Homework 5

Weiyu Li

I. Stein 2.7: the graph of a measurable function is zero-measurable

Proof: Denote $\Gamma_M = \Gamma \cap \big(B_M^d(0) \times [-M, M]\big)$. Since $\Gamma = \cup_{M=1}^{\infty} \Gamma_M$, it suffices to prove that $m(\Gamma_M) = 0$ for any $M \geq 1$.

Fix M, separate $B_M^d(0) \in \mathbb{R}^d$ into disjoint measurable sets

$$A_{k,i} = \{x \in B_M^d(0) : \frac{i}{2^k} \leq f(x) < \frac{i+1}{2^k}\}, \quad -2^k M \leq i \leq 2^k M.$$

[Claim: measurable set $E \subset \mathbb{R}^d$ with $m(E) \in [0, \infty)$, then in $\mathbb{R}^{d+1}$, $m(E \times [a, b]) = (b - a)m(E)$.] Therefore, $\Gamma_M \subset \bigsqcup_i \left(A_{k,i} \times \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]\right)$ with

$$m_*(\Gamma_M) \leq \sum_i m(A_{k,i}) \frac{1}{2^k} = m(B_M^d(0)) \frac{1}{2^k} \to 0, \quad k \to \infty.$$

Thus $m(\Gamma_M) = m_*(\Gamma_M) = 0$. ∎

II. Stein 2.8: distribution function of an integrable function is uniformly continuous

Hint: Use the absolute continuity property of integral.

III. Stein 2.9: Tchebyshev inequality

Proof:

$$\int f \overset{f \geq 0}{\geq} \int_{E_\alpha} f \geq \int_{E_\alpha} \alpha = \alpha m(E_\alpha).$$

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IV. Stein 2.10: Equivalent representations of integrability: sum of survival times

A. Equivalence of (1) f is integrable, (2) $\sum 2^k m(F_k) < \infty$, (3) $\sum 2^k m(E_{2^k}) < \infty$,

Proof: (1) $\Rightarrow$ (2). In F_k, $2^k < f(x)$, thus

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) \leq \sum_{k=-\infty}^{\infty} \int_{F_k} f(x)dx = \int f(x)dx < \infty.$$

(2) $\Rightarrow$ (3). Notice that $m(E_{2^k}) = \sum_{j=k}^{\infty} m(F_j)$, we have

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = \sum_{k=-\infty}^{\infty} 2^k \sum_{j=k}^{\infty} m(F_j) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j} m(F_j) 2^k = \sum_{j=-\infty}^{\infty} m(F_j) 2^{j+1} < \infty.$$

(3) $\Rightarrow$ (2). From $m(F_k) < m(E_{2^k})$, it is trivial.

(2) $\Rightarrow$ (1). In F_k, $f(x) \le 2^{k+1}$, thus

$$\int f(x)dx = \sum_{k=-\infty}^{\infty} \int_{F_k} f(x)dx \le \sum_{k=-\infty}^{\infty} \int_{F_k} 2^{k+1}dx = \sum_{k=-\infty}^{\infty} 2^{k+1} m(F_k) < \infty.$$

■

B. Check when is f or g integrable.

Proof: For

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and $k \ge 0$, $m(E_{2^k}) = m(B_{2^{-k/a}}(0)) = 2^{-k\times d/a} m(B_1(0))$. For $k < 0$, $m(E_{2^k}) = m(B_1(0))$. Then we verify (3) that

$$\sum 2^k m(E_{2^k}) = m(B_1(0)) \left(1 + \sum_{k=0}^{\infty} 2^{k(1-d/a)}\right) = m(B_1(0)) \left(1 + \sum_{k=0}^{\infty} (2^{(1-d/a)})^k\right),$$

which is finite if and only if $a < d$.

For

$$g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

and $k < 0$, $m(E_{2^k}) = m(B_{2^{-k/b}}(0) \setminus \bar{B}_1(0)) = 2^{-k\times d/b} m(B_1(0)) - m(B_1(0))$. For $k \ge 0$, $m(E_{2^k}) = 0$. Then we verify (3) that

$$\sum 2^k m(E_{2^k}) = m(B_1(0)) \left(\sum_{k=-\infty}^{-1} 2^{k(1-d/b)} - 1\right) = m(B_1(0)) \left(\sum_{k=1}^{\infty} (2^{(d/b-1)})^k - 1\right),$$

which is finite if and only if $b > d$. ■

V. Stein Problem 2.3: L_1-convergence implies m-convergence

Proof: For every $\epsilon > 0$, by Chebyshev's inequality (Ex 2.9),

$$m\left(\{x : |f_k(x) - f(x)| > \epsilon\}\right) \le \frac{1}{\epsilon} \int |f_k - f| dx,$$

where RHS$\to 0$ since $f_k \to f$ in L_1. From the definition, we have that $\{f_k\}$ converges in measure to f.

