

Modélisation, estimation, simulation des risques climatiques
Session 1

Introducing an Economic Block

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Technical requirements

- **Python** preferably with conda

if not yet installed, do it now it takes ~15min

- <https://anaconda.org/anaconda/conda>
- Exercises in python are done under **Jupyter notebooks**
- All materials are available on Moodle
<https://moodle.ip-paris.fr/course/view.php?id=18199>
- Main python packages:

[sys] [os] [math] [numpy] [pandas] [matplotlib] [pandas] [spicy]

Schedule of the day

Objectives

- Understanding the **basics** of macroeconomic model with saving/consumption.
- Simulating macroeconomic models .

Materials:

- notebook_session1.ipynb *optimal growth and saving decisions.*
- growth_model.pdf *mathematical formulation and interpretation.*
- OptimalGrowth.py *python package solving optimal growth models.*

Some proofs:

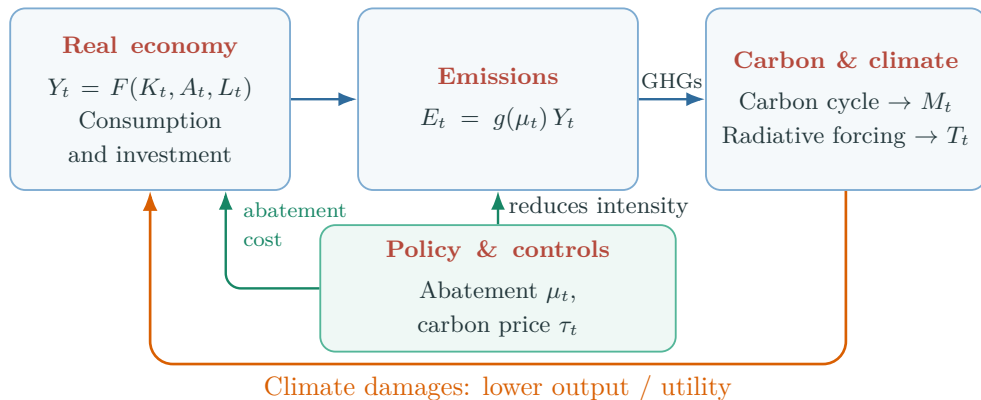
- Olivier Loisel's lectures on growth models [[Lecture I](#)] [[Lecture II](#)]

Introduction

Intro: what is an integrated assessment model?

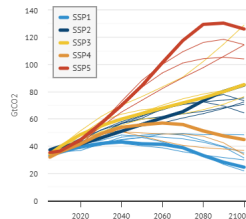
- **Definition:** IAMs are a combination of science from different fields: economics, climate, and energy systems into a unified quantitative framework.
- **Purpose:** Provide a tractable way to evaluate long-run interactions between the economy and the climate.
- **Institutional use:**
 - **IPCC** (Intergovernmental Panel on Climate Change) relies on IAMs to define emission scenarios (e.g. SSP pathways).
 - **Central banks, NGFS, OECD, IEA, World Bank** use IAM-based tools for stress testing, policy design, and energy forecasts.
- IAMs are **quantitative tools** used for scenario design, policy evaluation, and to inform international climate negotiations.

Intro: IAM schematic: economy-climate feedbacks

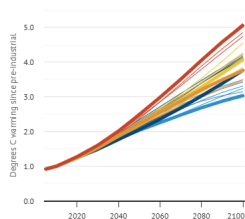


Intro: Illustration of IAM output

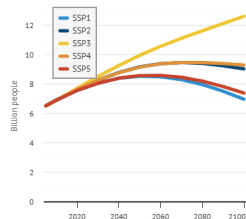
CO2 emissions for SSP baselines



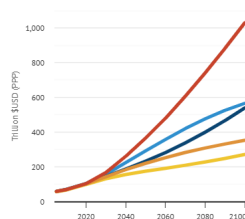
Global mean temperature



Global population



Global GDP



Notes: Illustrative SSP baseline trajectories across models.

Intro: What's inside an IAM?

- **Two core blocks** today we treat them as *separate objects*:

- **Economic block** (2 hours today)

Goal: optimal growth / social planner chooses saving (and later abatement).

Assumptions (standard in macro, exotic for mathematicians): Cobb-Douglas technology, CRRA preferences, competitive pricing, simple capital law.

Why we use them: analytical clarity, stability, (often) uniqueness of equilibrium, and transparent welfare.

- **Climate block** (2 hours today)

Goal: map emissions to concentrations, forcing, and temperature.

