

Modélisation, estimation, simulation des risques climatiques
Session 2

An Introduction to a Climate Model

Gauthier Vermandel
Ecole polytechnique

Schedule of the day

Objectives

- Understanding the **basics** of climate models to predict world temperatures.
- Simulating climate models on historical data and make some forecasts conditional on emission scenarios.

Materials:

- notebook_session2.ipynb *carbon-cycle and temperature dynamics.*
- climate_model.pdf *ODEs, initial conditions and discretization.*
- Climate_Models.py *package to simulate reduced form climate models.*

Introduction

Intro: Why should economists care about climate science?

- Climate change is driven by physical processes that can be **summarized in a simple causal chain**.
- Integrated Assessment Models (IAMs) like DICE or FAIR rely on this chain.
- To interpret or extend IAMs, we need to understand the building blocks.

The Climate-Economy Chain

Emissions → Concentrations → Radiative Forcing → Temperature → Impacts

- Each arrow corresponds to a physical process.
- IAMs implement reduced-form versions of these processes.

Intro: Climate science is about big models

How do climate scientists study the system?

- **General Circulation Models (GCMs)** and **Earth System Models (ESMs)**:
 - Simulate atmosphere, ocean, cryosphere, land surface, and biogeochemical cycles.
 - Built on physical laws (fluid dynamics, thermodynamics, radiative transfer).
 - Used in IPCC reports for future climate projections.
- Strengths:
 - High fidelity, spatial detail, many processes included.
 - Provide benchmarks for scientific understanding.
- Limitations for Integrated Assessment Modelling:
 - Extremely computationally intensive (supercomputers).
 - Results not easily integrated into optimization or policy models.

Intro: Focus of This Course

Today's focus: Economics of climate change

- We will work with **Reduced-form Climate Models (RCMs)**:
 - Tractable and transparent → usable in economic models.
 - Capture the key physical mechanisms without full Earth System complexity.

The Climate-Economy Chain

Emissions → Concentrations → Radiative Forcing → Temperature

Roadmap: We will follow this chain step by step, from *emissions inputs* to *temperature outcomes*, highlighting at each stage how reduced-form models approximate the underlying processes.

Emissions (the inputs)

Emissions: What are emissions?

- **Definition:** Anthropogenic greenhouse-gas (GHG) *flows* into the atmosphere (mainly CO₂, CH₄, N₂O).
- **Focus today:** CO₂ (fossil fuel + land use). Others can be added (e.g., CH₄) as separate inputs.
- Emissions are flows $\Rightarrow$ they feed atmospheric *stocks* in the carbon cycle.

Emissions: Three main green house gases

The three main sources:

- **Carbon dioxide (CO_2)**
 - From fossil fuel combustion, cement, land-use change.
 - Long-lived: persists for centuries.
 - $\sim 76\%$ of global GHG emissions ($\text{CO}_2\text{-eq}$).
- **Methane (CH_4)**
 - From agriculture (rice, livestock), fossil fuel extraction, waste.
 - Short lifetime (~ 10 years) but high warming power.
 - $\sim 16\%$ of global GHG emissions.
- **Nitrous oxide (N_2O)**
 - From fertilizers, biomass burning, industry.
 - Lifetime ~ 120 years.
 - $\sim 6\text{--}7\%$ of global GHG emissions.

To summarize this section, you only need to remember:

- In principle, models can take three distinct exogenous inputs:

$$\{E_t^{CO_2}\}_{t \geq 0}, \quad \{E_t^{CH_4}\}_{t \geq 0}, \quad \{E_t^{N_2O}\}_{t \geq 0}.$$

- In simplified reduced-form models, only CO₂ emissions are modeled explicitly; CH₄ and N₂O are often represented by an additional exogenous forcing term.

