

Modélisation, estimation, simulation des risques climatiques
Session 5

The Social Cost of Carbon

Gauthier Vermandel
Ecole polytechnique

Objectives

- Formulate the social planner's **optimal control** problem (Ramsey setup).
- See how the **SCC emerges as a shadow price** and maps to an optimal carbon tax (USD/tCO₂).

Materials:

- notebook_session5.ipynb *optimal saving, abatement and carbon pricing*.
- Next session: notebook_session6.ipynb, discounting and tipping points.
- Handout_DICE.pdf a synthetic presentation of the DICE model.

Introduction

Introduction: From what could happen to what should happen

- In the previous lecture: we built **transition scenarios** (SSPs, forcing pathways) — describing *possible futures*.
- But in principle, we may also ask: **what should the transition look like?**
 - Formalized as an **optimal control problem**:

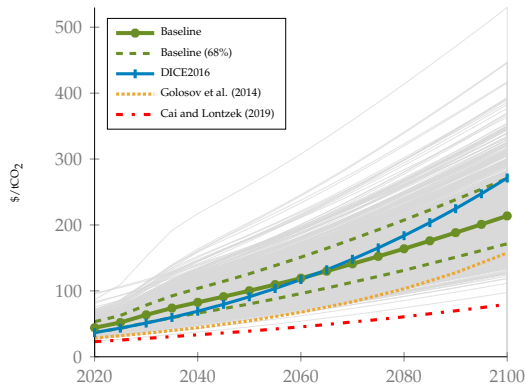
$$\max_{\{\mu_t, \dots\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \quad \text{s.t. climate-economy dynamics}$$

- Find the sequence of decisions that maximizes social welfare.
- Advantages:
 - Reduces an **infinite set of possible paths** to **one “optimal” path**.
 - Provides a **normative benchmark** for policy: what should be done.

Introduction: From Optimal Abatement to Optimal Carbon Price

- The key policy metric is not the abatement rate itself, but the **optimal carbon price** (or **Social Cost of Carbon, SCC**).
- Definition: the SCC is the **monetary value (USD per ton of CO₂)** of the marginal damage from one additional ton of emissions.
- Why it matters:
 - Abstract concept (model-based), but provides guidance on **what carbon price should be today**.
 - Forms the basis for actual carbon tax design.
 - Example: in the US, the Biden Administration used **five IAM-based SCC estimates** to guide federal policy.

Introduction: Example



Note: The figure displays the social cost of carbon (SCC) on the vertical axis. The baseline model from [Jondeau et al. \(2022\)](#) is represented by a solid green line. Uncertainty interval, computed from 2,000 simulations using alternative parameter draws from the Metropolis-Hastings algorithm, are depicted by solid gray lines. Alternative SCC are also presented: (i) DICE2016 (solid blue with vertical markers), (ii) [Golosov et al. \(2014\)](#) (orange dotted line), and (iii) [Cai and Lontzek, 2019](#) with a benchmark calibration assuming 1.5% total factor productivity growth (red dashed line).

Outline

1 Introduction

2 The Optimal Control Problem

3 The SSC in the Ramsey Problem

The Optimal Control Problem

Control Problem: the objective function

- **Individual preferences.** Utility from consumption $u(c)$ with $u'(c) > 0$ and $u''(c) < 0$ (diminishing marginal utility). We use CRRA:

$$u(c) = \begin{cases} \frac{c^{1-\gamma} - 1}{1-\gamma}, & \gamma \neq 1, \\ \log c, & \gamma = 1, \end{cases} \quad \gamma > 0.$$

- **Aggregation and units.** Let C_t be aggregate consumption (trillion USD) and L_t population (million). Per-capita consumption in *thousand USD* is

$$c_t = 1,000 \times \frac{C_t}{L_t}.$$

Instantaneous social welfare is $L_t u(c_t)$.

- **Intertemporal welfare.** With pure rate of time preference $\rho > 0$,

$$W_t = \sum_{i=0}^{\infty} \frac{L_{t+\Delta i} u(c_{t+\Delta i})}{(1+\rho)^{\Delta i}}.$$

Control Problem: the solution toolkit

Planner's problem. Consider a government that chooses decisions variables (savings/investment S_t and abatement μ_t) to maximize

$$\max_{\{s_t, \mu_t\}_{i \geq 0}} \sum_{i=0}^{\infty} \frac{L_{t+\Delta i} u(c_{t+\Delta i})}{(1 + \rho)^{\Delta i}} \quad \text{s.t. resource, capital, ...}$$

How to solve the problem?

