Homework 12

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I. Stein 6.12: Open sets in $\mathbb{R}^d - \{0\}$ can be countable union of open rectangles in polar coordinate form

Method 1: Follow the hint, then refer to the method in the proof of Theorem 1.4 in Chapter 1 (Stein P7-8).

Method 2: Denote

$$B_{j,k,l,n} := \left\{r_j < r < r'_k\right\} \times \left\{\gamma \in S^{d-1} : |\gamma - \gamma_\ell| < 1/n\right\}$$

as in the Hint. (Constructing a countable dense subset of S^{d-1} may be a bonus, but not our main focus here.) $\forall O$ open in $\mathbb{R}^d - \{0\}$, we claim that

$$O = \bigcup_{B_{j,k,l,n} \subset O} B_{j,k,l,n}.$$

II. Stein 6.5: Calculation about a d-dim ball

A. $\int_{\mathbb{R}^d} e^{-\pi|x|^2} dx = 1$ *for all* d*.*

Proof: When $d = 2$,

$$\int_{\mathbb{R}^2} e^{-\pi|x|^2} dx \overset{\text{polar}}{=} \int_{\mathbb{R}} e^{-\pi r^2} r dr 2\pi = 1,$$

where the last equality easily comes from the Riemann integral, but it is the same value in the sense of Lebesgue integral (by using MCT). Then,

$$1 = \int_{\mathbb{R}^2} e^{-\pi|x|^2} dx = \int_{\mathbb{R}_+ \times \mathbb{R}} e^{-\pi x_1^2} e^{-\pi x_2^2} dx_1 dx_2 = \left(\int_{\mathbb{R}} e^{-\pi x_1^2} dx_1\right)^2 \quad \Rightarrow \quad \int_{\mathbb{R}} e^{-\pi x^2} dx = 1.$$

For any d,

$$\int_{\mathbb{R}^d} e^{-\pi|x|^2} dx = \prod_{i=1}^{d} \int_{\mathbb{R}} e^{-\pi x_i^2} dx_i = 1.$$

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B. The surface area of the unit ball.

Proof:

$$1 = \int_{\mathbb{R}^d} e^{-\pi|x|^2} dx \overset{\text{polar}}{=} \left(\int_0^\infty e^{-\pi r^2} r^{d-1} dr\right) \sigma\left(S^{d-1}\right) \overset{t=\pi r^2}{=} \left(\int_0^\infty e^{-t} t^{d/2-1} dt\right) \frac{1}{2\pi^{d/2}} \sigma\left(S^{d-1}\right).$$

Consequently, $\sigma\left(S^{d-1}\right) = 2\pi^{d/2}/\Gamma(d/2)$. ■

C. The volume of the unit ball.

Proof:

$$v_d = m(B) = \left(\int_0^1 r^{d-1} dr \right) \sigma \left(S^{d-1} \right) = \frac{1}{d} \frac{2\pi^{d/2}}{\Gamma(d/2)} = \pi^{d/2} / \Gamma(d/2 + 1).$$

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