Structure: few ODEs / linear difference equations (carbon cycle boxes, 2-box temperature), calibrated to climate science.

Intro: The big picture: from optimal growth to modern IAMs

- **Optimal growth tradition** in economics:
 - **Ramsey (1928)**: how much to save vs. consume over time?
 - **Cass (1965) - Koopmans (1965)**: rigorous optimal growth with capital accumulation, stability and uniqueness of equilibrium.

Provides the *benchmark model* of a social planner maximizing intertemporal welfare.

- **Nordhaus (1992)**: integrated this framework with climate science.
 - Extended optimal growth to include *emissions, carbon cycle, temperature*.
 - Added *damages* and *abatement* as economic choices.
 - Created the **DICE model**, a prototype IAM (Integrated Assessment Model).
- **Big picture**: IAMs are the continuation of the optimal growth tradition, enriched with climate dynamics.

Core building blocks

Building Blocks: main ingredients

- **Discrete time** framework
 - Period length fixed (e.g. annual).
 - Easier to match with economic and climate data.
- **Notation:** X_t denotes the value of variable X in period t .
- **Centralized economy** (social planner)
 - A single agent (the planner) chooses allocations for society $\rightarrow$ deterministic optimal control problem.
- **Aggregate representation**
 - Output, capital, consumption, labor all modeled as aggregate variables.
 - Abstracts from heterogeneity to focus on long-run dynamics.

Building Blocks: Production function I

- **Gross Domestic Product (GDP)**: total value of goods and services produced in one period. → It is a **flow**, not a stock.
- GDP is calculated from a production function $F(\cdot)$ which we define as “Cobb-Douglas”:

$$Y_t = F(K_t, A_t L_t) = A_t K_t^\alpha L_t^{1-\alpha}$$

- $Y_t \geq 0$: output in period t (GDP),
- $A_t \geq 0$: productivity (technology),
- $K_t \geq 0$: capital,
- $L_t \geq 0$: labor,
- $\alpha \in [0, 1]$: capital share (~ 0.3 empirically).

Building Blocks: Production function II

Denoting by $k_t = \frac{K_t}{A_t L_t}$ the stock of capital per effective-labor unit, we get

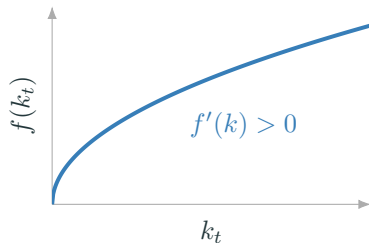
$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} F(K_t, A_t L_t) = F(k_t, 1) \equiv f(k_t)$$

where f has the following properties:

1. $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $k \mapsto f(k)$, with $f(0) = 0$,
2. f is **strictly increasing**: $\forall k \in \mathbb{R}^+$, $f'(k) > 0$,
3. f is **strictly concave**: $\forall k \in \mathbb{R}^+$, $f''(k) < 0$,
4. f satisfies the **Inada conditions**: $\lim_{k \rightarrow 0} f'(k) = +\infty$ and $\lim_{k \rightarrow +\infty} f'(k) = 0$:
 - $\lim_{k \rightarrow 0} f'(k)$ infinite $f'(k)$ guarantees that it is always worthwhile to accumulate at least some capital: avoid “poverty trap” at $k = 0$
 - $\lim_{k \rightarrow +\infty} f'(k)$ prevents infinite growth simply from capital accumulation alone.

Building Blocks: Production function III

Output increases with capital



Marginal product decreases

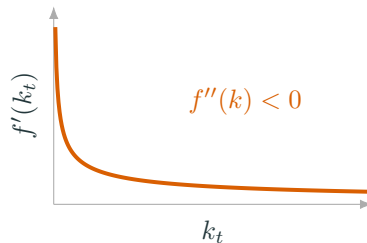


Illustration: $f(k) = \sqrt{k}$.

More capital raises output, but each additional unit contributes less.

Building Blocks: Why Cobb–Douglas?

- **Homogeneity of degree 1** (constant returns to scale):

$$Y(\lambda K, \lambda L) = \lambda Y(K, L)$$

→ doubling capital and labor doubles output.

- **Implications:**
 - Factor shares are constant over time.
 - Each factor has diminishing marginal productivity.
 - Guarantees stability and uniqueness of equilibrium in optimal growth models.
- Widely used in macroeconomics for simplicity and tractability.

