Homework 9

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I. Stein 3.13 (optional): Cantor-Lebesgue Function Is Not Absolutely Continuous

Proof: Denote F as the Cantor-Lebesgue function, and

1) $(a_1^1, b_1^1) = (0, 1)$.
2) $(a_1^2, b_1^2) = (0, \frac{1}{3}), (a_2^2, b_2^2) = (\frac{2}{3}, 1)$.
3) $(a_1^3, b_1^3) = (0, \frac{1}{9}), (a_2^3, b_2^3) = (\frac{2}{9}, \frac{1}{3}), (a_3^3, b_3^3) = (\frac{2}{3}, \frac{7}{9}), (a_4^3, b_4^3) = (\frac{8}{9}, 1)$.
4) .....

Then $\exists \epsilon = \frac{1}{2}, \forall \delta > 0, \exists n,\ s.t.\ (\frac{2}{3})^n < \delta$. Taking $\{(a_k^n, b_k^n)\}_{k=1}^{2^n}$, we have

$$\sum_{k=1}^{2^n} (b_k^n - a_k^n) = (\frac{2}{3})^n < \delta, \quad \text{but} \quad \sum_{k=1}^{2^n} |F(b_k^n) - F(a_k^n)| = 1 > \epsilon.$$

Therefore, $F \notin AC([0, 1])$. ■

II. Stein 3.19: Mapping Properties of Absolutely Continuous Functions

A. f maps zero-measure to zero-measure.

Proof: By the definition of AC functions, $\forall \epsilon > 0, \exists \delta > 0,\ s.t.\ \forall \sum_{k=1}^n |b_k - a_k| < \delta$, we have $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$. Suppose $m(Z) = 0$, then there exists open set $O,\ s.t.\ Z \subset O$, and $m(O) < \delta$. Notice that open set O can be written as $O = \cup_{k=1}^{\infty}(a_k, b_k)$ with a_k, b_k finite. Letting $m_k := \arg\min_{x \in [a_k, b_k]} f(x)$, $M_k := \arg\max_{x \in [a_k, b_k]} f(x)$, the continuity of f shows that $f(m_k), f(M_k)$ are both finite and then

$$f(Z) \subset f(O) = \cup_{k=1}^{\infty} f\big((a_k, b_k)\big) = \cup_{k=1}^{\infty} \big(f(m_k), f(M_k)\big).$$

Since $m\big(\cup_{k=1}^n (m_k, M_k)\big) \subset m(O) < \delta$ for any n,

$$m\big(f(Z)\big) \le m\big(f(O)\big) = \lim_{n \to \infty} m\bigg(\cup_{k=1}^n \big(f(m_k), f(M_k)\big)\bigg) \le \epsilon.$$

Letting $\epsilon \to 0$, we conclude that $m\big(f(Z)\big) = 0$. ■

B. f maps measurable to measurable.

Proof: Any measurable set E can be written as

$$E = (\cup_{k=1}^{\infty} F_k) \bigcup Z,$$

where $m(Z) = 0$ and F_k are compact sets. (F_k can first be closed by P23 Corollary 3.5, then notice that $F_k = \cup_{n=1}^{\infty}(F_k \cap [-n, n])$.) Since f is continuous, it maps compact sets to compact sets. From II-A, we know that $f(Z)$ is measurable with measure zero. Therefore,

$$f(E) = (\cup_{k=1}^{\infty} f(F_k)) \bigcup f(Z) \in \mathcal{M}.$$

■

Remark 1. *Continuous function f doesn't necessarily map closed sets to closed sets! For example: $f(x) = e^x$.*

III. Stein 3.20: Properties on Absolutely Continuous and Increasing Functions.

A. Construct a strictly increasing F with zero differentiation on a positive measure set.

Proof: Let C be a Cantor-like set on $[a, b]$ with $m(C) > 0$ (see P38 Exercise 1.4 for its construction on $[0, 1]$) and let $K = [a, b] \setminus C$. Construct $F(x) = \int_a^x \chi_K(x)dx$, then F is absolutely continuous with

$$F'(x) = \chi_K(x), \quad a.e.\ x \in [a, b],$$

which implies

$$F'(x) = 0, \quad x \in C \setminus Z,$$

where $m(Z) = 0$, and $m(C \setminus Z) = m(C) > 0$.

Also we should check that F is strictly increasing. Notice that C is closed and contains no open interval (Exercise 1.4.c), then K is open and $\forall a \le x_1 < x_2 \le b,\ K \cap (x_1, x_2) \neq \emptyset$. For an $x \in K \cap (x_1, x_2)$, there exists $(x - r, x + r) \subset K \cap (x_1, x_2)$, thus

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} \chi_K(x)dx \ge 2r > 0,$$

which implies that F is strictly increasing.

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B. There exists zero-measure set E, such that $F^{-1}(E) \notin \mathcal{M}$ for F in III-A.