Concentrations

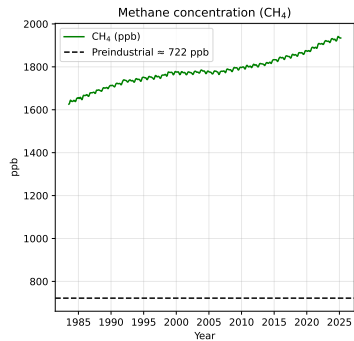
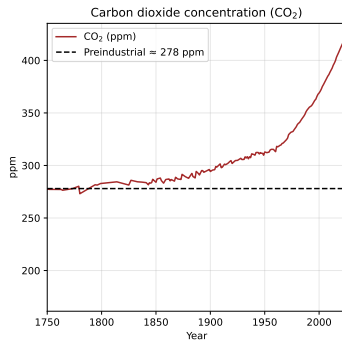
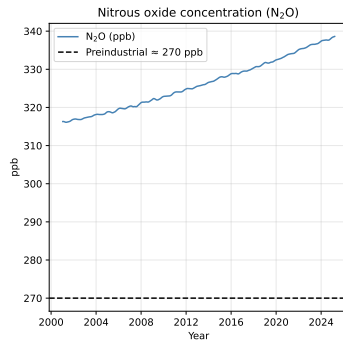
Concentrations: from to emissions to concetration

[Emissions (flows)] $\rightarrow$ [Atmospheric Concentration (stock)] $\rightarrow \dots$

From emissions (flows) to concentrations (stocks):

- Climate does not react directly to annual emissions, but to the **cumulative stock of gases** in the atmosphere.
- Emissions are measured in flows (e.g. GtC/yr).
- Once emitted, gases accumulate in the atmosphere.
- What matters for climate forcing is the concentration (ppm or ppb).

Concentrations: Atmospheric concentrations



Concentrations: ODE and discrete periods

Let s be calendar time, $s_t = s_0 + t\Delta$, and t the period index. The next observation is $t + 1$, separated by Δ years.

$$\frac{d\mathbf{M}_g(s)}{ds} = \mathbf{A}_g(\mathbf{M}_g(s) - \overline{\mathbf{M}}_g) + \mathbf{b}_g E^g(s).$$

$$\boxed{\mathbf{M}_{g,t+1} = \mathbf{M}_{g,t} + \Delta \mathbf{A}_g(\mathbf{M}_{g,t} - \overline{\mathbf{M}}_g) + \Delta \mathbf{b}_g E_t^g.}$$

- $\overline{\mathbf{M}}_g$: natural equilibrium; $\mathbf{M}_{g,0}$: given initial state. They need not coincide.
- $\mathbf{A}_g$: annual transfer/removal rates; $\mathbf{b}_g$: injection and unit conversion. E_t^g is an annual emissions rate.
- Forward Euler: evaluate all terms at the old state, then multiply rates by Δ . Zero input leaves the reference state stationary.

Cncentrations: Some precisions

Modeling choice. The number of boxes n_g determines precision:

- $n_g = 1$: a single decay process (e.g. CH_4 , N_2O).
- $n_g = 2$ – 3 : separates atmosphere, fast reservoirs (biosphere/upper ocean), and slow reservoirs (deep ocean) for CO_2 .
- Larger n_g : more realistic sequestration processes (soil carbon, permafrost, multiple ocean layers) at the cost of extra parameters.

Conservation:

- the system is physical, so total mass is conserved: transfers occur *between* boxes, with no loss outside the system (except for explicitly modelled sinks).

