Homework 11

Weiyu Li

Remark 1. *An important property:*

$$(\cup A_k)^c = \cap A_k,$$

so if we check a σ-algebra, we only need to check it is closed under "$\cdot^c$" and "$\cup_{k=1}^{\infty}$". (So we don't need to check closedness under countable intersections.)

Remark 2. *If $\mathcal{M}$ is closed under countable unions, it is closed under finite unions, since $\cup_{k=1}^{n} E_k = (\cup_{k=1}^{n} E_k) \bigcup (\cup_{k=1}^{\infty} E_n)$. The reverse is wrong. For example, let $\mathcal{M} = \{$finite subset of $\mathbb{N}\}$ and consider $E_k = \{k\}$.*

I. Stein 6.1: A σ-algebra is equivalent to a collection of subsets closed under complements and countable disjoint unions

This is a wrong proposition! A counterexample:

$$X = \{1, 2, 3, 4\}, \mathcal{M} = \{\varnothing, X, \{1, 2\}, \{3.4\}, \{1, 3\}, \{2, 4\}\}.$$

A modification: $\mathcal{M}$ being closed under complements and countable disjoint unions means that $\mathcal{M}$ is a **monotone class**. To be a σ-algebra, a sufficient condition is that $\mathcal{M}$ is an **algebra**, so we can prove $\mathcal{M}$ is a σ-algebra if we add another assumption that it is closed under **finite** intersections.

Proof: We only need to check that $\mathcal{M}$ is closed under countable unions, *i.e.*,

$$\forall E_1, E_2, \ldots \in \mathcal{M}, \quad \cup_{k=1}^{\infty} E_k \in \mathcal{M}.$$

Let $A_1 = E_1$, $A_k = E_k \setminus (\cup_{i=1}^{k-1} E_i) = E_k \setminus (\cup_{i=1}^{k-1} A_i)$, then $\{A_k\}$ are disjoint and $\cup_{k=1}^{\infty} E_k = \cup_{k=1}^{\infty} A_k$. So it remains to prove that $A_k = E_k \bigcap (\cup_{i=1}^{k-1} A_i)^c \in \mathcal{M}$. By deduction, it suffices to show $A_2 = E_2 \setminus E_1 \in \mathcal{M}$. This is because

$$E_2 \setminus E_1 = E_2 \cap (E_1)^c \in \mathcal{M}.$$

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II. Stein 6.2: completion of a measure space

A. $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ and $\{Z : Z \subset F$ with some $F \in \mathcal{M}, \mu(F) = 0\}$

Proof: For simplicity, we denote $N := \{Z : Z \subset F$ with some $F \in \mathcal{M}, \mu(F) = 0\}$.

Step 1. $\overline{\mathcal{M}}$ is a σ-algebra. First it is closed under countable unions.

$$\forall E_1 \cup Z_1, E_2 \cup Z_2, \ldots, \text{ where } E_k \in \mathcal{M}, Z_k \in N, \quad \text{we have } \cup_{k=1}^{\infty}(E_k \cup Z_k) = (\cup_{k=1}^{\infty} E_k) \bigcup (\cup_{k=1}^{\infty} Z_k).$$

Since $\mathcal{M}$ is a σ-algebra, $(\cup_{k=1}^{\infty} E_k) \in \mathcal{M}$. Second, $(\cup_{k=1}^{\infty} Z_k) \subset (\cup_{k=1}^{\infty} F_k)$ with $(\cup_{k=1}^{\infty} F_k) \in \mathcal{M}$, $\mu(\cup_{k=1}^{\infty} F_k) \leq \sum_{k=1}^{\infty} \mu(F_k) = 0$, so $(\cup_{k=1}^{\infty} Z_k) \in N$. From the definition of $\overline{\mathcal{M}}$, $\cup_{k=1}^{\infty}(E_k \cup Z_k) \in \overline{\mathcal{M}}$.

Then we prove that $\overline{\mathcal{M}}$ is closed under complements.

$$\forall E \cup Z \in \overline{\mathcal{M}}, \text{ where } E \in \mathcal{M}, Z \in N, \quad \text{we have } (E \cup Z)^c = ((E \cup F) \cap (F - Z)^c)^c = (E \cup F)^c \cup (F - Z).$$

Since $\mathcal{M}$ is a σ-algebra, $E, F \in \mathcal{M}$, $(E \cup F)^c \in \mathcal{M}$. Second, $(F - Z) \subset F$ with $F \in \mathcal{M}, \mu(F) = 0$, so $(F - Z) \in N$. From the definition of $\overline{\mathcal{M}}$, $(E \cup Z)^c \in \overline{\mathcal{M}}$.

Step 2. $\overline{\mathcal{M}}$ contains $\mathcal{M}$ and N. It's a simple task by setting E or $Z = \varnothing$ respectively.

Step 3. $\overline{\mathcal{M}}$ is the smallest. For any σ-algebra Σ containing $\mathcal{M}$ and N,

$$\forall E_1 = E \cup Z \in \mathcal{M}, \quad E, Z \in \Sigma \Rightarrow E_1 \in \Sigma.$$

Since E_1 is arbitrary chosen, $\overline{\mathcal{M}} \subset \Sigma$, which implies $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ and N. ■

B. $\overline{\mu}$ is a complete measure on $\overline{\mathcal{M}}$

Proof:

Step 1. $\overline{\mu}$ is well-defined. That is to say, if any $E \cup Z \in \overline{\mathcal{M}}$ can be decomposed in another way as $E \cup Z = E' \cup Z'$, then we have $\mu(E) = \mu(E')$, so the measure doesn't depend on the way of decomposition. On the one hand, $E \subset E' \cup Z' \subset E' \cup F'$, so

$$\mu(E) \leq \mu(E' \cup F') \leq \mu(E') + \mu(F') = \mu(E').$$

On the other hand, the similar argument shows that $\mu(E') \leq \mu(E)$, thus $\mu(E) = \mu(E')$.

Step 2. $\overline{\mu}$ is a measure. First $\overline{\mu} \geq 0$. Second, prove the countable additivity: if $E_1 \cup Z_1, E_2 \cup Z_2, \cdots \in \overline{\mathcal{M}}$ is a countable family of disjoint sets, then

$$\begin{aligned}\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup Z_n)\right) =& \overline{\mu}\left((\cup_{n=1}^{\infty} E_n) \bigcup (\cup_{n=1}^{\infty} Z_n)\right) \\ =& \mu(\cup_{n=1}^{\infty} E_n) \\ =& \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \overline{\mu}(E_n \cup Z_n).\end{aligned}$$

Step 3. $\overline{\mu}$ is complete. That is to say, $\forall E \cup Z \in \overline{\mathcal{M}}$ with $\overline{\mu}(E \cup Z) = \mu(E) = 0$, any $A \subset E \cup Z$ is stil $\overline{\mu}$-measurable. Notice that $Z \subset F$ with $\mu(F) = 0$, so $A \subset E \cup Z \subset E \cup F$ with $\mu(E \cup F) = 0$. Thus $A \in N \subset \overline{\mathcal{M}}$.

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