Homework 2

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I. Stein 1.5: Relationship between measurable set and its outer $1/n$-near open set

A. $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$ *for compact* E.

Proof: First prove that $\lim \mathcal{O}_n = \bar{E}$. Then check that $m(\mathcal{O}_1) < \infty$. Finally use Corollary 3.3 (P20). ■

B. *Counterexamples when boundedness or closedness doesn't hold.*

E not bounded: $\mathbb{Z}$.

E not closed: complement of a Cantor-like set $\hat{\mathcal{C}}$ with $m(\hat{\mathcal{C}}) > 0$. Then $m(\hat{\mathcal{C}}^c) = 1 - m(\hat{\mathcal{C}}) < 1$ but $RHS \geq m(\bar{\hat{\mathcal{C}}^c}) = m([0,1]) = 1$. (see Exercise 4 for more details)

Another example: $E := \cup_{i=1}^{\infty}(q_i - \frac{\epsilon}{2^i}, q_i + \frac{\epsilon}{2^i})$, where $\{q_1, q_2, \cdots\} = \mathbb{Q} \cap [0,1]$, then check $\mathcal{O}_n \supset [0,1]$.

II. Stein 1.7: dilations - $m(\delta E) = \delta_1 \dots \delta_d m(E)$, $\delta_i > 0$

Proof: Use the equivalent definition of rectangular covering (Exercise 15, which we have done last week). Prove that if $\cup_{i=1}^{\infty} Q_i$ covers E, then $\cup_{i=1}^{\infty} \delta Q_i$ covers δE with $m(\delta Q_i) = \delta_1 \cdots \delta_d m(Q_i)$. ■

III. Stein 1.8: linear transformation of measurable sets are measurable sets

A. $L(E)$ *is compact if* E *is compact. And changing "compact" into "an* F_σ *set", the argument also holds.*

Proof: Since L is continuous, it conserves compactness (see your notes in Mathematical Analysis B3 for a complete proof). For F_σ set E, decompose it into closed sets and truncate each set by closed balls. Mathematically speaking,

$$E = \cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} \left(F_i \cap B_j(0)\right),$$

where F_i is closed and $B_j(0)$ is closed j-radius balls centered at 0. Therefore, an F_σ set can be countable unions of compact sets. Compactness of $L(F_i \cap B_j(0))$ completes the proof. ■

B. $L(E)$ *is measurable if* E *is measurable.*

Proof: Follow the hints. But check (1)why $c_d = 2\sqrt{d}$, (2)show $m(L(E)) = 0$ and (3)how to use Corollary 3.5. ■

IV. STEIN 1.9: AN EXAMPLE OF AN OPEN SET WHOSE CLOSURE'S BOUNDARY HAS POSITIVE MEASURE

Proof: When constructing the Cantor-like set $\hat{\mathcal{C}}$ with positive measure, we bisect $F_{0,1} := [0,1]$ by $G_{0,1}$ and the remains are $F_{1,1}$ and $F_{1,2}$. Iteratively, we bisect $\{F_{k,i}, 1 \le i \le 2^k\}$ by $\{G_{k,i}\}$ and yields $\{F_{k+1,i}, 1 \le i \le 2^{k+1}\}$. Thus we have

$$[0,1] = \hat{\mathcal{C}} \sqcup \left(\bigsqcup_{k\text{odd}, i \le 2^k} G_{k,i} \right) \sqcup \left(\bigsqcup_{k\text{even}, i \le 2^k} G_{k,i} \right) := \hat{\mathcal{C}} \sqcup \mathcal{O} \sqcup \tilde{\mathcal{O}},$$

where $\sqcup$ denotes the disjoint unions.

Two claims should be checked:

1) $\bar{\mathcal{O}} = \hat{\mathcal{C}} \sqcup \mathcal{O}$.

2) $\partial\big(\hat{\mathcal{C}} \sqcup \mathcal{O}\big) \supset \hat{\mathcal{C}}$.

For 1), It is obvious that $\mathcal{O} \subset \bar{\mathcal{O}}$ and $\tilde{\mathcal{O}} \cap \bar{\mathcal{O}} = \varnothing$, so it suffices to prove that any $x \in \hat{\mathcal{C}}$ can be approximated by points in $\mathcal{O}$ as close as possible. A trick is used. I call it *ternary-like expansion*. Any $x \in \hat{\mathcal{C}}$ can be mapped to a sequence $(a_1 a_2 \ldots)$ as follows. If $x \in F_{1,1}$, that is, it is in the left part, let $a_1 = 0$, else let $a_1 = 2$. Every time when a closed interval $F_{k,i}$ which contains x is bisected, a_k is determined as 0 or 2. Therefore we get a mapping from x to $\big(a_1 a_2 \ldots, a_k \in \{0,2\}\big)$. Besides, denote the centre point of $G_{k,i}$ as $(a_1 a_2 \ldots a_{k-1} 10 \ldots)$, where $a_i (i < k)$ obeys the bisection rule as above. Now the only thing to do is to approximate $\big(a_1 a_2 \ldots, a_k \in \{0,2\}\big)$ by $(a_1 a_2 \ldots a_{k-1} 10 \ldots)$ and check the difference between them are small enough (some tedious calculation, please check on your own).