Concentrations: the three-box carbon cycle

$$\mathbf{M}_t = (M_{AT,t}, M_{UP,t}, M_{LO,t})^\top,$$

$$\mathbf{M}_{t+1} = \bar{\mathbf{M}} + \Phi(\mathbf{M}_t - \bar{\mathbf{M}}) + \Delta\Xi E_t^{CO_2}.$$

$$\Phi = I + \Delta\mathbf{A}, \quad \Phi\bar{\mathbf{M}} = \bar{\mathbf{M}}, \quad \Xi = (3/11, 0, 0)^\top.$$

- Stocks are in GtC; emissions are in GtCO₂/year. Only the atmosphere receives emissions directly.
- The equilibrium identity gives the equivalent level equation $\mathbf{M}_{t+1} = \Phi\mathbf{M}_t + \Delta\Xi E_t^{CO_2}$.
- Column sums of Φ equal one: internal transfers conserve total carbon, which rises by $\Delta(3/11)E_t^{CO_2}$.
- Reference stocks: $\bar{\mathbf{M}} = (588, 360, 1720)^\top$; default 2015 state: $\mathbf{M}_0 = (851, 460, 1740)^\top$.

Concentrations: methane decay relative to a baseline

$$M_{t+1}^{CH_4} = \overline{M}^{CH_4} + (1 - \Delta\delta_{CH_4})(M_t^{CH_4} - \overline{M}^{CH_4}) + \Delta\kappa_{CH_4}E_t^{CH_4}.$$

- $\overline{M}^{CH_4} = 722$ ppb is the preindustrial reference. The default initial concentration is $M_0^{CH_4} = 1860$ ppb.
- Only the *excess above the reference* decays. Without additional emissions, methane returns towards 722 ppb, not zero.
- Course calibration: $\delta_{CH_4} = 1/9 \text{ year}^{-1}$, so the characteristic lifetime is 9 years.
- For emissions in $\text{MtCH}_4/\text{year}$, $\kappa_{CH_4} = 1/2.78 \text{ ppb/MtCH}_4$. The notebook converts inputs to ppb/year before calling the module; the module therefore uses $\kappa_{CH_4} = 1$ on that converted input.

Radiative Forcing

Forcing: Definition

[Emissions (flows)] $\rightarrow$ [Concentration (stock)] $\rightarrow$ [Forcing (flow)] $\rightarrow \dots$

Key idea. Radiative forcing is the *side effect of rising concentrations*: it increases the thickness of the atmospheric “blanket” that traps heat.

- **Baseline:** in the preindustrial climate, incoming solar $\approx$ outgoing infrared (energy balance).
- **Forcing:** any perturbation that shifts this balance.
 - Positive forcing $\Rightarrow$ warming (extra blanket from GHGs).
 - Negative forcing $\Rightarrow$ cooling (reflective aerosols, volcanic dust).
- **Units.** Watts per square meter (W/m^2).

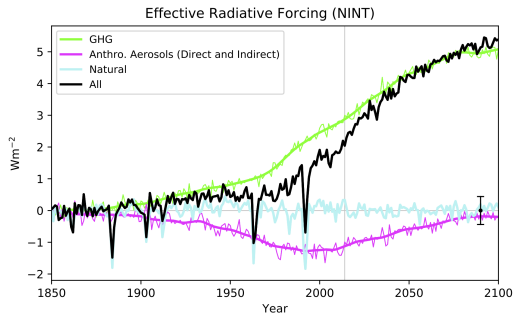
Forcing: reference levels and timing

Forcing is an energy imbalance relative to the reference climate, not an additional state variable.

$$F_t = \sum_{g \in \{CO_2, CH_4, N_2O\}} F_t^g + F_t^{\text{ex}}, \quad F_t^g = f_g(M_{AT,t-1}^g) - f_g(\overline{M}_{AT}^g).$$

- $\overline{M}_{AT}^g$ is the preindustrial level, not the initial concentration at the start of a modern simulation.
- Code convention: F_t acts during the transition from $t - 1$ to t and is computed from the *old* concentrations $M_{AT,t-1}^g$.
- Equivalently, $F_{t+1} = \sum_g [f_g(M_{AT,t}^g) - f_g(\overline{M}_{AT}^g)] + F_{t+1}^{\text{ex}}$.
- The course module includes CO_2 and CH_4 only; no overlap correction, explicit N_2O or other external forcing is included.

