1.

证：(a)

$$\begin{aligned}\mathrm{E}(X(t)) &= \mathrm{E}\sin Ut\\ &= \int_0^{2\pi}\frac{1}{2\pi}\sin UtdU\\ &= 0 \quad (t=1,2,\cdots)\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(X(t),X(s)) &= \mathrm{E}(\sin Ut\cdot\sin Us)\\ &= \frac{1}{2}\mathrm{E}\left(\cos(t-s)U-\cos(t+s)U\right)\\ &= \frac{1}{4\pi}\left\{\frac{1}{t-s}\sin(t-s)U\Big|_0^{2\pi}-\frac{1}{t+s}\sin(t+s)U\Big|_0^{2\pi}\right\}\\ &= 0 \quad (t\neq s)\end{aligned}$$

当 $t=s$ 时 $\mathrm{Cov}(X(t),X(s))=\mathrm{E}\sin^2 Ut=\frac{1}{2}$

$\therefore$ 是宽平稳

考虑 $F_t(x)=P(\sin Ut\leqslant x)$, 显然 $F_{t+h}=P(\sin U(t+h)\leqslant x)$ 与其不一定相同

$\therefore$ 不是严平稳

(b)

$$EX(t)=\frac{1}{2\pi t}(1-\cos 2\pi t)$$

$$DX(t)=\mathrm{E}\left(\sin Ut-\frac{1}{2\pi t}(1-\cos 2\pi t)\right)^2=\frac{1}{2}-\frac{\sin 4\pi t}{8\pi t}-\left(\frac{1-\cos 2\pi t}{2\pi t}\right)^2$$

都与 t 相关

$\therefore$ 不是宽平稳

若其严平稳, 则因二阶矩存在, 应为宽平稳, 矛盾.

$\therefore$ 不是严平稳.

2.

证：

$1^\circ \ell=0$ 时

$\mathrm{E}X_n$ 依定义为常数 C_0

$\mathrm{Cov}(X_n,X_m)$ 依定义为 $n-m$ 的函数 $f_0(n-m)$

成立

2° 设当 $\ell\leqslant k$ 时成立, 则当 $\ell=k+1$ 时

$$
\begin{aligned}
\mathrm{E}X_n^{(\ell)} &= \mathrm{E}(X_n^{(k)} - X_{n-1}^{(k)}) = C_k - C_k = 0 \\
\mathrm{Cov}(X^{(k+1)}, X_m^{(k+1)}) &= \mathrm{E}(X_n^{(k+1)} X_m^{(k+1)}) \\
&= \mathrm{E}\left[(X_n^{(k)} - X_{n-1}^{(k)})(X_m^{(k)} - X_{m-1}^{(k)})\right] \\
&= \mathrm{E}(X_n^{(k)} X_m^{(k)}) - \mathrm{E}(X_{n-1}^{(k)} X_m^{(k)}) - \mathrm{E}(X_n^{(k)} X_{m-1}^{(k)}) + \mathrm{E}(X_{n-1}^{(k)} X_{m-1}^{(k)}) \\
&= f_k(n-m) - f_k(n-1-m) - f_k(n-m+1) + f_k(n-m) \\
&= f_\ell(n-m)
\end{aligned}
$$

只与 $n-m$ 有关

$\therefore$ 是平稳的

7.

(i)

$$
\begin{aligned}
E(Z(t)W(t)) &= E(X(t+1)X(t-1)) \\
&= R(2) \\
&= 4e^{-4}
\end{aligned}
$$

$$
\begin{aligned}
E(Z(t)W(t))^2 &= E\left(X^2(t+1) + 2X(t+1)X(t-1) + X^2(t-1)\right) \\
&= 2EX^2(t) + 2R(2) \\
&= 2[DX(t) - E^2X(t)] + 8e^{-4} \\
&= 2R(0) + 8e^{-4} \\
&= 8(1+e^{-4})
\end{aligned}
$$

(ii)$Z(t) = X(t+1) \sim N(0, 2^2)$

$\therefore f_Z(z) = \frac{1}{\sqrt{2\pi \cdot 2^2}} e^{-\frac{z^2}{2\cdot 2^2}} = \frac{1}{\sqrt{8\pi}} e^{-\frac{z^2}{8}}$