1. **Numerical solver:** use solver to find sequences $\{S_t, \mu_t\}_{t=0}^T$ that maximizes objective.
2. **Ramsey solution:** maximize social welfare based on N_y equations and $N_y + N_z$ control variables.
3. **Dynamic programming:** use Bellman equation and value function iterations or projections possibly with fancy neural networks.

Control Problem: model summary (a recap)

- **Exogenous variables** each driven by an exogenous process (growth/decline path).

$$x_t = \{A_t, \sigma_t, \theta_{1,t}, F_t^{EX}, E_t^{\text{land}}, L_t\}$$

- **Endogenous state/flow variables**

$$y_t = \{M_t^{AT}, M_t^{UP}, M_t^{LO}, T_t^{AT}, T_t^{LO}, Y_t, K_t, C_t\}$$

- **Decision variables (controls)**

$$z_t = \{\mu_t, s_t\}$$

- **Horizon and time step:** $t \in \{2015, 2020, 2025, \dots, T\}$, $\Delta = 5$ years (quinqennial).

Control Problem: a sketch of the numerical solution

- Because the social welfare is forward looking, the system is both forward and backward looking as follows:

$$z_t = \arg \min_{\{s_t, \mu_t\}_{i \geq 0}} - \sum_{i=0}^{\infty} (1 + \rho)^{-\Delta i} L_{t+\Delta i} u(c_{t+\Delta i})$$

- When a sequence $\{z_{t+\Delta i}\}_{i=0}^{\infty}$ is found, one can compute:

$$\begin{aligned} x_t &= g(x_{t-1}), & g : \mathbb{R}^{N_x} &\rightarrow \mathbb{R}^{N_x}, \\ y_t &= f(y_{t-1}, x_t, z_t), & f : \mathbb{R}^{N_y} \times \mathbb{R}^{N_x} \times \mathbb{R}^2 &\rightarrow \mathbb{R}^{N_y}. \end{aligned}$$

- $x_t \in \mathbb{R}^{N_x}$: **exogenous variables** ($A_t, \sigma_t, \dots, L_t$).
- $y_t \in \mathbb{R}^{N_y}$: **endogenous states/flows** ($M_t^{AT}, M_t^{UP}, \dots, Y_t, K_t$).
- $z_t = [\mu_t, s_t]' \in \mathbb{R}^2$: **decision variables** (abatement, saving).

Control Problem: a sketch of the numerical solution

- **Idea.** Approximate the **infinite-horizon** objective with a **finite** one. Let $\beta_{\Delta} \equiv (1 + \rho)^{-\Delta}$. Because $\lim_{i \rightarrow \infty} \beta_{\Delta}^i = 0$, the tail contributes little.
- **Pick the horizon I^*** so that the tail weight is below a tolerance ε :

$$\beta_{\Delta}^{I^*} \leq \varepsilon$$

Trade-off: smaller $I^* \Rightarrow$ faster solve but larger truncation error.

- **Finite-horizon problem** with $T = t + I^* \Delta$, solve

$$\max_{\{s_{t+i}, \mu_{t+i}\}_{i=0}^{I^*}} \sum_{i=0}^{I^*} \beta_{\Delta}^i L_{t+i\Delta} u \left(1000 \times \frac{C_{t+i\Delta}}{L_{t+i\Delta}} \right)$$

s.t. dynamics (resource, capital, emissions, climate) and bounds.

Control Problem: Two Policies

- The **mitigation policy**:

$$\max_{\{s_{t+i}, \mu_{t+i}\}_{i=0}^{I^*}} \sum_{i=0}^{I^*} \beta_{\Delta}^i L_{t+i\Delta} u\left(1000 \times \frac{C_{t+i\Delta}}{L_{t+i\Delta}}\right)$$

s.t. dynamics (resource, capital, emissions, climate) and bounds.

- The *laissez-faire* / Business As Usual (BAU) policy:

$$\max_{\{s_{t+i},\}_{i=0}^{I^*}} \sum_{i=0}^{I^*} \beta_{\Delta}^i L_{t+i\Delta} u\left(1000 \times \frac{C_{t+i\Delta}}{L_{t+i\Delta}}\right)$$

s.t. dynamics (resource, capital, emissions, climate) and bounds.

