{2}$$

$$\hat{R}_{\mathbf{n}}(\alpha)\hat{S}_{\mathbf{n}'}\hat{R}_{\mathbf{n}}^\dagger(\alpha) = \hat{S}_{\mathbf{n}''} \qquad (\mathbf{n}'' = \mathcal{R}_{\mathbf{n}}(\alpha)\mathbf{n}')$$

Tensors

$$\langle v\otimes w, \tilde{v}\otimes\tilde{w}\rangle = \langle v, \tilde{v}\rangle\langle w, \tilde{w}\rangle \qquad \hat{A}\otimes\hat{B} = (A_{ij}B) \qquad (\hat{A}\otimes\hat{B})(\hat{C}\otimes\hat{D}) = \hat{A}\hat{C}\otimes\hat{B}\hat{D} \qquad \sum_{ij} A_{ij}e_i\otimes f_j \text{ entangled} \Leftrightarrow \det(A) \ne 0.$$

Bell states: $\left|\Phi^\pm\right\rangle = |\Phi_{0,3}\rangle = \frac{|{+}{+}\rangle \pm |{-}{-}\rangle}{\sqrt{2}}$; $\left|\Psi^+\right\rangle = |\Phi_1\rangle = \frac{|{+}{-}\rangle + |{-}{+}\rangle}{\sqrt{2}}$, $\left|\Psi^-\right\rangle = |\Phi_2\rangle = i\frac{|{+}{-}\rangle - |{-}{+}\rangle}{\sqrt{2}}$

Schrödinger and Heisenberg Pictures

$$\hat{H}(t) = i\hbar\frac{\partial \hat{U}}{\partial t}\hat{U}^\dagger = i\hbar\hat{\Lambda}(t) \qquad \hat{A}_H(t) \equiv \hat{U}^\dagger \hat{A}_S \hat{U} \qquad i\hbar\frac{\mathrm{d}\hat{A}_H(t)}{\mathrm{d}t} = \left[\hat{A}_H(t), \hat{H}_H(t)\right] + i\hbar\frac{\partial \hat{A}_S(t)}{\partial t}$$

$$\left[\hat{H}(t), \hat{H}(t')\right] = 0 \Rightarrow \hat{U}(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\hat{H}(t')\right] \ \& \ \hat{H}_H(t) = \hat{H}_S(t),$$

$$\hat{U}(t,t_0) = \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t'\hat{H}(t')\right] = \sum_k\left(-\frac{i}{\hbar}\right)^k\int_{t_0}^{t}\mathrm{d}t_1\hat{H}(t_1)\int_{t_0}^{t_1}\cdots$$

Angular Momentum

$$\left[\hat{J}_i, \hat{\mathcal{O}}_j\right] = i\hbar\varepsilon_{ijk}\hat{\mathcal{O}}_k \qquad \left[\mathbf{n}\cdot\hat{\mathbf{J}}, \hat{\mathcal{O}}\right] = -i\hbar\mathbf{n}\times\hat{\mathcal{O}} \qquad \left[\hat{J}_i, \hat{\mathcal{O}}_1\cdot\hat{\mathcal{O}}_2\right] = 0 \quad \left(\hat{\mathcal{O}} = \hat{\mathbf{r}}, \hat{\mathbf{p}}, \hat{\mathbf{J}}\right) \qquad \left[\hat{J}_i, \hat{\mathbf{J}}^2\right] = 0 \qquad \hat{R}_{\mathbf{n}}(\theta) = e^{-i\theta\mathbf{n}\cdot\hat{\mathbf{J}}/\hbar}$$

$$\hat{\mathbf{J}}^2\left|j,m\right\rangle = \hbar^2 j(j+1)\left|j,m\right\rangle \qquad \hat{J}_z\left|j,m\right\rangle = \hbar m\left|j,m\right\rangle \qquad \hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \qquad \left[\hat{J}_z, \hat{J}_\pm\right] = \pm\hbar\hat{J}_\pm \qquad \hat{\mathbf{J}}^2 = \hat{J}_\pm\hat{J}_\mp + \hat{J}_z^2 \mp \hbar\hat{J}_z$$

