

Climate Dynamics as an Initial-Value Problem

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This note describes how an emissions path generates carbon concentrations, radiative forcing and temperature changes. The model is deliberately small: three carbon reservoirs, one methane concentration and two temperatures. We first pose a continuous-time ordinary differential equation (ODE) system, then derive the explicit time-stepping scheme used in `Climate_Models.py`. Every stock, flow and initial condition has a distinct role.

1 What is given, what is solved?

Unlike the economic block, the stand-alone climate block contains no optimization. Emissions are prescribed inputs. Once parameters, initial states and an emissions scenario are specified, the task is to solve an initial-value problem, also called a Cauchy problem.

Time t is measured in years on a finite interval $[t_0, t_f]$. Let M_a, M_u, M_ℓ denote atmospheric, upper-ocean and lower-ocean carbon stocks in gigatonnes of carbon (GtC). Let q be atmospheric methane concentration in parts per billion (ppb), and T_a, T_ℓ temperature anomalies relative to the model's reference climate in degrees Celsius. The state and input vectors are

$$x = (M_a, M_u, M_\ell, q, T_a, T_\ell)^\top, \quad e = (e_{\text{CO}_2}, e_{\text{CH}_4})^\top. \quad (1)$$

Interpretation. The six states summarize the system's inherited physical condition. The prescribed inputs are a carbon-dioxide mass flow e_{CO_2} in GtCO₂/year and a methane concentration source e_{CH_4} in ppb/year. A mass-based methane series must therefore be converted before it can enter this model.

For fixed parameters θ , the complete mathematical task is

$$\boxed{\dot{x}(t) = f(x(t), e(t); \theta), \quad x(t_0) = x^0, \quad t \in [t_0, t_f].} \quad (2)$$

Interpretation. The derivative is the instantaneous change of the state per year. Six first-order equations require six initial values. No terminal temperature target or terminal state is imposed: future climate is the output. An inverse problem (find emissions meeting a target) or a control problem (choose emissions to optimize welfare) would be a different task.

All components of x^0 must be specified at the same calendar date. Carbon stocks are positive, methane concentration is positive, and temperature anomalies may have either sign. Stocks cannot be replaced by emissions, nor can an anomaly be interpreted as an absolute temperature.

2 Physical equations and their interpretation

The model is triangular: emissions affect carbon and methane; these determine radiative forcing; forcing drives temperatures. There is no temperature feedback on the carbon cycle in this particular benchmark. That omission is a modeling restriction, not a general physical claim.

Carbon exchange. Let (m_a, m_u, m_ℓ) be reference carbon stocks, with each entry positive. The annual transfer coefficients are $b_{12}, b_{23} \geq 0$. Define reverse rates and the CO₂-to-carbon conversion by

$$b_{21} = b_{12} \frac{m_a}{m_u}, \quad b_{32} = b_{23} \frac{m_u}{m_\ell}, \quad \xi = \frac{12}{44} = \frac{3}{11}. \quad (3)$$

Interpretation. Reverse rates balance the exchange flows at the reference stocks. The factor ξ converts a mass of CO₂ into its carbon content; it is not the fraction of emissions that remains in the atmosphere.

Writing $M = (M_a, M_u, M_\ell)^\top$ and $v_a = (1, 0, 0)^\top$, carbon dynamics are

$$\dot{M} = BM + \xi e_{\text{CO}_2} v_a, \quad B = \begin{pmatrix} -b_{12} & b_{21} & 0 \\ b_{12} & -(b_{21} + b_{23}) & b_{32} \\ 0 & b_{23} & -b_{32} \end{pmatrix}. \quad (4)$$

Interpretation. Each diagonal entry is an outflow rate from a reservoir; each off-diagonal term transfers carbon from another reservoir. Emissions enter the atmosphere. There is no direct atmosphere–lower-ocean exchange. The upper ocean links the fast atmospheric reservoir to the slow deep one.

Two identities are useful for understanding and checking the system:

$$B \begin{pmatrix} m_a \\ m_u \\ m_\ell \end{pmatrix} = 0, \quad \frac{d}{dt}(M_a + M_u + M_\ell) = \xi e_{\text{CO}_2}. \quad (5)$$

Interpretation. Without emissions, the reference stocks are stationary, and internal transfers conserve total carbon. Zero emissions do not generally restore the reference climate: an initially elevated total stock remains inside these three modeled reservoirs and is redistributed among them. The zero eigenvalue associated with carbon conservation is not a numerical instability.

