

Optimal Growth as a Discrete-Time Control Problem

Gauthier Vermandel
École polytechnique

This note formulates the saving decision as a constrained dynamical system. The aim is to distinguish what is given, what the planner chooses, and what the equations determine. It then connects the mathematical problem to `OptimalGrowth.py`. Basic multivariable calculus and discrete-time recursions are the only prerequisites.

1 The general finite-horizon problem

A control problem combines a state equation with a criterion for ranking admissible trajectories. It is not defined by an objective alone: the initial state, feasible controls, time horizon and terminal treatment are also data.

Let $t = 0, \dots, H - 1$ index decision periods of length $\Delta > 0$ years. The state $x_t \in X \subseteq \mathbb{R}^n$ records inherited quantities; the control $a_t \in U_t(x_t) \subseteq \mathbb{R}^m$ is chosen at date t . A known path z_t contains exogenous inputs. An admissible trajectory satisfies

$$x_{t+1} = F_t(x_t, a_t; z_t), \quad x_0 = \bar{x}, \quad a_t \in U_t(x_t), \quad x_t \in X. \quad (1)$$

Interpretation. The transition map propagates today's state and decision into tomorrow's state. The initial condition is data, not a decision. A control sequence is feasible only if its entire induced state path is admissible.

For a period reward r_t , terminal payoff Φ and annual discount rate $\rho \geq 0$, the problem is

$$\sup_{(a_0, \dots, a_{H-1}) \text{ admissible}} J_H, \quad J_H = \sum_{t=0}^{H-1} \beta^t r_t(x_t, a_t; z_t) + \beta^H \Phi(x_H), \quad \beta = (1 + \rho)^{-\Delta}. \quad (2)$$

Interpretation. The sum values rewards during the H decision periods; Φ values the state left after the final decision. Setting $\Phi = 0$ is a substantive assumption, not an innocuous omission. Multiplying all rewards and the terminal payoff by the same positive constant leaves the optimizer unchanged. With a fixed Δ , this covers a common factor Δ when rewards are recorded as annual flows.

We use a supremum because existence is not automatic. For a finite horizon, compact feasible control sets, continuous transitions and rewards, and a nonempty closed feasible set ensure that a maximum is attained. Differentiability is additionally needed for the first-order conditions below.

2 Specialization: households, production and aggregation

The economic question is how much output to consume now and how much to invest for later. There is one good, one capital stock and an exogenous population. A social planner internalizes the intertemporal allocation; there is no government, tax system or decentralized price-setting problem in this benchmark.

Households and welfare. Let $L_t > 0$ be population and $C_t > 0$ aggregate consumption. A factor $\kappa > 0$ converts model consumption units into the units used in utility. Constant-relative-risk-aversion (CRRA) utility is

$$c_t = \frac{\kappa C_t}{L_t}, \quad u(c) = \begin{cases} (c^{1-\gamma} - 1)/(1-\gamma), & \gamma > 0, \gamma \neq 1, \\ \log c, & \gamma = 1, \end{cases} \quad r_t = L_t u(c_t). \quad (3)$$

Interpretation. Welfare is the sum of identical individual utilities. Marginal utility is $u'(c) = c^{-\gamma} > 0$ and decreases with consumption. In this deterministic model γ governs consumption smoothing; the intertemporal elasticity of substitution is $1/\gamma$. Its usual risk-aversion interpretation matters only when uncertainty is introduced.

Firms and technology. For productivity $A_t > 0$, capital $K_t > 0$ and $0 < \alpha < 1$,

$$Y_t = f_t(K_t) := A_t K_t^\alpha L_t^{1-\alpha}, \quad f'_t(K_t) = \alpha \frac{Y_t}{K_t}. \quad (4)$$

Interpretation. Production uses inherited capital and the available population. The marginal product f'_t measures the extra annual output produced by one additional unit of capital; diminishing returns make this marginal benefit fall as capital increases.