Forcing: Some illustration



- **Pink line:** Anthropogenic aerosols $\Rightarrow$ *negative forcing*.
e.g. sulfate aerosols (from burning coal and oil- SO_2 emissions), very reflective: scatter sunlight back to space, short lifetime (days-weeks).
- **Cyan line:** Natural forcings (solar variability, volcanic eruptions).

Logarithmic forcing relative to preindustrial carbon

$$F_t^{CO_2} = F_{2 \times CO_2} \frac{\ln(M_{AT,t-1}^{CO_2} / \overline{M}_{AT}^{CO_2})}{\ln 2}.$$

- Use the same units in numerator and denominator. In the module, carbon is in GtC and $\overline{M}_{AT}^{CO_2} = 588$ GtC.
- Course calibration: $F_{2 \times CO_2} = 3.3613$ W/m². At twice the reference stock, the forcing is exactly $F_{2 \times CO_2}$.
- With the default initial stock $M_{AT,0}^{CO_2} = 851$ GtC, the first-interval CO₂ forcing is

$$F_1^{CO_2} = 3.3613 \frac{\ln(851/588)}{\ln 2} \simeq 1.79 \text{ W/m}^2.$$

Square-root forcing relative to preindustrial methane

$$F_t^{CH_4} = \alpha_{CH_4} \left(\sqrt{M_{t-1}^{CH_4}} - \sqrt{\overline{M}^{CH_4}} \right).$$

- Reference: $\overline{M}^{CH_4} = 722$ ppb; default initial concentration: $M_0^{CH_4} = 1860$ ppb.
- $\alpha_{CH_4} = 0.036 \text{ W/m}^2/\sqrt{\text{ppb}}$. The forcing vanishes at the reference, not at the modern initial state.
- First-interval methane forcing:

$$F_1^{CH_4} = 0.036 \left(\sqrt{1860} - \sqrt{722} \right) \simeq 0.59 \text{ W/m}^2.$$

Temperatures (output)

[Emissions (flows)] $\rightarrow$ [Concentration (stock)] $\rightarrow$ [Forcing (flow)] $\rightarrow$ [Temperature (stock)]

- **Key point:** Radiative forcing $\neq$ instantaneous warming.
Temperatures behave as a *state variable*: they accumulate the effect of past forcings, with strong inertia.
- **Sources of inertia:**
 - The ocean absorbs most of the excess heat, releasing it slowly.
 - Ice sheets and cryosphere respond on multi-decadal to millennial scales.
 - Feedbacks (clouds, carbon cycle) operate with delays.
- **Illustration:** Even if CO₂ concentrations stabilized today, global temperatures would still rise for decades (“committed warming”).

Temperatures: ODE and explicit time stepping

Temperature anomalies are measured relative to the reference climate, so $\bar{\mathbf{T}} = \mathbf{0}$ (not necessarily $\mathbf{T}_0$).

$$\frac{d\mathbf{T}(s)}{ds} = \mathbf{A}_T(\mathbf{T}(s) - \bar{\mathbf{T}}) + \mathbf{B}_T F(s).$$

$$\boxed{\mathbf{T}_t = \mathbf{T}_{t-1} + \Delta \left[\mathbf{A}_T(\mathbf{T}_{t-1} - \bar{\mathbf{T}}) + \mathbf{B}_T F_t \right].}$$

- All temperature terms on the right use $t - 1$; F_t is computed from the gas stocks at $t - 1$.
- Equivalently, advance from $\mathbf{T}_t$ to $\mathbf{T}_{t+1}$ using F_{t+1} . The index advances by one, not by Δ .
- The reference is already subtracted in an anomaly. Initial anomalies are given: zero for the preindustrial experiment, $(0.85, 0.007)^\top$ in the default 2015 calibration.

