Homework 6

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I. Stein 2.15: An a.e. converging series that is a.e. unbounded in any interval

Proof: Step 1. Prove that $\int f dx = 2$.

Denote $f_k = f\chi_{[\frac{1}{k},1]}$, then $\int^L f_k dx = \int^R f_k dx = 2(1 - \sqrt{\frac{1}{k}})$. By MCT (Monotone Convergence Theorem),

$$\int f dx = \lim_{k\to\infty} \int f_k dx = 2.$$

Step 2. Prove that $F(x) = \sum_{n=1}^{\infty} 2^{-n} f\left(x - r_n\right)$ is integrable.

This comes from the translation invariance of integral and MCT (considering $F_N(x) = \sum_{n=1}^{N} 2^{-n} f\left(x - r_n\right)$ and letting $N \to \infty$).

Step 3. Prove that any function $\tilde{F}$ that agrees with F a.e is unbounded in any interval $(a, b), a < b$.

$\forall M > 0, \exists r_n \in (a, b)$, $\exists x \in (a, b)$, s.t. $0 < x - r_n < (2^n M)^{-2}$. Then $F(y) \geq 2^{-n} f(y - r_n) > M$ for all $r_n < y < x$. Thus $\tilde{F} \geq M$ a.e. on an open interval (r_n, x). Since M is arbitrary, we have that $\tilde{F}$ is unbounded in any interval. ■

Remark 1. *Proving that F is not bounded does not imply that $\tilde{F}$ is unbounded, since $\tilde{F}$ may be different with those unbounded points in F.*

II. Stein 2.17: Fubini fails when not integrable

Just check.

III. Stein 2.18: Show f integrable if $|f(x) - f(y)|$ integrable on $[0, 1] \times [0, 1]$.

Proof: By Fubini's Theorem, that $|f(x) - f(y)|$ is integrable on $[0, 1] \times [0, 1]$ implies it is integrable on $x \in [0, 1]$ with y fixed and satisfying $|f(y)| < \infty$. Thus,

$$\left|\int_{[0,1]} f(x)dx\right| \leq \left|\int_{[0,1]} f(y)dx\right| + \left|\int_{[0,1]} \big(f(x) - f(y)\big)dx\right| \leq |f(y)| + \int_{[0,1]} |f(x) - f(y)|dx < \infty,$$

which shows that f is integrable on $[0, 1]$. ■

IV. Stein 2.19: (Survival time) $\int |f|dx = \int_0^{\infty} m(E_\alpha)d\alpha$.

Proof: Using Tonelli Theorem,

$$\int_0^{\infty} m\left(E_\alpha\right) d\alpha = \int_0^{\infty} \int_{\mathbb{R}^d} \chi_{\{|f(x)|>\alpha\}} dx d\alpha = \int_{\mathbb{R}^d} \int_0^{\infty} \chi_{\{\alpha<|f(x)|\}} d\alpha dx = \int_{\mathbb{R}^d} |f(x)|dx.$$

■

V. Stein 2.14: Evaluation of the volume of the unit ball in $\mathbb{R}^d$.

A. Calculate v_2.

Proof: Use Corollary 3.8 to the upper half circle in $\mathbb{R}^2$, and notice that the Lebesgue integral is the same as the Riemann integral. Thus,

$$\begin{aligned} v_2 =& 2\int_{[-1,1]} \left(1-x^2\right)^{1/2} dx = 2\int_{[-1,1]}^{R} \left(1-x^2\right)^{1/2} dx \\ \overset{u=\sin x}{=}& 2\int_{[-\frac{\pi}{2},\frac{\pi}{2}]}^{R} \cos^2 u du = \pi. \end{aligned}$$

■

B. The recursive expression of v_d.

Proof: Use Corollary 3.8 to the upper half ball in $\mathbb{R}^d$, that is,

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \le y \le (1-|x|^2)^{1/2}\}$$

and notice that $(1-|x|^2)^{1/2}$ is integrable. Thus,

$$\begin{aligned} v_d =& 2\int_{B_1^{d-1}} \left(1-|x|^2\right)^{1/2} dx \\ \overset{\text{Fubini}}{=}& 2\int_{[-1,1]} \left(\int_{x_1 B_1^{n-2}} \left((1-x_1^2)-|y|^2\right)^{1/2} dy \right) dx_1 \\ =& 2\int_{[-1,1]} \frac{1}{2} v_{d-1} (1-x_1^2)^{(d-1)/2} dx \\ =& 2v_{d-1} \int_0^1 \left(1-x^2\right)^{(d-1)/2} dx. \end{aligned}$$

■

C. Calculate v_d.

Proof: Change of variables (since Lebesgue integral is equal to Riemann integral in this case) and the definition of beta functions yields

$$\begin{aligned} \int_0^1 \left(1-x^2\right)^{(d-1)/2} dx \overset{y=x^2}{=}& \int_0^1 (1-y)^{(d-1)/2} y^{-1/2} dy \\ =& Beta\left(\frac{1}{2}, \frac{d+1}{2}\right) = \frac{\Gamma(1/2)\Gamma((d+1)/2)}{\Gamma(d/2+1)} \\ =& \frac{\pi\Gamma((d-1)/2+1)}{\Gamma(d/2+1)}. \end{aligned}$$

Induction and the value of v_2 complete the proof. ■

VI. Stein 2.21: properties of convolution $f * g$.

A. Measurability of $f(x-y)g(y)$ on $\mathbb{R}^{2d}$.

Proof: By Corollary 3.7 and Proposition 3.9, $f(x-y)$ and $g(y)$ are measurable on $\mathbb{R}^{2d}$. By Property 5 in Page 29, $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2d}$. ■

B. Integrability of $f(x-y)g(y)$ on $\mathbb{R}^{2d}$.

Proof: The measurability in VI-A guarantees that we can calculate $\int |f(x-y)||g(y)|$ (may be infinity). Applying Tonelli Theorem (Theorem 3.2) to $|f(x-y)||g(y)|$, we obtain

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |f(x-y)||g(y)|dxdy = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x-y)|dx\right)|g(y)|dy = \|f\|\|g\| < \infty,$$

thus $f(x-y)g(y)$ is integrable. ■

C. Well-definedness of $f * g(x)$.

Proof: This proof require the assumption that f and g are integrable.

From the integrability proved in VI-B and Fubini Theorem, $\int (f*g)dx < \infty$, thus well defined for a.e. x. ■

D. Integrability of $f * g$ and (Young's Inequality)$\|f*g\| \le \|f\|\|g\|$.

Proof: Fubini Theorem guarantees the integrability, as proved in the previous problem. Then,

$$\|f*g\|_1 = \int |f*g|dx = \int \left|\int f(x-y)g(y)dy\right| dx \le \int\int |f(x-y)||g(y)|dydx = \|f\|_1\|g\|_1.$$

If f and g are non-negative, then $\left|\int f(x-y)g(y)dy\right| = \int |f(x-y)||g(y)|dy$, which gives the equality. (The condition is sufficient but not necessary.) ■