Aggregation and the state transition. The saving share s_t allocates output between consumption and investment:

$$I_t = s_t f_t(K_t), \quad C_t = (1 - s_t) f_t(K_t), \quad Y_t = C_t + I_t. \quad (5)$$

Interpretation. Saving is not a separate good: it is the part of output invested rather than consumed. Once K_t , s_t , A_t and L_t are known, Y_t , I_t and C_t are derived quantities, not independent controls.

Write $d = (1 - \delta)^\Delta$, where $0 \leq \delta < 1$ is annual depreciation. The economic state evolves according to

$$K_{t+1} = F_t(K_t, s_t) := dK_t + \Delta s_t f_t(K_t), \quad K_0 = \bar{K} > 0. \quad (6)$$

Interpretation. Next period's capital is surviving capital plus investment over Δ years. Output, consumption and investment are annual flows; capital is a stock. This is the course's discrete accounting convention, not the exact solution of a continuous-time capital equation with within-period depreciation of new investment.

Exogenous inputs. With initial values $A_0, L_0 > 0$, the module constructs

$$A_{t+1} = \frac{A_t}{1 - g_A e^{-\delta_A \Delta(t+1)}}, \quad L_{t+1} = L_t \left(\frac{L_\infty}{L_t} \right)^{\ell_g}. \quad (7)$$

Interpretation. Productivity growth declines over time if $0 < g_A < 1$ and $\delta_A > 0$; it vanishes in the baseline $g_A = 0$. For $0 < \ell_g < 1$, the logarithmic population gap to L_∞ contracts by $1 - \ell_g$ each step. This is geometric convergence in logs, not the continuous logistic equation. The baseline $\ell_g = 0$ freezes population. Both g_A and ℓ_g are step-specific parameters; changing Δ requires attention to their calibration.

Substituting the derived quantities gives the complete reduced problem:

$$\begin{aligned} & \sup_{s_0, \dots, s_{H-1}} \sum_{t=0}^{H-1} \beta^t R_t(K_t, s_t) + \beta^H \Phi(K_H), \\ & R_t(K, s) = L_t u \left(\frac{\kappa(1-s)f_t(K)}{L_t} \right), \\ & K_{t+1} = dK_t + \Delta s_t f_t(K_t), \quad K_0 = \bar{K}, \\ & K_t > 0, \quad 0 \leq s_t < 1, \quad t = 0, \dots, H-1. \end{aligned} \quad (8)$$

Interpretation. There is one endogenous state, K_t , and one control, s_t . The known pair (A_t, L_t) enters through the time-dependent maps f_t and R_t . The strict upper bound keeps consumption positive. For a compact numerical feasible set one can impose $s_t \leq 1 - \varepsilon_s$ with fixed $\varepsilon_s > 0$.

3 Optimality and economic interpretation

Dynamic programming makes the trade-off in the objective explicit: a saving choice sacrifices current utility but changes the value of the capital inherited tomorrow. The Euler equation is a local expression of this trade-off, not a substitute for the constraints or terminal data.

Let $V_t(K)$ be optimal remaining welfare, discounted to date t . The Bellman recursion is

$$V_H(K) = \Phi(K), \quad V_t(K) = \sup_{0 \leq s < 1} \{R_t(K, s) + \beta V_{t+1}(F_t(K, s))\}. \quad (9)$$

Interpretation. The planner compares immediate utility with the discounted continuation value. With $\Phi = 0$, saving in the last period gives no future reward, so $s_{H-1} = 0$. This finite-horizon end effect is why the numerical planning window must extend beyond the dates we want to plot.

At a differentiable interior optimum, define the shadow value $\lambda_{t+1} = V'_{t+1}(K_{t+1})$. Differentiating with respect to saving gives

$$-\kappa u'(c_t) f_t(K_t) + \beta \lambda_{t+1} \Delta f_t(K_t) = 0, \quad \kappa u'(c_t) = \beta \Delta \lambda_{t+1}. \quad (10)$$

Interpretation. One extra unit of annual investment costs $\kappa u'(c_t)$ in current aggregate utility and creates Δ units of capital with discounted value $\beta \lambda_{t+1}$. At a lower-bound optimum the derivative of the Bellman objective with respect to s is nonpositive; at an imposed upper bound it is nonnegative. Equality applies only to an interior saving choice.