$$s.t. \mu_t = 0.03$$

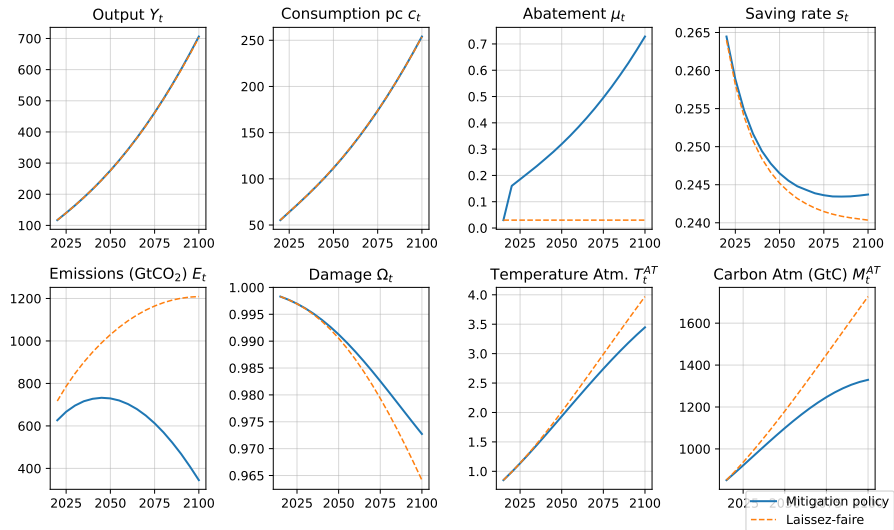


Figure. Simulations under two policy scenarios

Some striking but controversial results from DICE:

- “Optimal” to warm the planet up to 3.3°C ;
- Mitigation policy saves about 1% of climate damage;
- Net zero emission optimal by 2130, 80 years later than in Paris-Agreement;
- Main critic from DICE’s policy output: those results have scientifically grounded climate inaction in policy circles...

... but DICE has also the merit to set the climate issue to the agenda of (macro-)economists.

The SSC in the Ramsey Problem

- Why is it optimal to warm so much the planet according to DICE?
- An optimal carbon tax = how much the society is willing to pay to reduce emissions (\$ per ton of carbon).
- This is usually referred to as the Social Cost of Carbon (SCC) in literature.
- SCC reflects the society's gains and losses from implementing the carbon tax.
- What is the model implied SCC? Let's use Ramsey problem!

The **Ramsey (1927)** problem:

$$\begin{aligned}
& \max_{\{c_t, Y_t, K_t, T_t, M_t, \mu_t, s_t\}} \sum_{i=0}^{\infty} \beta_{\Delta}^i L_{t+\Delta i} u(c_{t+\Delta i}) \\
& + \beta_{\Delta}^i \lambda_{1,t+\Delta i} \left[(1 - s_{t+\Delta i}) Y_{t+\Delta i} \left(1 - \theta_{1,t+\Delta i} \mu_{t+\Delta i}^{\theta_2} \right) \Omega \left(T_{t+(i-1)\Delta} \right) - L_{t+\Delta i} c_{t+\Delta i} / 1,000 \right] \\
& + \beta_{\Delta}^i \lambda_{2,t+\Delta i} \left[K_{t+\Delta i} - (1 - \delta_K) K_{t+(i-1)\Delta} - s_{t+\Delta i} Y_{t+\Delta i} \left(1 - \theta_{1,t+\Delta i} \mu_{t+\Delta i}^{\theta_2} \right) \Omega \left(T_{t+(i-1)\Delta} \right) \right] \\
& + \beta_{\Delta}^i \lambda_{3,t+\Delta i} \left[Y_{t+\Delta i} - A_{t+\Delta i} K_{t+(i-1)\Delta}^{\gamma} L_{t+\Delta i}^{1-\gamma} \right] \\
& + \beta_{\Delta}^i \lambda_{4,t+\Delta i} \left[T_{t+\Delta i} - \Phi_T T_{t+(i-1)\Delta} - \Xi_T F_{t+\Delta i} (M_{t+\Delta i}) \right] \\
& + \beta_{\Delta}^i \lambda_{5,t+\Delta i} \left[M_{t+\Delta i} - \Phi_M M_{t+(i-1)\Delta} - \Xi_M \left(\sigma_{t+\Delta i} (1 - \mu_{t+\Delta i}) Y_{t+\Delta i} + E_{t+\Delta i}^{\text{land}} \right) \right]
\end{aligned}$$

Summary: optimal control problem with 7 control variables, 5 constraints $\rightarrow$ 12 variables/equations.

