Homework 1

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I. Stein 1.2: Cantor set described in ternary expansions

Some notations are shown as follows:

On construction of Cantor set $\mathcal{C}$: Let $C_0 = [0,1]$. This interval is divided into three parts: the middle third open interval of C_0 is $E_{1,1} = (1/3, 2/3)$ which is excluded, while $I_{1,1} = [0, 1/3]$, $I_{1,2} = [2/3, 1]$ is included to obtain $C_1 = I_{1,1} \cup I_{1,2}$. Generally, when we get $C_k = \cup_{i=1}^{2^{k-1}} I_{k,i}$, each $I_{k,i}$ is divided into three parts with the middle open one denoted as $E_{k+1,i}$, the other two are $I_{k+1,2i-1}, I_{k+1,2i}$. Then $\mathcal{C} := \cap_{k=0}^{\infty} C_k$.

A. Cantor sets are points represented in 0 and 2.

Proof: Notice that $x \notin E_{1,1}$ ($i.e.$, $x \in C_1$) if and only if x has a decomposition with $a_1 \neq 1$. By deduction, we claim that C_k consists of x that has a decomposition with $a_j \neq 1, j \leq k$. The definition of $\mathcal{C}$ completes the proof. ■

B. Well-definedness and continuity of Cantor-Lebesgue function.

Proof: First, to show the well-definedness, it is sufficient to show that the ternary expansion of 0 and 2 is unique. Suppose that

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} = \sum_{k=1}^{\infty} b_k 3^{-k}, \quad a_k, b_k \in \{0, 2\}.$$

Let $k_0 := \inf\{k : a_k \neq b_k\} < \infty$, and WLOG (without loss of generality), $a_{k_0} = 0, b_{k_0} = 2$. Then

$$0 = \sum_{k=1}^{\infty} (b_k - a_k) 3^{-k} \geq 2 \cdot 3^{-k_0} - \sum_{k=k_0+1}^{\infty} 2 \cdot 3^{-k} = 3^{-k_0},$$

which is a contradiction! This argument also shows that $\forall |x - y| < 3^{-k_0}, x, y \in \mathcal{C}$, the first k_0 expansions are the same.

Second, the continuity. $\forall \epsilon > 0,\ \exists k_0,\ s.t.\ 2^{-k_0} < \epsilon$. Therefore, for any $x \in \mathcal{C}$, choose $\delta = 3^{-k_0}$, then $\forall y \in \mathcal{C}, |x-y| < \delta$, the first k_0 expansions of x and y are the same, thus $|F(x)-F(y)| \leq \sum_{k_0+1}^{\infty} 2^{-k} = 2^{-k_0} < \epsilon$. From the definition, we know that F is continuous.

Moreover, $F(0) = 0, F(1) = 1$ follows directly from $0 = \sum_{k=1}^{\infty} 0 \cdot 3^{-k}$ and $1 = \sum_{k=1}^{\infty} 2 \cdot 3^{-k}$. ■

C. Surjectiveness of Cantor-Lebesgue function.

Proof: Every $y \in [0,1]$ has a binary expression $y = \sum_{k=1}^{\infty} b_k \cdot 2^{-k}, b_k \in \{0, 1\}$, from which we can recover $x = \sum_{k=1}^{\infty} 2b_k \cdot 3^{-k} \in \mathcal{C}$. ■

D. Continuity of extended Cantor-Lebesgue function.

Proof: Note that $F|_{\mathcal{C}}$ is non-decreasing and the extended definition can be interpreted as $F(x) := F(\sup\{y : y \in \mathcal{C}, y \leq x\} = \sup\{F(y) : y \in \mathcal{C}, y \leq x\}$ for any $x \in [0,1]$, or $F(x) := \inf\{F(y) : y \in \mathcal{C}, y \geq x\}$ due to the monotonicity of $F|_{\mathcal{C}}$ and closedness of $\mathcal{C}$.

