1. $$\begin{cases} u_{tt} = a^2 u_{xx} \\ u(t,0) = u_x(t,l) = 0 \\ u(0,x) = \varphi(x) \quad u_t(0,x) = \psi(x) \end{cases} \qquad (*)$$

解：分离变量：$u(t,x) = X(x)T(t)$. 代入原方程 $(*)$

易有 $\dfrac{T''}{a^2 T} = \dfrac{X''}{X}$ 设等式值为 $-\lambda$.

则 $T'' + \lambda a^2 T = 0$; $X'' + \lambda X = 0$

且 $u(t,0) = X(0)T(t) = 0$. $u_x(t,l) = X'(l)T(t) = 0$.

$\therefore X(0) = 0$, $X'(l) = 0$.

I: $$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0,\ X'(l) = 0 . \end{cases} \qquad (**)$$

讨论：① $\lambda = 0$ 时 $X'' = 0$ 则 $X = Ax + B$ $X' = A$

$\therefore B = A = 0$. 此时 $X \equiv 0$ $\therefore \lambda \neq 0$

② $\lambda < 0$ 时 令 $\lambda = -k^2$ $\therefore X'' - k^2 X = 0$.

通解 $X'' - k^2 X = 0$ 为 $X = Ae^{kx} + Be^{-kx}$

代入边界条件 $(**)$

$$\begin{cases} A + B = 0 \\ Ake^{kl} - Bke^{-kl} = 0 . \end{cases} \Rightarrow A = B = 0 \qquad \therefore \lambda \not< 0 .$$

③ $\lambda > 0$ 时. 令 $\lambda = k^2$ $\therefore X'' + k^2 X = 0$.

通解为 $X = A\cos kx + B\sin kx$ 由于 $X(0) = 0$.

$\therefore A = 0$ 又 $X' = kB\cos kx$ $\therefore kB\cos kl = 0$.

$\therefore k = \dfrac{\frac{2n+1}{2}\pi}{l}$ $\therefore \lambda = \left(\dfrac{(n+\frac{1}{2})\pi}{l}\right)^2$ $n = 0, 1, 2 \cdots$

易错处:

$$\therefore\ u_n(t,x)=\sum_{n=0}^{\infty}\left(C_n\cos\frac{\boxed{(n+\frac{1}{2})}\pi}{l}at+D_n\sin\frac{(n+\frac{1}{2})\pi}{l}at\right)\sin\frac{(n+\frac{1}{2})\pi}{l}x$$

$$C_n=\frac{2}{l}\int_0^l\varphi(\alpha)\sin\frac{(n+\frac{1}{2})\pi}{l}\alpha\,d\alpha$$

$$D_n=\frac{2}{\boxed{(n+\frac{1}{2})\pi a}}\int_0^l\psi(\alpha)\sin\frac{(n+\frac{1}{2})\pi}{l}\alpha\,d\alpha.$$

2. $$\begin{cases}u_{tt}=a^2u_{xx}\\ u_x(0,t)=u_x(0,l)=0\\ u(x,0)=\varphi(x)\quad u_t(x,0)=\psi(x)\end{cases}$$

$\therefore$

※(思考一下!)

$$u(t,x)=\frac{A_0}{2}+B_0t+\sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi a}{l}t+B_n\sin\frac{n\pi a}{l}t\right)\cos\frac{n\pi}{l}x$$

$$A_n=\frac{2}{l}\int_0^l\varphi(\alpha)\cos\frac{n\pi\alpha}{l}d\alpha\qquad(n=0,1,\cdots)$$

$$B_n=\frac{2}{n\pi a}\int_0^l\psi(\alpha)\cos\frac{n\pi\alpha}{l}d\alpha\quad(n=1,2\cdots)\qquad B_0=\frac{1}{l}\int_0^l\psi(\alpha)\,d\alpha\ ※$$

3. $$\begin{cases}u_{tt}=a^2u_{xx}\\ u_x(0,t)=u(l,t)=0\\ u(x,0)=\varphi(x)\quad u_t(x,0)=\psi(x).\end{cases}$$

$$\therefore\ u(t,x)=\sum_{n=0}^{\infty}\left(A_n\cos\frac{\boxed{(n+\frac{1}{2})}\pi}{l}at+B_n\sin\frac{(n+\frac{1}{2})\pi}{l}at\right)\cos\frac{(n+\frac{1}{2})\pi}{l}x.$$

$$A_n=\frac{2}{l}\int_0^l\varphi(\alpha)\cos\frac{n+\frac{1}{2}}{l}\pi\alpha\,d\alpha$$

$$B_n=\frac{2}{\boxed{(n+\frac{1}{2})\pi a}}\int_0^l\psi(\alpha)\cos\frac{(n+\frac{1}{2})\pi}{l}\alpha\,d\alpha$$

(后2个自推试下)