# Homework 4

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## I. Stein 2.4: check that $\int_0^b \int_x^b \frac{f(t)}{t} dtdx = \int_0^b f(t)dt$

*Proof:* WLOG, suppose $f \geq 0$, otherwise separate $f = f^+ - f^-$. Using Tonelli's Theorem, we have

$$\begin{aligned}\int_0^b \int_x^b \frac{f(t)}{t} dtdx &= \int_0^b \int_0^b \frac{f(t)}{t} \chi_{\{0\leq x\leq t\leq b\}} dtdx \\ &= \int_0^b \frac{f(t)}{t} \int_0^b \chi_{\{0\leq x\leq t\leq b\}} dxdt \\ &= \int_0^b f(t)dt.\end{aligned}$$

From the integrability of $f$ on $[0, b]$, we have also proved that $g$ is integrable. ■

**Remark 1.** *There's difference between Tonelli/Fubini theorem and changing the upper/lower limit of integral.*

## II. Stein 2.6: Integrability does not necessarily imply vanishing at infinity

### A. An counterexample

We construct a function that looks like an "impulse" at $n \in \mathbb{N}$. Consider

$$f(x) = \begin{cases} n^4 x + (n - n^5), & x \in [n - 1/n^3, n], \forall n > 1 \\ -n^4 x + (n + n^5), & x \in [n, n + 1/n^3], \forall n > 1 \\ 0, & else \end{cases}.$$

Since $\sum 1/n^2 < \infty$, $f$ is integrable, but $\limsup_{x\to\infty} f(x) = \infty$.

### B. Further assuming uniformly continuity guarantees vanishing

*Proof:* We prove by contradiction. Assume $\exists \epsilon > 0, \{x_n\}$ satifying $|x_n| \to \infty$ and

$$|f(x_n)| > \epsilon.$$

WLOG, $x_n > 0$ is increasing with $x_n - x_{n-1} > 1$ and $f(x_n) > 0$. As $f$ is uniformly continuous, $\exists 1 > \delta > 0$, such that $|f(x) - f(y)| > \epsilon/2, \forall x, y$. Therefore, $f(x) > \epsilon/2, \forall x \in (x_n - \delta, x_n + \delta)$, thus $f^+$ is not integrable, which is a contradiction with the integrability of $f$. ■

**Remark 2.** *The distance between $x_n$ and $x_{n-1}$ is important to avoid overlap of $(x_n - \delta, x_n + \delta)$ and $(x_{n-1} - \delta, x_{n-1} + \delta)$.*

## III. STEIN 2.11: $\int_E f dx \geq (=) 0, \forall E$ IMPLIES $f \geq (=) 0, a.e.$

*Proof:* We prove this fact by contradiction. Define $A_n = \{x : f(x) < -1/n\}$. If $m(\{x : f(x) < 0\}) > 0$, then $\lim_{m\to\infty} m(A_n) > 0$. Thus $\exists N$, s.t. $m(A_N) > 0$. On this measurable set, we have $\int_{A_N} f dx < -m(A_N)/N < 0$, which is a contradiction.

As a result, if $\int_E f dx = 0$, that is, $\int_E f dx \geq 0$ and $\int_E (-f) dx \geq 0$. We have proved that $f \geq 0, a.e.$ and $-f \geq 0, a.e.$, which results in $f = 0, a.e.$. ■

**Remark 3.** *One can not directly say that*

$$m(\{f < 0\}) > 0 \quad \Rightarrow \quad \int_{\{f<0\}} f dx < 0.$$

*The truncation technique (construction of $A_n$) is necessary.*