$\therefore P(Z(t) < 1) = \int_{-\infty}^{1} f_Z(z)\,dz = \frac{1}{\sqrt{8\pi}} \int_{-\infty}^{1} e^{-\frac{z^2}{8}}\,dz$

(iii) 显然 $f_{Z,W}(z,w)$ 为二维正态分布概率密度函数

协方差矩阵

$$
C = \begin{pmatrix} 4 & 4e^{-4} \\ 4e^{-4} & 4 \end{pmatrix}
$$

其逆矩阵

$$C^{-1}=\begin{pmatrix}\frac{1}{4(1-e^{-8})} & -\frac{e^{-4}}{41-e^{-8}}\\ -\frac{e^{-4}}{41-e^{-8}} & \frac{1}{4(1-e^{-8})}\end{pmatrix}$$

其行列式 $|C|=16(1-e^{-8})$

期望向量 $\bar{\mu}=(0,0)$

$$\begin{aligned}\therefore f_{Z,W}(z,w)&=\frac{1}{2\pi\sqrt{|C|}}\exp\left\{-\frac{1}{2}\Big((z,w)-\bar{\mu}\Big)C^{-1}\Big((z,w)-\bar{\mu}\Big)^{T}\right\}\\&=\frac{1}{8\pi\sqrt{1-e^{-8}}}\exp\left\{-\frac{z^2+w^2-2e^{-4}wz}{8(1-e^{-8})}\right\}\end{aligned}$$

13.

证：$EX^{(n)}(t)=\left[EX(t)\right]^{(n)}=0$

$\mathrm{Cov}\big(X^{(n)}(t),X^{(n)}(t+\tau)\big)=(-1)^nR^{(2n)}(\tau)$

$\therefore \{X^{(n)}(t)\}$ 是平稳过程.

15.

证：

取固定的 $\tau\in\mathbb{Z}$, 记 $X_{n+\tau}X_n\triangleq Y_n$, 则

$$\begin{aligned}\mathrm{E}Y_n&=R_X(\tau)(const)\\ \mathrm{Cov}(Y_{n+\tau_1},Y_n)&=\mathrm{E}Y_{n+\tau_1}Y_n-R_X^2(\tau)\\&=\mathrm{E}X_{n+\tau_1+\tau}X_{n+\tau_1}X_{n+\tau}X_n-R_X^2(\tau)\\&=R_X^2(\tau)+R_X^2(\tau_1)+R_X(\tau_1+\tau)R_X(\tau_1-\tau)-R_X^2(\tau)\\&=R_X^2(\tau_1)+R_X(\tau_1+\tau)R_X(\tau_1-\tau)\\&=R_Y(\tau_1)\end{aligned}$$

$\therefore \{Y_n\}$ 是平稳过程.

又易见 $X=\{X_n,n\in\mathbb{Z}\}$ 的协方差函数遍历性成立的充要条件是 $Y=\{Y_n,n\in\mathbb{Z}\}$ 的均值遍历性成立.

而我们有

$$\begin{aligned}\left|\frac{1}{N}\sum_{\tau_1=0}^{N-1}R_Y(\tau_1)\right| &\leqslant \frac{1}{N}\sum_{\tau_1=0}^{N-1}\left|R_Y(\tau_1)\right| \\ &\leqslant \frac{1}{N}\sum_{\tau_1=0}^{N-1}\left[R_X^2(\tau_1)+\left(R_X^2(\tau_1+\tau)+R_X^2(\tau_1-\tau)\right)/2\right]\to 0,(N\to+\infty)\end{aligned}$$

由均值遍历性定理 (i) 可知, $Y=\{Y_n,n\in\mathbb{Z}\}$ 的均值遍历性成立, 即 $X=\{X_n,n\in\mathbb{Z}\}$ 的协方差函数遍历性成立.

16.

