Ex 3.4. (注："identically 0"为"$f\neq 0$ 在a.e.意义下")
(不恒为0)

定义：$f^*(x)=\sup_{x\in B}\frac{1}{m(B)}\int_B |f|dy$

(1) $\exists r_0$, s.t. $\int_{B_{r_0}(0)}|f(y)|dy=C_0>0$. 【记$A_m=\{x:|f(x)|>\frac{1}{m}$, 则$\lim m(A_m)>0$
$\Rightarrow \exists m(A_{m_0})>0$. 再由 $m(A_{m_0})=\lim_{r\to\infty} m(A_{m_0}\cap B_r(0))$
可取得r_0.】

$\forall\ 1\le|x|<r_0$, $f^*(x)\ge\frac{1}{m(B_{r_0}(0))}\int_{B_{r_0}(0)}|f(y)|dy\triangleq C_1>0$. $\left(C_1\ge\frac{C_1}{|x|^d}\right)$

$\forall\ |x|\ge r_0$, $f^*(x)\ge\frac{1}{m(B_{|x|}(0))}\int_{B_{|x|}(0)}|f(y)|dy\ge\frac{C_0}{|B_1(0)|\cdot|x|^d}\triangleq\frac{C_2}{|x|^d}$

Let $C=\min\{C_1,C_2\}$, then $f^*(x)\ge\frac{C}{|x|^d}$, $\forall\ |x|\ge 1$.

(2) the weak type estimate comes from 书p101 Th 1.1(iii)

(3) r_0 in (i) can be set as 1 now.

$\forall\ |x|\ge 1$, $f^*(x)\ge\frac{1}{|B_1(0)|}\cdot\frac{1}{|x|^d}$

if $\frac{1}{|B_1(0)|}\cdot\frac{1}{|x|^d}>\alpha$, then $f^*(x)>\alpha$

$\Leftrightarrow\quad |x|^d<\frac{1}{\alpha\cdot|B_1(0)|}$

for $\alpha\ll 1$, $m(\{x:f^*(x)>\alpha\})\gtrsim m\left(\left\{x:1\le|x|^d<\frac{1}{\alpha|B_1(0)|}\right\}\right)$

$\gtrsim m\left\{x:\frac{1}{2|B_1(0)|\alpha}\le|x|^d<\frac{1}{|B_1(0)|\alpha}\right\}$

$=\frac{1}{2}\cdot\frac{1}{\alpha}$.

#.

Ex 3.5. (a) Everyone knows that $\int_0^{\frac{1}{2}}\frac{1}{x(\log 1/x)^2}dx=-\frac{1}{\log x}\Big|_0^{\frac{1}{2}}=\frac{1}{\log 2}$

in Riemann Integral, but why we can use this calculation in Lebesgue Integral?

$\int_{\frac{1}{m}}^{\frac{1}{2}}\frac{1}{x(\log\frac{1}{x})^2}dx=\frac{1}{\log 2}-\frac{1}{\log m}$ is correct in Lebesgue integral.

Notice that $\frac{1}{x(\log\frac{1}{x})^2}\chi_{(0,\frac{1}{m})}$ is monotone,

so we can use Monotone Convergence Theorem to conclude that

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$$\int_0^{\frac{1}{2}} \frac{1}{x\log(\frac{1}{x})^2}\,dx = \lim_{m\to\infty}\int_{\frac{1}{m}}^{\frac{1}{2}} \frac{1}{x\log(\frac{1}{x})^2}\,dx = \frac{1}{\log 2}.$$

(b) $\forall x\in[0,\frac{1}{4}]$,

$$f^*(x) \geq \frac{1}{(0,2x)}\int_{(0,2x)}|f(x)|dx = \frac{1}{2x\log\frac{1}{2x}} \geq \frac{c_1}{x\log\frac{1}{x}} \quad \text{for some } c>0.$$

$\forall x\in(\frac{1}{4},\frac{1}{2}]$

$f^*(x) \geq \int_{[\frac{1}{2},\frac{1}{2}]} f(y)dy \geq \frac{c_2}{x\log\frac{1}{x}}$, since $x\log\frac{1}{x}$ is bounded in $[m,M]$ for some $m,M>0$.

$$\therefore f^*(x) \geq \frac{\min\{c_1,c_2\}}{|x|\log\frac{1}{|x|}}, \quad \forall |x|\leq\frac{1}{2}. \qquad —$$

$$\because \int \frac{1}{x\log\frac{1}{x}} = -\log\log\frac{1}{x} \to -\infty \quad (x\to 0+)$$

using the MCT argument again concludes that f^* is not locally integrable #.

Ex 3.7. From Cor 1.5., a.e. $x\in E^c$ is ~~not~~ a point of density of E^c. Suppose such an x exists.

$$\therefore \lim_{\substack{m(B)\to 0\\ x\in B}} \frac{m(B\cap E^c)}{m(B)} = 1 \qquad \exists B, \text{ s.t. } m(B\cap E^c) > (1-\alpha)m(B).$$

On the other hand, $m(B\cap E) \geq \alpha\, m(B)$

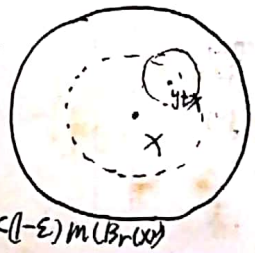
$\Rightarrow m(B) = m(B\cap E^c) + m(B\cap E) > m(B)$, which is a contradiction!

$\therefore$ every point in E^c is not a point of density of E^c

$\Rightarrow m(E^c) = 0 \Rightarrow m(E) = 1.$ #.

Ex 3.9. $\forall x\in F$ a point of density of F: $\lim\limits_{\substack{m(B)\to 0\\ x\in B}} \frac{m(B\cap F)}{m(B)} = 1$

$\forall \varepsilon>0, \exists r_\varepsilon$, s.t. $\frac{m(B_r(x)\cap F)}{m(B_r(x))} \geq 1-\varepsilon, \forall r\leq r_\varepsilon$



$\forall y\in B_{\frac{r}{2}}(0)\setminus B_{\frac{r}{2}}(0)$, if $B_r(x+y)\cap F = \phi$ for some $\varepsilon r < r' \leq \frac{r}{2}$

then $m(B_r(x)\cap F) = m(B_r(x)\setminus B_r(x+y))\cap F) \leq m(B_r(x)\setminus B_{r'}(x+y)) < (1-\varepsilon)m(B_r(x))$

which is a contradiction! $\Rightarrow B_{r'}(y)\cap F \neq \phi$, $\forall \varepsilon r < r' \leq \frac{r}{2}$

$\Rightarrow \delta(x+y) \leq \varepsilon r \Rightarrow \frac{\delta(x+y)}{|y|} \leq 4\varepsilon$, $\forall |y|\leq\frac{r_\varepsilon}{2}$ (由 r 任意性) $\Rightarrow \delta(x+y) = o(|y|)$ #

Ex 3.14: https://math.stackexchange.com/questions/1387604/prove-that-jump-functions-are-measurable

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