Building Blocks: GDP allocation

- Recall: **GDP** Y_t is the total value of goods and services produced in one period. → It is a **flow**, not a stock.
- From the **demand side**, GDP is allocated to:

$$Y_t = C_t + I_t$$

- C_t : consumption (households, public),
 - I_t : investment (new machines, buildings, infrastructure).
- Example:
 - A household buying food → consumption.
 - A firm building a new factory → investment.

Building Blocks: Capital accumulation

- **Capital dynamics:**

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where:

- K_t : stock of capital (machines, factories, infrastructure),
- δ : depreciation rate (typically $\sim 5\%$ per year),
- I_t : new investment.

- **Interpretation:**

- Each period, a fraction δ of capital "wears out".
- Investment replenishes and grows the capital stock.
- Today's saving $\rightarrow$ tomorrow's productive capacity.

- This law of motion is the **state equation** of the optimal growth model.

Building Blocks: Preferences (setup)

- In growth models, we evaluate **per capita consumption**:

$$c_t = \frac{C_t}{L_t}$$

where C_t is aggregate consumption, L_t is population.

- **Utility at the individual level**: captures satisfaction from consumption of goods at date t .
- Standard assumption: **CRRA utility** (constant relative risk aversion):

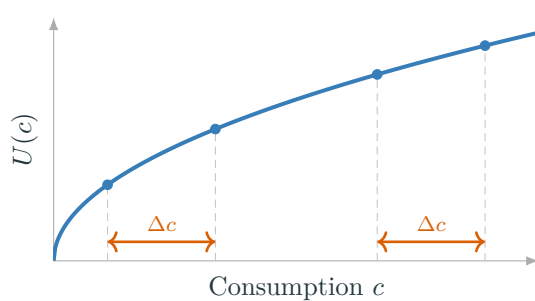
$$U(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \quad \gamma > 0$$

- γ : curvature parameter = risk aversion = $1/(\text{intertemporal elasticity of substitution})$.

Building Blocks: Preferences II

- $U: \mathbb{R}^+ \rightarrow \mathbb{R}$ (U can take negative values, what matters is the relative difference)
- U is strictly increasing: $\forall c \in \mathbb{R}^+, u'(c) > 0$
- U is strictly concave: $\forall c \in \mathbb{R}^+, u''(c) < 0$
- U satisfies “Inada conditions”: $\lim_{c \rightarrow 0} u'(c) = +\infty$ and $\lim_{c \rightarrow +\infty} u'(c) = 0$
 - Ensures an interior optimum in dynamic optimization problems (no corner solutions).
 - Interpretation:
 $\lim_{c \rightarrow 0} u'(c)$ formalizes the idea that **starving is infinitely painful** (so agents avoid it),
 $\lim_{c \rightarrow +\infty} u'(c)$ formalizes satiation: more and **more consumption eventually adds negligible utility**.

Building Blocks: Preferences III



$U'(c) > 0$
More is better.

$U''(c) < 0$
Marginal utility
decreases.

Equal increases in consumption yield smaller utility gains at higher c .

Illustration: $U(c) = \sqrt{c}$, up to affine normalization.

Building Blocks: Intertemporal welfare

- The planner values utility flows across time:

$$W = \sum_{t=0}^{\infty} \beta^t L_t U(c_t)$$

- **Discounting:**

$$\beta = \frac{1}{1 + \rho}, \quad \rho > 0$$

- ρ : pure rate of time preference.
 - Low ρ : future generations valued more (Stern).
 - High ρ : more weight on present (Nordhaus).
- Preferences combine two dimensions:
 1. *Across goods*: concavity (γ) governs curvature of $U(c)$.
 2. *Across time*: discounting (β, ρ) governs how much future is valued.

Building Blocks: Introducing the saving rate

- Recall the resource identity with $c_t = C_t/L_t$:

$$Y_t = c_t L_t + I_t$$

- Define the **saving rate** s_t :

$$I_t = s_t Y_t, \quad C_t = (1 - s_t) Y_t$$

- Interpretation:
 - s_t is the **balance-sheet view of GDP**: it tells us how much of income is devoted to *future capacity* (investment) vs *current use* (consumption).
 - This simple choice variable links today's output to tomorrow's capital.

Building Blocks: model recap

- A set of **exogenous variables** (independent of controls)

$$x_t = \{A_t, L_t\}$$

- A set of **endogenous variables**

$$y_t = \{c_t, Y_t, K_t\}$$

- A set of **control variables**

$$z_t = \{s_t\}$$

Optimal growth problem

Optimal growth problem: setup

- **Objective function:**

$$\max_{0 \leq \{s_t\}_{t=0}^{\infty} \leq 1} W_t = \sum_{s=0}^{\infty} \beta^s L_{t+s} U(c_{t+s}),$$

- **Constraints:**

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \quad (\text{production})$$

$$Y_t = L_t c_t + I_t \quad (\text{resource allocation})$$

$$K_{t+1} = (1 - \delta) K_t + I_t \quad (\text{capital accumulation})$$

- **Control:**

$$s_t \in [0, 1] \quad (\text{saving rate})$$

determines the split: $I_t = s_t Y_t$, $C_t = (1 - s_t) Y_t$.

