Homework 10

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I. Stein 3.1: A Special Family of Good Kernels

A. $\{K_\delta\}$ is a family of good kernels.

Proof: We check the three conditions in the definition.

1) $\int_{\mathbb{R}^d} K_\delta(x)dx = \int_{\mathbb{R}^d} \phi(x)dx = 1$.
2) $\int_{\mathbb{R}^d} |K_\delta(x)|\, dx = \int_{\mathbb{R}^d} |\phi(x)|\, dx =: A < \infty$, since ϕ is integrable.
3) For every $\eta > 0$, $\int_{|x|\geq\eta} |K_\delta(x)|\, dx = \int_{|x|\geq\frac{\eta}{\delta}} |\phi(x)|\, dx \to 0$ as $\delta \to 0$, since ϕ is integrable (Stein P65 Proposition 1.12(i)).

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B. $\{K_\delta\}$ is an approximation to the identity if ϕ is BSF.

Proof: Suppose that $\phi \leq M$ and $supp(\phi) \subset B_0(R)$. We check the three conditions in the definition.

1) The same as in the definition of good kernels.
2) $|K_\delta(x)| \leq M\delta^{-d}$ for all $\delta > 0$.
3) $supp(K_\delta) \subset B_0(\delta R)$, then $|K_\delta(x)| \leq \begin{cases} \frac{M}{\delta^d} \leq \frac{MR^{d+1}\delta}{|x|^{d+1}}, & x \in B_0(\delta R) \\ 0 \leq \text{any positive value}, & x \notin B_0(\delta R) \end{cases} \leq \frac{MR^{d+1}\delta}{|x|^{d+1}}$ for all $\delta > 0$ and $x \in \mathbb{R}^d$.

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C. L^1 Convergence Theorem holds for good kernels.

Proof: Similar to how we prove the convergence for good kernels,

$$\begin{aligned}
\|f * K_\delta - f\| &= \int_{\mathbb{R}^d} |f * K_\delta(x) - f(x)|\, dx \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-y)K_\delta(y)dy - \int_{\mathbb{R}^d} f(x)K_\delta(y)dy \right| dx \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y) - f(x)|\, |K_\delta(y)|\, dydx \\
&\overset{\text{Tonelli}}{=} \int_{|y|\leq\eta}\int_{\mathbb{R}^d} |f(x-y) - f(x)||K_\delta(y)|dxdy + \int_{|y|>\eta}\int_{\mathbb{R}^d} |f(x-y) - f(x)||K_\delta(y)|dxdy \\
&=: I_1 + I_2.
\end{aligned}$$

For the first term, notice the continuity of translation property (Stein P74 Proposition 2.5), $\|f(\cdot - y) - f\|_1 = \|f_y - f\|_1 \to 0$ as $y \to 0$. Then $\forall \epsilon > 0$, $\exists \eta,\ s.t.\ \|f_y - f\|_1 < \epsilon/A$ for all $|y| \le \eta$, so we have

$$I_1 \le \int_{|y|\le\eta} \|f_y - f\|_1 |K_\delta(y)| dy \le \epsilon,$$

which implies that $I_1 \to 0,\ \text{as } \delta \to 0$.

On the other hand, the second term is bounded as

$$I_2 \le \int_{|y|>\eta} 2\|f\|_1 |K_\delta(y)| dy \to 0, \quad \text{as } \delta \to 0.$$

Combining the two parts together yields $\|(f * K_\delta) - f\|_1 \to 0,\ \text{as } \delta \to 0.$ ■

II. Stein 3.2: A Special Family of Kernels Approximating Zero

Proof: Writing $0 = \int f(x) K_\delta(y) dy$, we prove in a similar way to that in Stein P112 Theorem 2.1.

$$\begin{aligned}
|f * K_\delta(x)| =& \left| \int_{\mathbb{R}^d} f(x-y) K_\delta(y) dy - \int_{\mathbb{R}^d} f(x) K_\delta(y) dy \right| \\
&\le \int_{\mathbb{R}^d} |f(x-y) - f(x)|\, |K_\delta(y)|\, dy \\
&= \int_{|y|\le\delta} |f(x-y) - f(x)| |K_\delta(y)| dy + \sum_{k=1}^{\infty} \int_{2^{k-1}\delta < |y| \le 2^k \delta} |f(x-y) - f(x)| |K_\delta(y)| dy \\
&=: I_0 + \sum_{k=1}^{\infty} I_k.
\end{aligned}$$

Then the bounding strategy is the same as in the proof of Theorem 2.1. ■

III. Stein 6.3: Equivalence between Caratheodory and Lebesgue measurability of exterior Lebesgue measure

Proof:

1) C⇒L. If E is Caratheodory measurable, and from the definition of m_*,

$$\forall \frac{1}{n},\ \exists \text{ open set } O_n \supset E,\ s.t.\ m_*(O_n) < m_*(E) + \frac{1}{n}.$$

Then $G := \cap_n O_n$ is a G_δ set with $E \subset G$ and $m_*(E) = m_*(G)$. From the definition of Caratheodory,

$$m_*(G) = m_*(G \cap E) + m_*(G \cap E^c) = m_*(E) + m_*(G \setminus E),$$

so $m_*(G \setminus E) = 0$, which implies that $G \setminus E$ is Lebesgue measurable and so is E.

2) C⇐L. Suppose E is Lebesgue measurable. $\forall A \subset \mathbb{R}^d$, $\exists G_\delta$ set G with $A \subset G$ and $m_*(A) = m(G)$. Since

$$m_*(E \cap A) + m_*(E^c \cap A) \le m_*(E \cap G) + m_*(E^c \cap G) = m(G) = m_*(A)$$

and the other direction $m_*(A) \le m_*(E \cap A) + m_*(E^c \cap A)$ is trivial, E is Caratheodory measurable.

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