$$\hat{J}_\pm\left|j,m\right\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\left|j,m\pm1\right\rangle \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \qquad \mathcal{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\mathrm{d}\Omega = \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi \qquad \int\mathrm{d}\Omega = \int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^{2\pi}\mathrm{d}\phi \qquad \left\langle\Omega \mid \Omega'\right\rangle = \delta\left(\Omega - \Omega'\right) = \delta\left(\cos\theta - \cos\theta'\right)\delta\left(\phi - \phi'\right)$$

$$\left|\Omega\right\rangle = \left|\theta\phi\right\rangle \qquad Y_{\ell m}(\theta,\phi) = \left\langle\Omega\middle|\ell,m\right\rangle \qquad \sum_{\ell\geq0}\sum_{m=-\ell}^{\ell}\left|\ell,m\right\rangle\left\langle\ell,m\right| = \mathbf{1} \qquad \hat{\mathbf{L}}^2 = \hat{\mathbf{r}}^2\hat{\mathbf{p}}^2 - (\hat{\mathbf{r}}\cdot\hat{\mathbf{p}})^2 + i\hbar\hat{\mathbf{r}}\cdot\hat{\mathbf{p}} \qquad \mathbf{r}\cdot\mathbf{p} = \frac{\hbar}{i}r\frac{\partial}{\partial r}$$

$$j_1\otimes j_2 = (j_1+j_2)\oplus(j_1+j_2-1)\oplus\cdots\oplus\left|j_1 - j_2\right|$$

Central Potentials

$$\left[\hat{\mathbf{L}}, \hat{H}\right] = 0 \qquad \psi_{E\ell m}(r,\theta,\phi) = \frac{u_{E\ell}(r)}{r}Y_{\ell m}(\theta,\phi) \qquad \frac{\mathrm{d}^2u}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2}u - \frac{2m}{\hbar^2}V(r)u = -\frac{2m}{\hbar^2}Eu \qquad \frac{\mathbf{p}^2}{2m} = \frac{1}{2mr^2}\mathcal{L}^2 - \hbar^2\frac{1}{r}\frac{\partial^2}{\partial r^2}r$$

$$\hat{H} = -\frac{h^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{1}{2mr^2}\mathcal{L}^2 + V(r) \qquad \int_0^\infty\mathrm{d}r\ \left|u_{E\ell}(r)\right|^2 = 1 \qquad V_{\mathrm{eff}}(r) = V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}$$

Density Matrices

$$\rho = \sum_k p_k\left|\psi_k\right\rangle\left\langle\psi_k\right| \qquad \left\langle\hat{Q}\right\rangle = \mathrm{Tr}\left(\hat{Q}\rho\right) \qquad i\hbar\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \left[\hat{H}, \rho\right] \qquad \tilde{\rho} = \sum_i(\left|i\right\rangle\left\langle i\right|)\rho(\left|i\right\rangle\left\langle i\right|) \qquad \rho_A = \mathrm{Tr}_B\rho_{AB}$$

$$\xi(\rho) = \mathrm{Tr}(\rho^2) \qquad \overline{\rho} = \frac{\mathbf{1}}{N} \qquad \rho = \frac{1}{2}\left(\mathbf{1} + \mathbf{a}\cdot\boldsymbol{\sigma}\right) \quad \left(s = \frac{1}{2}, \left|\mathbf{a}\right| \leq 1\right) \qquad \frac{\mathrm{d}\xi}{\mathrm{d}t} = 0 \qquad \mathrm{Tr}_A(A\otimes B) = \mathrm{Tr}(A)B$$

$$\rho(t) = \hat{U}\rho\hat{U}^\dagger \qquad \left|\psi_{AB}\right\rangle = \sum_{k=1}^{r}\sqrt{p_k}\left|k_A\right\rangle\otimes\left|k_B\right\rangle \qquad \left(\rho_A = \sum_{k=1}^{r}p_k\left|k_A\right\rangle\left\langle k_A\right|, \quad \rho_B = \sum_{k=1}^{r}p_k\left|k_B\right\rangle\left\langle k_B\right|\right)$$

$$\rho_{\mathrm{th}} = \frac{1}{Z}\exp\left[-\frac{\hat{H}}{k_BT}\right] \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{i\hbar}\left[\hat{H}, \rho\right] + \sum_k\left(L_k\rho L_k^\dagger - \frac{1}{2}\left\{L_k^\dagger L_k, \rho\right\}\right)$$