证：

$$\mathrm{E}X_0=\int_0^1 2x^2\,dx=\frac{2}{3}$$

$$\mathrm{E}X_0^2=\int_0^1 2x^3\,dx=\frac{1}{2}$$

$$\begin{aligned}\mathrm{E}X_{n+1}&=\mathrm{E}[\mathrm{E}(X_{n+1}|X_N)]\\ &=\mathrm{E}\left[\int_{1-x_n}^1\frac{x_{n+1}}{x_n}\,dx_{n+1}\right]\\ &=\mathrm{E}(1-\frac{1}{2}X_n)\\ &=1-\frac{1}{2}\mathrm{E}X_n\end{aligned}$$

$$\because \mathrm{E}X_0=\frac{2}{3}\qquad \therefore \mathrm{E}X_n\equiv\frac{2}{3}$$

又有

$$\begin{aligned}\mathrm{E}X_{n+1}^2&=\mathrm{E}\left[\mathrm{E}(X_{n+1}^2|X_n)\right]\\ &=\mathrm{E}\left[\int_{1-x_n}^1\frac{x_{n+1}^2}{x_n}\,dx_{n+1}\right]\\ &=1-\mathrm{E}X_n+\frac{1}{3}\mathrm{E}X_n^2\end{aligned}$$

$$\because \mathrm{E}X_0^2=\frac{1}{2}\qquad \therefore \mathrm{E}X_n^2\equiv\frac{1}{2}$$

$$
\begin{aligned}
\mathrm{E}(X_nX_{n+m}) &= \mathrm{E}\left[\mathrm{E}(X_nX_{n+m}|X_n)\right] \\
&= \mathrm{E}\left[X_n\mathrm{E}(X_{n+m}|X_n)\right] \\
&= \mathrm{E}\left[X_n\left(1-\frac{1}{2}\mathrm{E}(X_{n+m-1}|X_n)\right)\right] \\
&= \mathrm{E}X_n - \frac{1}{2}\mathrm{E}\left[\mathrm{E}(X_nX_{n+m-1}|X_n)\right] \\
&= \frac{2}{3} - \frac{1}{2}\mathrm{E}(X_nX_{n+m-1})
\end{aligned}
$$

$$
\therefore \mathrm{E}(X_nX_{n+m}) - \frac{4}{9} = -\frac{1}{2}\left(\mathrm{E}(X_nX_{n+m}) - \frac{4}{9}\right) = \cdots = \left(-\frac{1}{2}\right)^m\left(\mathrm{E}X_n^2 - \frac{4}{9}\right) = \frac{1}{18}\left(-\frac{1}{2}\right)^m
$$

$$
\therefore R_X(n,n+m) = \mathrm{E}\left(X_n - \frac{2}{3}\right)\left(X_{n+m} - \frac{2}{3}\right) = \mathrm{E}(X_nX_{n+m}) - \frac{4}{9} = \frac{1}{18}\left(-\frac{1}{2}\right)^m = R(m)
$$

$\therefore \{X_n\}$ 是平稳序列

又 $\because \lim_{m\to+\infty} R(m) = 0$

$\therefore$ 是均值遍历的

17.

解：

$$
\mathrm{E}X_n = \sum_{k=0}^{+\infty} \alpha^k \mathrm{E}\varepsilon_{n-k} = 0
$$

$$
\begin{aligned}
R_X(n,n+m) &= \mathrm{Cov}(X_n, X_{n+m}) \\
&= \mathrm{E}\left(\sum_{k=0}^{+\infty}\alpha^k\varepsilon_{n-k}\right)\left(\sum_{\ell=0}^{+\infty}\alpha^\ell\varepsilon_{m+n-\ell}\right) \\
&= \sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\alpha^{k+\ell}\mathrm{E}\varepsilon_{n-k}\varepsilon_{m+n-\ell} \\
&= \sum_{k=0}^{+\infty}\alpha^{2k+m}\mathrm{E}\varepsilon_{n-k}^2 \\
&= \alpha^m\frac{\sigma^2}{1-\alpha^2}
\end{aligned}
$$

$\therefore \{X_n\}$ 为平稳序列

又 $\lim\limits_{m\to+\infty} R(m) = 0,\ \therefore$ 是均值遍历的

由于协方差遍历性涉及到四阶矩，很难验证，此处仅考虑了其均值遍历性。

20.(1)