Then, 2) is a easy task, since we can analogously approximate $\hat{\mathcal{C}}$ by $\tilde{\mathcal{O}}$. ■

Remark 1. *Some examples must be wrong: Cantor-like set $\hat{\mathcal{C}}$, $\cup_{i=1}^{\infty}(q_i - \epsilon/2^i, q_i + \epsilon/2^i)$.*

I think the correct example is so technical that one can hardly find another correct one.

V. STEIN 1.16: BOREL-CANTELLI LEMMA

This problem is MUCH MUCH easier than 1.9! The hint is enough.

VI. STEIN 1.20: EXAMPLES OF $m(A+B) > 0$ WITH ZERO-MEASURE SETS

Please check that (1)what does $A + B$ look like and (2)$m(A) = m(B) = 0$ but $m(A + B) > 0$.

Remark 2. *Many students don't realize the motivation of these two examples, and who never check the measure are WRONG!*

VII. Stein 1.13: Relationships between G_δ and F_σ sets

A. Closed sets are G_δ and open sets are F_σ.

Proof: Follow the hint to prove that a closed set F equals $\cup_{n=1}^{\infty}\mathcal{O}_n$. Then for the second part, the complement of an open set is closed, and $(G_\delta)^c = F_\sigma$. ■

B. An F_σ set which is not a G_δ.

Proof: Prove that $\mathbb{Q}$ is not G_δ. An easy proof is using Baire Theorem, see Zhou Minqiang P43 Example 13. Or a direct proof is as follows.

Suppose $\mathbb{Q} = \cap_{n=1}^{\infty} G_n$, where G_n are open. Define $\tilde{G}_n = G_n / r_n$, where r_n is the n-th rational in some enumeration. Note that the $\tilde{G}_n$ are also open and dense, but their intersection is the empty set, which is a contradiction. To be specific, as $\tilde{G}_1$ is open, there exists closed interval $[a_1, b_1] \subset \tilde{G}_1$. As $\tilde{G}_2$ is open, there exists closed interval $[a_2, b_2] \subset [a_1, b_1] \cap \tilde{G}_2$. Continue this process, there exists at least a point x belonging to all G_n, that is, $x \in \cup \tilde{G}_n$, where we meet the contradiction. ■

C. A Borel set neither F_σ nor G_δ.

Proof: $((0,1) \cap \mathbb{Q}) \bigcup ([2,3] \cap \mathbb{Q}^c)$. Notice that $\mathbb{Q}$ is not G_δ and $\mathbb{Q}^c$ is not F_σ. ■

VIII. Stein 1.19: properties of $A + B$

You must be familiar with this properties when studying Mathematical Analysis B3. But here's another hint for (b): if A is compact and B is closed, then $A + B$ is closed. Then truncate A by closed balls and follow the hint after the problem.

A counterexample for (c): $A = \mathbb{Z}$ and $B = \cup_{n=1}^{\infty}\{n + 1/n\}$, then the integers are not in $A + B$ but as close as possible.

IX. Stein 1.25: Equivalent definition of measurability by inner closed approximation

Proof: In this problem, we call our original definition as *open-measurable* and the alternative as *close-measurable*. If E is close-measurable, then E^c is open-measurable, since $E - F = F^c - E^c$. From Property 5 (P18), E is open-measurable. On the other hand, if E is open-measurable, from Theorem 3.4 (ii) (P21), E is close-measurable. ■

X. Stein 1.26: If $m(A) = m(B)$ and $A \subset E \subset B$, then E is measurable

Proof: Since $E - A \subset B - A$, we have that

$$m_*(E - A) \leq m_*(B - A) = m(B - A) = m(B) - m(A) = 0.$$

As any set of exterior measure 0 is measurable, $E - A$ is measurable. Consequently, $E = A \cup (E - A)$ is measurable. ■

XI. Stein 1.27: Existence of a compact set between two compact sets with an arbitrary measure between those two

Proof: Since E_1 is measurable, there is an open set $G \supset E_1$ with $m(U - E_1) < b - c$. Then $E_2 - U$ is compact with measure more than $c - a$. If we can find a compact subset $K \subset E_2 - U$ with $m(K) = c - a$, then $E := K \cup E_1$ will satisfy the conditions we want. Hence the problem is reduced to: given a compact set $F \subset \mathbb{R}^d$ with $m(F) = b$ and $c \in (0, b)$, find a compact subset $K \subset F$ with $m(K) = c$.

This can be solved as follows: Let $f(y) := m(F \cap B_y(0))$, where $B_y(0)$ is the close y-radius ball centered at 0. Then $f(0) = 0$, $f(y) = b$ for sufficiently large y since F is bounded. Next, we prove that f is continuous. For any y and $\epsilon > 0$, there exists δ such that $\forall |z - y| < \delta$, $|m\big(B_z(0)\big) - m\big(B_y(0)\big)| < \epsilon$. Then $|f(z) - f(y)| = m\Big(F \cap \big(B_z(0) \Delta B_y(0)\big)\Big) < \epsilon$. Hence f is continuous, and using the Intermediate Value Theorem completes the proof. ■

Remark 3. *One cannot directly consider*

$$f(t) = m\left((E_2 - E_1) \cap B_t(0)\right),$$

since $(E_2 - E_1) \cap B_t(0) + E_1$ *may not be compact!*