Optimal growth problem: Ways to solve the optimal growth problem

1. **Dynamic programming (Bellman)** Value function $V(K_t)$, recursive solution. Guarantees existence/uniqueness but often costly numerically.
2. **Ramsey approach (Euler)** Derive first-order conditions (Euler eqs.), solve with Newton methods. Provides analytical intuition on intertemporal trade-offs.
3. **Brute-force / direct optimization** Parameterize $\{s_t\}$ over horizon, use a nonlinear solver. Easy to implement, but less efficient.

In class: we will use the *brute-force approach* in Python because it is very flexible, but we connect back to the Ramsey conditions to explain mechanism.

Why concavity matters

- **Concavity of utility** $U(c)$ and production $Y(K, L)$ ensures:
 - The planner's objective $W = \sum \beta^t L_t U(c_t)$ is *strictly concave* in the control sequence $\{s_t\}$.
 - The constraints (capital accumulation, resource constraint) are linear/convex.
- **Implications:**
 - The optimization problem is a **convex program**.
 - Existence of an optimal solution is guaranteed.
 - Strict concavity $\Rightarrow$ **uniqueness** of the optimal path (c_t, K_t) .
- **For dynamic programming:** Concavity of the Bellman operator $\Rightarrow$ value function is concave $\Rightarrow$ optimal policy is well-defined and stable.

Optimal growth problem: the Ramsey approach I

We use the **Ramsey approach**: Optimal-control theory with setting up the Lagrangian with two constraints (capital law and resource allocation via s_t):

$$\begin{aligned}\mathcal{L} = & \sum_{i=0}^{\infty} \beta^i L_{t+i} U(c_{t+i}) \\ & + \sum_{i=0}^{\infty} \beta^i \lambda_{1t+i} \left[K_{t+1+i} - (1 - \delta) K_{t+i} - s_{t+i} A_{t+i} K_{t+i}^{\alpha} L_{t+i}^{1-\alpha} \right] \\ & + \sum_{i=0}^{\infty} \beta^i \lambda_{2t+i} \left[(1 - s_{t+i}) A_{t+i} K_{t+i}^{\alpha} L_{t+i}^{1-\alpha} - L_{t+i} c_{t+i} \right].\end{aligned}$$

We call $\lambda_{1t} \geq 0$ and $\lambda_{2t} \geq 0$ costate variables, while $s_t \geq 0$ is a control and $K_{t+1} \geq 0$ is a state variable

First-order conditions (interior solution):

$$c_t : \quad L_t U'(c_t) - \lambda_{2t} L_t = 0$$

$$s_t : \quad -\lambda_{1t} + \lambda_{2t} = 0$$

$$K_{t+1} : \quad \lambda_{1t} = \beta \lambda_{1,t+1} \left[(1 - \delta) + \alpha s_{t+1} \frac{Y_{t+1}}{K_{t+1}} \right] - \beta \lambda_{2,t+1} \left[(1 - s_{t+1}) \alpha \frac{Y_{t+1}}{K_{t+1}} \right].$$

Optimal growth problem: the optimal condition

Remarks

- Two multipliers enforce two constraints; at the optimum they collapse to $\lambda_t = U'(c_t)$.
- Strict concavity of U and Cobb-Douglas production ensures existence and uniqueness of the optimal path.

Ramsey rule:

$$c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[(1 - \delta) + \alpha \frac{Y_{t+1}}{K_{t+1}} \right].$$

Interpretation:

- c_{t+1} , Y_{t+1} are forward-looking variables.
- The system links today's decisions to both future and past states: a mix of backward- and forward-looking dynamics.