$\forall \epsilon > 0, x \in [0,1]$, choose $\delta = 3^{-k_0}$ same as in (I-B), $(s.t.\ 2^{-k_0} < \epsilon,)$ then $\forall |x-y| < \delta$, if $x, y \in \mathcal{C}$, $|F(x) - F(y)| \leq \sum_{k_0+1}^{\infty} 2^{-k} = 2^{-k_0} < \epsilon$; else, WLOG $y > x$, let $x' = \inf\{z : z \in \mathcal{C}, z \geq x\}$ and $y' = \inf\{z : z \in \mathcal{C}, z \leq y\}$, then $|F(x) - F(y)| = |F(x') - F(y')| < \epsilon$. From the definition, F is continuous. ■

Remark 1. *A simpler proof: prove that the extended F is monotone on $[0,1]$. Then by monotonicity and surjectiveness, F is continuous.*

II. Stein 1.3: Cantor sets $\mathcal{C}_\xi$

A. Interpretation of $\mathcal{C}_\xi^c$.

Proof: Same notations as in the construction of $\mathcal{C}$. In the space of C_0, $|C_1| = 1 - \xi$, and by deduction $|C_k| = (1-\xi)^k$. Also note that C_k^c is unions of finite open intervals. Therefore, $\mathcal{C}_\xi^c = (\cap_{k=0}^{\infty} C_k)^c = \cup_{k=0}^{\infty} C_k^c$ is the union of open intervals with total length no less than $1 - |C_k| \to 1$. Constrained in $[0,1]$, the total length equals to 1. ■

B. $m_*(\mathcal{C}_\xi) = 0$.

Proof: From the definition,

$$0 \leq m_*(\mathcal{C}_\xi) = \inf\{\sum_{j=1}^{\infty} |Q_j| \big| \mathcal{C}_\xi \subset \cup_{j=1}^{\infty} Q_j,\ Q_j \text{ are closed intervals}\} \leq \inf |C_k| = 0.$$

Thus, $m_*(\mathcal{C}_\xi) = 0$. ■

III. Stein 1.14: Outer Jordan content - finite covering intervals

A. $J_*(E) = J_*(\bar{E})$.

Proof: If $\bar{E} \subset \cup_{j=1}^{N} I_j$, then $E \subset \bar{E} \subset \cup_{j=1}^{N} I_j$, thus $J_*(E) \leq J_*(\bar{E})$. If $E \subset \cup_{j=1}^{N} I_j$, then $\bar{E} \subset \cup_{j=1}^{N} \bar{I}_j$ with $\sum |I_j| = \sum |\bar{I}_j|$, thus $J_*(E) \geq J_*(\bar{E})$. ■

B. $J_*(E) = 1, m_*(E) = 0$.

$E = \mathbb{Q} \cap [0,1]$ is an example.

IV. STEIN 1.15: OUTER MEASURE m_*^R BY COUNTABLE COVERING RECTANGLES

Proof: Given E, since cubes are rectangles, $m_*(E) \geq m_*^R(E)$.

To prove $m_*(E) \leq m_*^R(E)$, WLOG, $m := m_*^R(E) < \infty$. For any $\epsilon > 0$, there exists a countable covering $E \subset \cup R_j$, such that $m \leq \sum_{j=1}^{\infty} |R_j| < m + \epsilon$. For each $R_j = [a_{1j}, b_{1j}] \times \ldots \times [a_{dj}, b_{dj}]$, consider the grid method in proof of Theorem 1.4, but with side length small enough and accept all the tentatively accepted cubes. The almost disjoint accepted cubes $\{Q_{ji},\ i = 1, \ldots, n_j\}$ make up of a rectangle $Q_j \supset R_j$ and from Lemma 1.1 satisfy $|Q_j| = \sum_{i=1}^{n_j} |Q_{ji}| \leq |R_j| + \epsilon/2^j$. In summary, we have $\cup_{j \geq 1, 1 \leq i \leq n_j} Q_{ji} \supset \cup R_j \supset E$ with $\sum_{ij} |Q_{ji}| \leq \sum_j (|R_j| + \epsilon/2^j) \leq m + 2\epsilon$. Since ϵ is arbitrary chosen, $m_*(E) \leq m$. ■

Remark 2. *Lemma 1.1 is the finite sum. If you want to use the infinity summation, please use Observation 5 in P15 instead.*