Temperatures: the two-box model

$$T_t^{AT} = T_{t-1}^{AT} + \Delta c_1 \left[F_t - \lambda T_{t-1}^{AT} - c_3 (T_{t-1}^{AT} - T_{t-1}^{LO}) \right],$$

$$T_t^{LO} = T_{t-1}^{LO} + \Delta c_4 (T_{t-1}^{AT} - T_{t-1}^{LO}), \quad \lambda = \frac{F_{2 \times CO_2}}{T_{2 \times CO_2}}.$$

- T^{AT}, T^{LO} are anomalies relative to the preindustrial reference; there is no additional subtraction of the modern initial temperature.
- $c_1 F_t$ heats the surface; $c_1 \lambda T_{t-1}^{AT}$ is radiative restoration; the difference $T_{t-1}^{AT} - T_{t-1}^{LO}$ drives ocean heat uptake.
- $F_t = F_t^{CO_2} + F_t^{CH_4}$ uses gas stocks at $t - 1$. The deep ocean also uses the old surface temperature, not T_t^{AT} .
- Give both initial anomalies at the chosen start date. Only a preindustrial initial state has $T_0^{AT} = T_0^{LO} = 0$.

Temperatures: equilibrium under doubled CO₂

Hold atmospheric CO₂ at twice its reference level and set all other forcing contributions to zero.

$$M_{AT}^{CO_2} = 2\overline{M}_{AT}^{CO_2} \implies F = F_{2\times CO_2} = 3.3613 \text{ W/m}^2.$$

$$T_t^{AT} = T_{t-1}^{AT} + \Delta c_1 \left[F_{2\times CO_2} - \lambda T_{t-1}^{AT} - c_3(T_{t-1}^{AT} - T_{t-1}^{LO}) \right],$$

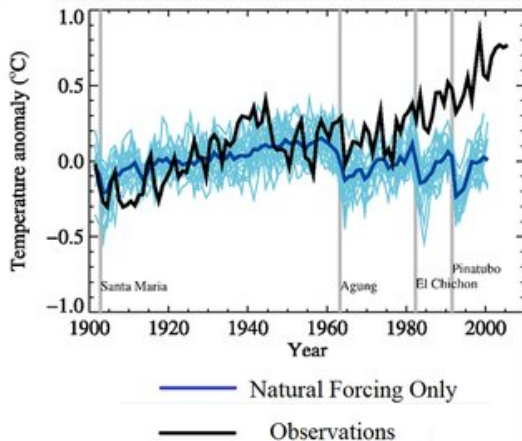
$$T_t^{LO} = T_{t-1}^{LO} + \Delta c_4(T_{t-1}^{AT} - T_{t-1}^{LO}).$$

At equilibrium both temperature changes vanish:

$$T^{AT,*} = T^{LO,*} = \frac{F_{2\times CO_2}}{\lambda} = T_{2\times CO_2} = 3.1 \text{ }^\circ\text{C}.$$

This is the long-run anomaly, not an immediate jump in temperature. The experiment holds concentration fixed; it is not a zero-emissions run.

Temperatures: Human vs. Natural Forcings



To wrap up: stocks, forcing and temperature

Consider an instantaneous addition of 1 GtC to the atmosphere, starting from the preindustrial reference $\overline{M}_{AT} = 588$ GtC.

$$M_{AT} : 588 \mapsto 589 \text{ GtC}, \quad \Delta F = 3.3613 \frac{\ln(589/588)}{\ln 2} \simeq 0.0082 \text{ W/m}^2.$$

- Atmospheric carbon changes first; forcing responds to the stock relative to its reference. Temperature does not jump instantaneously.
- A real pulse is then redistributed between the three carbon boxes; its subsequent forcing is not constant.
- Only if the extra atmospheric stock were *held fixed* would the long-run temperature change be

$$\Delta T^* = \frac{\Delta F}{\lambda} \simeq 0.0076 \text{ }^\circ\text{C}, \quad \lambda = 3.3613/3.1.$$

Thank you!

gauthier.vermandel@polytechnique.edu