(note that climate variables include more equations + Lagrangian multipliers)

- Recall that compact form for climate block with

$$T_t = [T_t^{AT}, T_t^{LO}]^\top \text{ and } M_t = [M_t^{AT}, M_t^{UP}, M_t^{LO}]^\top$$

- The corresponding multipliers:

$$\lambda_{4,t} = [\lambda_{5,t}^{AT}, \lambda_{5,t}^{LO}]^\top \text{ and } \lambda_{5,t} = [\lambda_{5,t}^{AT}, \lambda_{5,t}^{UP}, \lambda_{5,t}^{LO}]^\top$$

- Damages and forcing functions as function of vectors T_t and M_t :

$$\Omega(T_t) = 1 / \left(1 + a_2 \left(T_t^{AT} \right)^2 \right)$$

$$F_t(M_t) = \eta \log \left(M_t^{AT} / M_{1750} \right) / \log(2) + F_t^{EX}$$

- Corresponding gradients:

$$\nabla_T \Omega(T_t) = \begin{bmatrix} -\Omega(T_t)^2 2a_2 (T_t^{AT}) & 0 \end{bmatrix}^\top$$

$$\nabla_M F(M_t) = \begin{bmatrix} \eta / (M_t^{AT} \ln 2) & 0 & 0 \end{bmatrix}^\top$$

- Get back to FOCs of the Ramsey-planner:

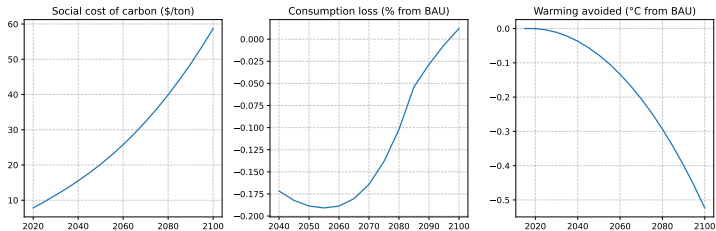
$$T_t : \lambda_{4,t} = \beta_{\Delta} \left[\Phi_T^{\top} \lambda_{4,t+\Delta} + \lambda_{2,t+\Delta} s_{t+\Delta} Y_{t+\Delta} \left(1 - \theta_{1,t+\Delta} \mu_{t+\Delta}^{\theta_2} \right) \nabla_T \Omega(T_t) \right]$$
$$M_t : \lambda_{5,t} = \beta_{\Delta} \left[\Phi_M^{\top} \lambda_{5,t+\Delta} + \left(\Xi_T^{\top} \lambda_{4,t+\Delta} \right) \nabla_M F(M_t) \right]$$

$\lambda_{5,t}$ ($\lambda_{4,t}$) is the marginal loss from carbon (temperature) increase in each climate box.

- SCC expresses the social loss into numeraire equivalents (here consumption):

$$SSC_t = -1000 \times \lambda_{5,t}^{AT} / \lambda_{1,t} \simeq -1000 \times \left(\partial W_t / \partial M_t^{AT} \right) / \left(\partial W_t / \partial C_t \right)$$

where $\lambda_{1,t}$ is marginal utility of consumption while $\lambda_{5,t}^{AT}$ measures the planner's value for a marginal change in atmospheric carbon.



- SCC reflect the planner's relative gains in terms of welfare from controlling M_t against its consumption losses C_t from such action;
- Optimal to cut consumption now by 0.2% in order to avoid 0.55°C of warming in 2100;
- As well off between scenarios by 2090 on current consumption grounds (but not in welfare terms).

Social cost of carbon

- How to retrieve the SCC if we don't have any lagrangian multiplier?
- One can use the firm's static problem (given output Y_t):

$$\min_{\mu_t \in [0,1]} \underbrace{\theta_{1,t} \mu_t^{\theta_2} Y_t}_{\text{abatement cost}} + \tau_t \underbrace{\sigma_t (1 - \mu_t) Y_t}_{\text{taxed residual emissions}}.$$

- First-order condition in μ_t :

$$\theta_2 \theta_{1,t} \mu_t^{\theta_2-1} Y_t = \tau_t \sigma_t Y_t \quad \Rightarrow \quad \boxed{\tau_t = \frac{\theta_2 \theta_{1,t}}{\sigma_t} \mu_t^{\theta_2-1}}$$

- This tax is in trillion USD per gigatonne carbon. The carbon tax $\tau_t^{\$}$ in US\$/tCO₂:

$$\tau_t^{\$} = \frac{10^{12}}{10^9} \times \tau_t = 1,000 \times \tau_t$$

as $1GtC = 10^9$ and 1 *trillion* \$ = 10^{12} .

References

- Cai, Y. and Lontzek, T. (2019). The social cost of carbon with economic and climate risks. *Journal of Political Economy*, 127:2684–2734.
- Golosov, M., Hassler, J., Krusell, P., and Tsyvinski, A. (2014). Optimal taxes on fossil fuel in general equilibrium. *Econometrica*, 82:41–88.
- Jondeau, E., Levieuge, G., Sahuc, J.-G., and Vermandel, G. (2022). Environmental subsidies to mitigate transition risk. *Swiss Finance Institute Research Paper*, (22-45).
- Ramsey, F. P. (1927). A contribution to the theory of taxation. *The economic journal*, 37(145):47–61.