证：

$$\begin{aligned}R(\tau)&=\mathrm{E}\left[X(t+a)-X(t-a)\right]\left[X(t-\tau+a)-X(t-\tau-a)\right]\\&=\mathrm{E}\left[X(t+a)X(t-\tau-a)\right]-\mathrm{E}\left[X(t+a)-X(t-\tau-a)\right]\\&\quad-\mathrm{E}\left[X(t-a)X(t-\tau-a)\right]+\mathrm{E}\left[X(t-a)-X(t-\tau-a)\right]\\&=R_X(\tau)-R_X(\tau+2a)-R_X(\tau-2a)+R_X(\tau)\\&=2R_X(\tau)-R_X(\tau+2a)-R_X(\tau-2a)\end{aligned}$$

22. 题目不够严谨，此处$a>0$， 23、24 同

$$S(\omega)=\frac{a^2\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{2ab^2}{\omega^2+a^2}$$

23.

(1)

$$a\sigma^2\left(\frac{1}{a^2+(\omega+b)^2}+\frac{1}{a^2+(\omega-b)^2}\right)$$

$$=\frac{2a\sigma^2(a^2+\omega^2+b^2)}{(a^2+\omega^2+b^2)^2-4\omega^2b^2}\qquad(a>0)$$

(2)

$$\frac{a\omega\sigma^2}{b}\left(\frac{-1}{a^2+(\omega+b)^2}+\frac{1}{a^2+(\omega-b)^2}\right)$$

$$=\frac{4a\sigma^2\omega^2}{(a^2+\omega^2+b^2)^2-4\omega^2b^2}\qquad(a>0)$$

24.

(2) $\frac{1}{4}(1+|\tau|)e^{-|\tau|}$ 绝对值：约当引理要求参数大于 0，对

于小于 0 的时候可采用换元或是下半平面围道积分

(3)

$$\sum_{k=1}^{N}\frac{a_k}{2b_k}e^{-b_k|\tau|} \qquad (a_k, b_k > 0)$$

补充题：

1. 设 Z_1, Z_2 独立，都服从 $U(-1,1)$，定义

$$X(t)=Z_1\cos\lambda t+Z_2\sin\lambda t \quad (t\in R,\lambda\neq 0)$$

(1)证明：X(t)是宽平稳的；

(2)X(t)是严平稳的吗，为什么？

(3)证明 X(t)的均值遍历性成立。

解：

(1) $EX(t)=E(Z_1\cos\lambda t+Z_2\sin\lambda t)=0$

$$\begin{aligned}EX(t)X(s)&=E[(Z_1\cos\lambda t+Z_2\sin\lambda t)(Z_1\cos\lambda s+Z_2\sin\lambda s)]\\&=EZ_1^2\cos\lambda t\cos\lambda s+EZ_2^2\sin\lambda t\sin\lambda s\\&=\sigma^2\cos\lambda(t-s)\\&=\frac{1}{3}\cos\lambda(t-s) \qquad \text{仅与(t-s)相关}\end{aligned}$$

又 $EX^2(t)=\sigma^2<\infty$

故宽平稳

(2) 显然 $X(t)$ 与 $X(t+h)$ 不一定同分布，非严平稳

(3) $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^{2T}(1-\frac{\tau}{2T})R(\tau)d\tau=\lim\limits_{T\to\infty}\frac{1}{T}\int_0^{2T}(1-\frac{\tau}{2T})\frac{1}{3}\cos\lambda\tau\,d\tau=0$

均值遍历性成立。

2. 设 $X(t)=A\cos(\omega t+\Theta)$,其中

$$A\sim f(x)=\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} \quad x>0$$

$$\Theta\sim U(0,2\pi)$$

A 与 Θ 独立， $\omega\neq 0$

证明 X(t)为平稳过程且有均值遍历性。

证明：

$$E\cos(\omega t+\Theta)=\int_0^{2\pi}\frac{1}{2\pi}\cos(\omega t+\Theta)d\Theta=0$$

$$EX(t)=EA*E\cos(\omega t+\Theta)=0$$

$$EA^2=2\sigma^2$$

$$EX(t)X(t+\tau)=EA^2\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta)=\sigma^2\cos\omega\tau=R(\tau)$$

平稳过程得证；均值遍历性证明同上题

$$\lim_{T\to\infty}\frac{1}{T}\int_0^{2T}(1-\frac{\tau}{2T})R(\tau)d\tau=0$$