Optimal growth problem: summary

Assumptions (for simplicity):

$$A_t = 1 \quad (\text{constant productivity}), \quad L_t = 1 \quad (\text{constant population}).$$

Dynamic system:

$$(\text{Ramsey}) \quad c_t^{-\gamma} = \beta c_{t+1}^{-\gamma} \left[(1 - \delta) + \alpha K_{t+1}^{\alpha-1} \right],$$

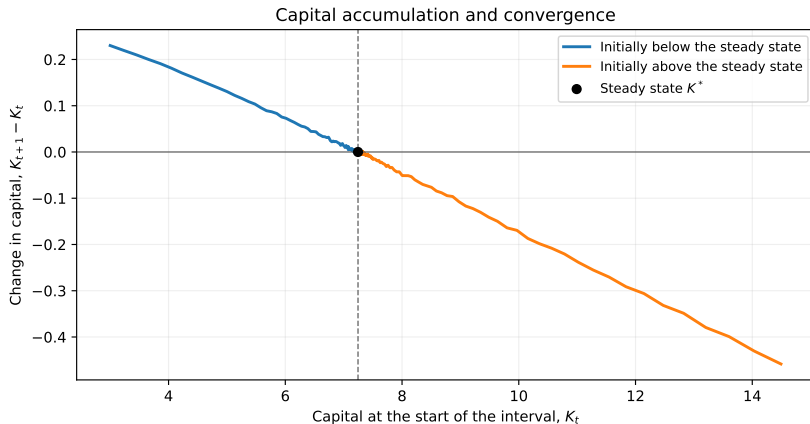
$$(\text{Capital law}) \quad K_{t+1} = (1 - \delta)K_t + K_t^\alpha - c_t.$$

Properties:

- Given $K_0 \geq 0$, the system generates a path $\{c_t, K_t\}$ converge to a unique *steady state*:

$$\bar{K} = \left(\frac{1}{\alpha} \left[\frac{1}{\beta} - (1 - \delta) \right] \right)^{\frac{1}{\alpha-1}}.$$

Optimal growth problem: Diagram



$$\alpha = 0.30, \beta = 1/(1 + 0.015), \delta = 0.06, \gamma = 2, A = L = 1, \Delta = 1.$$

Same calibration and solver as the notebook; $K_0 = 3$ and $K_0 = 2K^*$.

Optimal growth problem: Diagram Interpretation

Interpretation:

- Case $K_0 \leq \bar{K}$
 - Capital is too low relative to its steady-state.
 - This requires higher investment (a larger share of output devoted to I_t) and thus lower consumption in the short run and capital accumulates $\Delta K > 0$
 - Over time, capital rises toward $\bar{K}$.
- Case $K_0 \geq \bar{K}$
 - Opposite mechanism: overaccumulation of capital $\rightarrow$ more consumption.
 - Depreciation dominates $\Delta K < 0$
- Case $K_0 = \bar{K}$
 - ΔK capital stock stationary. Consumption and investment stable.

If you want the proof of convergence with saddle path see: [\[Lecture II\]](#)

Control Problem: a sketch of the numerical solution I

- Because the social welfare is forward looking, the system is both forward and backward looking as follows:

$$s_t = \arg \min_{\{s_t\}_{i \geq 0}} - \sum_{i=0}^{\infty} (1 + \rho)^{-i} L_{t+i} u(c_{t+i})$$

- When a sequence $\{s_{t+i}\}_{i=0}^{\infty}$ is found, one can compute:

$$\begin{aligned} x_t &= g(x_{t-1}), & g : \mathbb{R}^{N_x} &\rightarrow \mathbb{R}^{N_x}, \\ y_t &= f(y_{t-1}, x_t, z_t), & f : \mathbb{R}^{N_y} \times \mathbb{R}^{N_x} \times \mathbb{R}^2 &\rightarrow \mathbb{R}^{N_y}. \end{aligned}$$

- $x_t \in \mathbb{R}^{N_x}$: **exogenous variables** (A_t, L_t).
- $y_t \in \mathbb{R}^{N_y}$: **endogenous states/flows** (c_t, Y_t, K_t).
- $z_t = [s_t]' \in \mathbb{R}$: **decision variables** (saving).

Control Problem: a sketch of the numerical solution II

- **Idea.** Approximate the **infinite-horizon** objective with a **finite** one. Let $\beta \equiv (1 + \rho)^{-1}$. Because $\lim_{i \rightarrow \infty} \beta^i = 0$, the tail contributes little.
- **Pick the horizon I^*** so that the tail weight is below a tolerance ε :

$$\beta^{I^*} \leq \varepsilon$$

Trade-off: smaller $I^* \Rightarrow$ faster solve but larger truncation error.

- **Finite-horizon problem** that solves

$$\max_{\{s_{t+i}, \mu_{t+i}\}_{i=0}^{I^*}} \sum_{i=0}^{I^*} \beta^i L_{t+i} U(c_{t+i})$$

s.t. (resource, capital) and bounds.

Thank you!

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References

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