Methane. Let $\bar{q} > 0$ be the reference concentration and $\delta_q > 0$ the removal rate. The one-box methane equation is

$$\dot{q} = -\delta_q(q - \bar{q}) + e_{\text{CH}_4}, \quad \tau_q = \delta_q^{-1}. \quad (6)$$

Interpretation. Excess methane decays with characteristic lifetime τ_q , while the prescribed source replenishes it. The restoring term includes an implicit background source maintaining $\bar{q}$; setting the additional input to zero returns methane toward $\bar{q}$, not toward zero.

Radiative forcing. Let $F_{2\times} > 0$ denote forcing from doubled atmospheric CO₂, and $\alpha_q \geq 0$ the methane forcing coefficient. The forcing diagnostic is

$$F(M_a, q) = F_{2\times} \frac{\log(M_a/m_a)}{\log 2} + \alpha_q(\sqrt{q} - \sqrt{\bar{q}}). \quad (7)$$

Interpretation. Forcing, in W/m², is the radiative energy imbalance induced by the gas concentrations. Doubling M_a from its reference value contributes $F_{2\times}$; the methane term measures forcing relative to $\bar{q}$. Forcing is an algebraic output, not an additional state requiring its own initial condition. This simplified formula omits gas-overlap effects, aerosols and other external forcing components.

Temperature response. Let $S > 0$ be equilibrium climate sensitivity, and set $\lambda = F_{2\times}/S$. With $c_1, c_4 > 0$ and $c_3 > 0$,

$$\begin{aligned}\dot{T}_a &= c_1[F - \lambda T_a - c_3(T_a - T_\ell)], \\ \dot{T}_\ell &= c_4(T_a - T_\ell).\end{aligned}\tag{8}$$

Interpretation. Surface temperature rises with forcing, falls with radiative restoration λT_a , and loses heat to a colder deep ocean. Deep-ocean temperature adjusts more slowly toward the surface temperature. Here λ and c_3 have units $\text{W/m}^2/\text{K}$, c_1 has units $\text{K}/(\text{year W/m}^2)$, and c_4 has units year^{-1} .

For constant forcing and an equilibrium in both temperature boxes,

$$T_a^* = T_\ell^* = \frac{F}{\lambda}, \quad F = F_{2\times} \implies T_a^* = S.\tag{9}$$

Interpretation. Equilibrium climate sensitivity is the eventual response to sustained doubling forcing after ocean adjustment. It is not the temperature reached immediately, nor the warming at a fixed near-term date under a changing emissions scenario.

3 Well-posedness and discretization

Writing an ODE is not enough: its domain and assumptions must allow a solution, and the numerical scheme must respect the properties we intend to study. We separate the continuous problem from its approximation.

Existence, uniqueness and dependence on inputs. Suppose the emissions inputs are bounded and piecewise continuous on $[t_0, t_f]$, reference stocks are strictly positive, and the initial carbon and methane states are strictly positive. On the domain where $M_a > 0$ and $q > 0$, the right-hand side is locally Lipschitz in the state. The linear reservoir equations and the two-box temperature equations therefore give a unique solution on any finite interval that stays in this domain; solutions depend continuously on initial data and inputs.

For nonnegative emissions, the carbon matrix has nonnegative off-diagonal entries and the methane background source is positive. These properties preserve nonnegative stocks; strictly positive initial stocks stay positive on finite intervals. Bounded inputs then keep stocks finite and the forcing well-defined, so the solution extends through t_f . If negative emissions are allowed, positivity must be checked: an input that exhausts atmospheric carbon takes the logarithm outside the model's domain.

Forward Euler on a calendar grid. Let $t_n = t_0 + n\Delta$, with $n = 0, \dots, N$. Holding inputs constant over each interval gives the explicit approximation

$$x_{n+1} = x_n + \Delta f(x_n, e_n; \theta), \quad x_0 = x^0.\tag{10}$$

Interpretation. A derivative is a rate per year; multiplying by Δ converts it into a change over the interval. Forward Euler evaluates every right-hand side at the old state. Under usual smoothness assumptions its global error is first order in Δ ; halving the step is a basic convergence check, not a change in the physical model.

In particular, the carbon and methane steps are

$$\begin{aligned}M_{n+1} &= (I + \Delta B)M_n + \Delta \xi_{e_{\text{CO}_2, n} v_a}, \\ q_{n+1} &= \bar{q} + (1 - \Delta \delta_q)(q_n - \bar{q}) + \Delta e_{\text{CH}_4, n}.\end{aligned}\tag{11}$$

Interpretation. If emissions are stored as annual rates, both source terms require Δ . If instead the input is already a period total, it must not be multiplied by Δ again. The revised stand-alone module consistently uses annual rates for both inputs.