The envelope condition, combined with (10) at two consecutive interior dates, yields

$$u'(c_t) = \beta u'(c_{t+1}) [d + \Delta f'_{t+1}(K_{t+1})], \quad 0 \leq t \leq H - 2. \quad (11)$$

Interpretation. Giving up consumption today earns surviving capital plus its marginal product tomorrow. Along an optimal interior path, the utility cost equals the discounted utility benefit. Population does not appear as a separate multiplier because welfare is population-weighted and consumption is per capita using the same contemporaneous L_t .

For CRRA utility, an equivalent growth-rate expression is

$$\frac{c_{t+1}}{c_t} = \left[\beta \left(d + \Delta \alpha \frac{Y_{t+1}}{K_{t+1}} \right) \right]^{1/\gamma}. \quad (12)$$

Interpretation. Consumption rises when the discounted gross return exceeds one. A larger γ dampens the consumption-growth response to a given return. This equality need not hold at control bounds or across successive re-optimizations that use different terminal dates.

4 Numerical implementation and checks

The module uses direct optimization (also called single shooting): given a vector of saving rates, it simulates the state recursion and evaluates welfare. A bound-constrained solver then varies the saving vector. This differs from solving the Bellman equation on a capital grid.

Dates and array rows. Mathematical t labels the beginning of an interval. In the revised module, row zero stores only initial states; row $j \geq 1$ stores the decision and flows for the interval ending at that row. Precisely,

$$K[j] = K_j, \quad s[j] = s_{j-1}, \quad (Y, I, C)[j] = (Y_{j-1}, I_{j-1}, C_{j-1}). \quad (13)$$

Interpretation. Production in row j uses $K[j-1]$, $A[j-1]$ and $L[j-1]$. Its investment produces $K[j]$. The initial control and flows remain NaN and are never inputs to the transition. State and flow curves therefore refer respectively to dates and intervals ending at those dates.

Objective and horizon. The code sets $\kappa = 1000$, uses $\Phi = 0$, and minimizes negative welfare with the limited-memory BFGS bound-constrained algorithm (L-BFGS-B). Its current bounds allow $s = 1$ and its utility evaluation floors consumption at 10^{-12} . That numerical regularization is not identical to the strictly positive-consumption problem (8); check that the floor is inactive in a reported solution.

For positive ρ and a threshold $0 < \tau < 1$, the implemented window is

$$H = 1 + \min\{m \in \mathbb{N} : \beta^m \leq \tau\}, \quad \tau = \text{p.toly}. \quad (14)$$

Interpretation. The algorithm optimizes H decisions, retains the first one, moves forward one period and optimizes again. This is a rolling-horizon policy, not one optimization over the displayed sample. The threshold limits a discount factor; by itself it is not a bound on welfare truncation error. The loop requires $\rho > 0$, although a fixed-horizon mathematical problem can also use $\rho = 0$.

Calibration, not empirical estimation. The baseline isolates capital accumulation by freezing productivity and population. These values are supplied inputs, not parameters estimated from observations in this notebook.

Parameter	Value	Meaning
$\Delta, \bar{K}$	1, 3	Years per step; initial capital
α, δ	0.30, 0.06	Capital elasticity; annual depreciation
A_0, g_A, δ_A	1, 0, 0.005	Productivity and its growth parameters
L_0, ℓ_g	1, 0	Normalized population; convergence speed
L_∞	10500	Inactive population target in the baseline
ρ, γ	0.015, 2	Annual discount rate; utility curvature
τ	10^{-3}	Discount-factor threshold

If population growth is activated, L_0 and L_∞ must share a consistent unit: a normalized population of one cannot be interpreted as millions without an explicit conversion.

Verification and experiments. First verify positive finite consumption and capital, admissible saving, the resource identity (5), and the transition (6) on every simulated interval. Then compare interior Euler residuals, alternative starting guesses and longer planning windows. A successful solver return is useful but not a proof of global optimality. Finally, vary depreciation or the capital share one at a time and interpret the saving response through the return in (12). Population and productivity experiments should keep units and initial conditions explicit.