Conversely, there's a counterexample that $f_k \overset{m}{\to} f$ but $f_k \overset{L_1}{\not\to} f$. Suppose $k = m + 2^n, 0 \le m < 2^n$, then check $f_k = 2^n \chi_{[\frac{m}{2^n}, \frac{m+1}{2^n}]}$ satisfies. ■

VI. STEIN 2.2: DILATION CONTINUITY OF LEBESGUE INTEGRAL

Proof: Since $C_c(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, $\forall \epsilon > 0$, $\exists g \in C_c(\mathbb{R}^d)$, $s.t.$ $\|f-g\|_{L^1} < \epsilon$. From Exercise 2.16, we know that $\|f^\delta - g^\delta\|_{L^1} = \|(f-g)^\delta\|_{L^1} = \prod_{i=1}^d |\delta_i|^{-1}\|f-g\|_{L^1} < \prod_{i=1}^d |\delta_i|^{-1}\epsilon \le 2^d\epsilon$ when $|\delta_i - 1| \le \frac{1}{2}$. Notice that g is uniform continuous with support $supp(g) \subset B_R(0)$, then $\exists \eta > 0$, such that $|g(x)-g(y)| < \epsilon, \forall \|x-y\| < \eta$. If $\|\delta - 1\| < \frac{\eta}{R}$, then $|g^\delta(x) - g(x)| < \epsilon$, and if we further set $|\delta_i - 1| < \frac{1}{2}$ ($\|\delta - 1\| < \frac{1}{2}$ is sufficient),

$$\|g^\delta - g\|_{L^1} = \int |g^\delta(x) - g(x)|dx < m(B_{1.5R}(0))\epsilon.$$

In summary, $\forall \epsilon > 0$, there exists R, η, such that for any $\|\delta - 1\| < \min\{\frac{1}{2}, \frac{\eta(\epsilon)}{R(\epsilon)}\}$,

$$\|f^\delta - f\|_{L^1} \le \|f^\delta - g^\delta\|_{L^1} + \|g^\delta - g\|_{L^1} + \|g - f\|_{L^1} < \left(2^d + m(B_{1.5R}(0)) + 1\right)\epsilon.$$

From the definition, we know that $\|f^\delta - f\| \to 0$ as $\delta \to 1$. ■

VII. STEIN 2.16: DILATION PROPERTY OF LEBESGUE INTEGRAL

Proof: The proof follows the path how we define Lebesgue integral. For character function χ_E, notice that

$$m(\{x : \delta x \in E\}) = m(\delta^{-1}E) = \prod_{i=1}^d |\delta_i|^{-1} m(E).$$

For simple function ϕ with its canonical form $\phi = \sum a_k \chi_{E_k}$,

$$\int \phi^\delta dx = \sum a_k m(\delta^{-1}E_k) = \prod_{i=1}^d |\delta_i|^{-1} \sum a_k m(E_k) = \prod_{i=1}^d |\delta_i|^{-1} \int \phi dx.$$

For $f \in BSF$(Bounded, finite-measure Support Functions), if $\phi_k \to f$, then $\phi_k^\delta(x) = \phi(\delta x) \to f(\delta x) = f^\delta(x)$, thus

$$\int f^\delta dx = \lim_{\phi_k^\delta \to f^\delta} \int \phi_k^\delta dx = \lim_{\phi_k \to f} \prod_{i=1}^d |\delta_i|^{-1} \int \phi_k dx = \prod_{i=1}^d |\delta_i|^{-1} \int f dx.$$

In stage three for nonnegative function f, $\{g \in BSF : 0 \le g \le f\} = \{g \in BSF : 0 \le g^\delta \le f^\delta\} = \{g' \in BSF : 0 \le g' \le f^\delta\}$, thus

$$\int f^\delta dx = \sup_{g' \in BSF : 0 \le g' \le f^\delta} \int g' dx = \sup_{g \in BSF : 0 \le g^\delta \le f^\delta} \int g^\delta dx = \sup_{g \in BSF : 0 \le g \le f} \prod_{i=1}^d |\delta_i|^{-1} \int g dx = \prod_{i=1}^d |\delta_i|^{-1} \int f dx.$$

Last, for any integrable f, separating it into $f = f^+ - f^-$ completes the proof. ■