Sufficient conditions for nonnegative reservoir weights are

$$\Delta \max\{b_{12}, b_{21} + b_{23}, b_{32}\} \leq 1, \quad \Delta \delta_q \leq 1. \quad (12)$$

Interpretation. A discrete step should not remove more of a reservoir than it contains. These conditions prevent negative weights in the explicit gas updates; they are sufficient conditions, not necessary ones for every particular trajectory.

For temperatures, write $T = (T_a, T_\ell)^\top$. The update is

$$T_{n+1} = (I + \Delta D)T_n + \Delta \begin{pmatrix} c_1 \\ 0 \end{pmatrix} F(M_{a,n}, q_n), \quad D = \begin{pmatrix} -c_1(\lambda + c_3) & c_1 c_3 \\ c_4 & -c_4 \end{pmatrix}. \quad (13)$$

Interpretation. The forcing and exchange terms are evaluated at the beginning of the interval. For a stable homogeneous discrete temperature response, each eigenvalue η of D must satisfy $|1 + \Delta\eta| < 1$. Thus a stable continuous ODE can still have an unstable explicit discretization if the time step is too large.

4 From the equations to the notebook

The notebook first considers a stationary reference climate, then feeds historical emissions into the model and compares temperature anomalies. Future scenarios are conditional simulations, not forecasts independent of their emissions assumptions.

Code mapping. The names in `Params` and in the simulation matrix correspond to the mathematical objects as follows.

Mathematical object	Python name	Unit
M_a, M_u, M_ℓ	<code>M_AT, M_UP, M_LO</code>	GtC
q	<code>M_CH4</code>	ppb (not a mass)
T_a, T_ℓ	<code>T_AT, T_LO</code>	°C anomaly
e_{CO_2}	<code>E</code>	GtCO ₂ /year
e_{CH_4}	<code>E_CH4</code>	ppb/year
F	<code>F</code>	W/m ²
m_a, m_u, m_ℓ	<code>mat, mup, mlo</code>	GtC
$S, F_{2\times}$	<code>T2XC02, F2XC02</code>	°C; W/m ²
$\bar{q}, \delta_q, \alpha_q$	<code>mch4, delta_ch4, alpha_ch4</code>	As in (6)–(7)

Row zero stores the initial climate state. To compute row $j \geq 1$, `update_path` reads states and emissions from row $j - 1$. The requested final date is included. Consequently, an emissions value recorded at year 2020 acts over the interval starting in 2020; it is not the emissions value for the preceding interval.

The forcing stored in `F[j]` is evaluated from the states in row $j - 1$ and used for that interval's temperature update. It therefore represents interval forcing, whereas `T_AT[j]` is the end-of-interval temperature. The initial forcing entry can remain `NaN`: it is not an independent state. Emissions at the initial date, in contrast, must be provided because they drive the first transition.

Calibration and initial conditions. The default simulation uses $\Delta = 5$ years and begins in 2015. The historical exercise overrides the step to one year and resets the states to a preindustrial reference. Those are different initial-value problems and must not be combined accidentally.

Quantity	Default value	Role
(m_a, m_u, m_ℓ)	(588, 360, 1720)	Reference carbon stocks
(M_a^0, M_u^0, M_ℓ^0)	(851, 460, 1740)	Initial carbon stocks
(b_{12}, b_{23})	(0.024, 0.0014)	Annual transfer rates
$(\bar{q}, q^0, \delta_q)$	(722, 1860, 1/9)	Methane reference, initial value, decay
$(F_{2\times}, S)$	(3.3613, 3.1)	Forcing scale and climate sensitivity
(c_1, c_3, c_4)	(0.0201, 0.0176, 0.005)	Temperature-response coefficients
(T_a^0, T_ℓ^0)	(0.85, 0.007)	Initial temperature anomalies
α_q	0.036	Methane forcing coefficient

Data are not model states. The historical CO₂ series is an emissions flow: convert tonnes to gigatonnes before applying ξ . Do not feed CO₂-equivalent methane emissions directly into a concentration equation: the warming-potential conversion and the conversion from methane mass to ppb are separate assumptions. Match all series by calendar year rather than array position. For temperature comparisons, harmonize reference periods for anomalies; otherwise a level difference may be a baseline mismatch rather than a model error. These simplified exercises do not estimate all model parameters or reproduce a full climate-model calibration.

Reproducible checks. Start at $(m_a, m_u, m_\ell, \bar{q}, 0, 0)$ with zero inputs: every state should remain stationary. Next add a CO₂ pulse and verify that the change in total carbon equals $\Delta \xi e_{\text{CO}_2}$ at each step. Then run a constant-forcing temperature experiment and check convergence to (9). Finally compare annual and finer-step simulations at common dates, preserving the same annual emissions inputs. These tests distinguish equation errors, unit errors and discretization errors before interpreting differences between scenarios.