Proof: First, prove that $m(F(C)) = 0$. Since K is open, $K = \sqcup_{n=1}^{\infty}(a_n, b_n)$, then by the increasing property,

$$m(F(K)) = m\big(\sqcup_{n=1}^{\infty} F((a_n, b_n))\big) = \sum_{n=1}^{\infty}(F(b_n) - F(a_n)) = \sum_{n=1}^{\infty}(b_n - a_n) = m(K) = m(F([a, b])).$$

Thus, $m(F(C)) = m(F([a, b])) - m(F(K)) = 0$.

Second, find the E. We claim that any set with positive measure has non-measurable subset. (This construction is the same as what we constructed in Chapter 1.) So there exists non-measurable set $N \subset C$, and we set

$$E = F(N).$$

Since $F(N) \subset F(C)$, $m(E) = 0$. Since F is strictly increasing, $F^{-1}(E) = N$ is non-measurable. ■

C. Prove that $F^{-1}(E) \cap \{F'(x) > 0\} \in \mathcal{M}$ for any increasing $F \in AC([a,b])$ and measurable $E \subset [A,B]$.

Proof: Let $A_n = F^{-1}(E) \cap \left\{F'(x) > \frac{1}{n}\right\}$. If we could prove that $A_n \in \mathcal{M}$, then $F^{-1}(E) \cap \{F'(x) > 0\} = \cup_{n=1}^{\infty} A_n$ is measurable. It remains to show that A_n is measurable.

The Hint claims that $m(O) = \int_{F^{-1}(O)} F'(x)dx$ for any open O, which can be checked by noticing $F^{-1}(O)$ is still open (in (a,b)) due to the continuity and writing the open sets as disjoint unions of open intervals. Then write

$$E = (\cap_{i=1}^{\infty} G_i) \setminus Z,$$

where G_i are open and $m(Z) = 0$. First, we have

$$\begin{aligned}
&\forall \epsilon > 0, \exists \text{open set } O \supset Z, m(O) < \epsilon \\
\Rightarrow\ & \epsilon > \int_{F^{-1}(O)} F'(x)dx \geq \int_{F^{-1}(O)\cap\{F'(x)>\frac{1}{n}\}} F'(x)dx \geq \frac{1}{n} m(F^{-1}(Z) \cap \{F'(x) > \frac{1}{n}\}) \\
\Rightarrow\ & m(F^{-1}(Z) \cap \{F'(x) > \frac{1}{n}\}) < n\epsilon, \forall \epsilon > 0 \\
\Rightarrow\ & m(F^{-1}(Z) \cap \{F'(x) > \frac{1}{n}\}) = 0. \qquad (1)
\end{aligned}$$

Second, by the continuity of F, $F^{-1}(G_i)$ is open. Since F' is integrable, it is measurable, which means that $\{F'(x) > \frac{1}{n}\}$ is measurable. Thus, $F^{-1}(G_i) \cap \{F'(x) > \frac{1}{n}\} \in \mathcal{M}$. Consequently,

$$A_n = \cap_{i=1}^{\infty} \left(F^{-1}(G_i) \cap \{F'(x) > \frac{1}{n}\}\right) \setminus \left(F^{-1}(Z) \cap \{F'(x) > \frac{1}{n}\}\right) \in \mathcal{M}.$$

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IV. Stein 3.21: Change of Variable Formula.

A. Measurability of $f(F(x))F'(x)$ on $[a,b]$.

Proof: WLOG, suppose $f > 0$. (Otherwise, separate f into f^+ and f^-.) For any $a > 0$,

$$\begin{aligned}
\{f(F(x))F'(x) > a\} &= \bigcup_{k=1}^{\infty} \left(\{F'(x) > r_k\} \cap \{f(F(x)) > \frac{a}{r_k}\} \right) \\
&= \bigcup_{k=1}^{\infty} \left\{ F'(x) > r_k, F(x) \in f^{-1}((\frac{a}{r_k}, +\infty)) \right\}.
\end{aligned}$$

Since $f^{-1}((\frac{a}{r_k}, +\infty))$ is measurable, following what we proved in III-C, we know that $\left\{F'(x) > r_k, F(x) \in f^{-1}((\frac{a}{r_k}, +\infty))\right\}$ is measurable, thus $\{f(F(x))F'(x) > a\}$ is measurable. From the definition, $f(F)F'$ is measurable. ■

Remark 2. *The Note in the problem says that $f(F(x))$ is not necessarily measurable, so we cannot have $\{f(F(x)) > \frac{a}{r_k}\} = \{F(x) \in f^{-1}((\frac{a}{r_k}, +\infty))\}$ measurable.*

B. Prove the change of variable formula.

Proof: WLOG, suppose $f > 0$. Since $f \in L^1$, it can be approximated by increasing simple functions $\{\phi_k\}$ with $0 \leq \phi_k \leq f$. Monotone Convergence Theorem makes the approximation valid, so it remains to prove that

$$\int_A^B \chi_E(y)dy = \int_a^b \chi_E(F(x))F'(x)dx, \quad E \subset [A, B], E \in \mathcal{M}.$$

By the Hint, it holds for $E = O$, where O is open. By (1), we can derive that the above formula holds for $E = Z$. Thus, it holds for all measurable E. ■