Sequences with bounded lcm for consecutive elements

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Abstract

Let $1 \leq a_1 < a_2 < .. < a_k \leq n$ be a sequence of positive integers, such that $\text{lcm}(a_{i-1}, a_i) \leq n$ for all i with $2 \leq i \leq k$. In [1, p. 34], it is conjectured that $k = O(\sqrt{n})$. In this short note, we will provide a proof of this conjecture.

Main result and proof

We can immediately state our main theorem.

Theorem. *For any sequence $1 \leq a_1 < a_2 < .. < a_k \leq n$ of positive integers with $lcm(a_{i-1}, a_i) \leq n$ for all i, we have $k < c\sqrt{n} + \log(2n)$, where the constant c is equal to* $\displaystyle\sum_{j=1}^{\infty} \frac{1}{(j+1)\sqrt{j}} \approx 1.86$.

Proof. For $n \leq 4$, our upper bound for k is trivially true since we then have $c\sqrt{n} + \log(2n) > n \geq k$. So we may safely assume that n is at least 5. Define B_j to be $\max(a_i : a_i - a_{i-1} \leq j)$, if this exists and 0 otherwise. Note that $B_n = a_k \leq n$. We have the following upper bound on k, in terms of the B_j:

$$\begin{aligned} k &\leq \sum_{j=1}^{n} \frac{B_j - B_{j-1}}{j} \\ &= \frac{B_n}{n} + \sum_{j=1}^{n-1} \frac{B_j}{j(j+1)} \\ &\leq 1 + \sum_{j=1}^{n-1} \frac{B_j}{j(j+1)} \end{aligned}$$

On the other hand, we also have an upper bound on B_j; if $a_{i-1} \geq \sqrt{jn}$, then:

$$\begin{aligned} n &\geq \text{lcm}(a_{i-1}, a_i) \\ &= \frac{a_{i-1}a_i}{\text{gcd}(a_{i-1}, a_i)} \\ &> \frac{a_{i-1}^2}{a_i - a_{i-1}} \\ &\geq \frac{jn}{a_i - a_{i-1}} \end{aligned}$$

implying that $a_i - a_{i-1} > j$, and thus we must have that $B_j < \sqrt{jn} + j$. Using this estimate, we obtain:

$$\begin{aligned} k &< 1 + \sum_{j=1}^{n-1} \frac{\sqrt{jn} + j}{j(j+1)} \\ &= 1 + \sum_{j=1}^{n-1} \frac{\sqrt{jn}}{j(j+1)} + \sum_{j=1}^{n-1} \frac{j}{j(j+1)} \\ &= \sqrt{n} \sum_{j=1}^{n-1} \frac{1}{(j+1)\sqrt{j}} + \sum_{j=1}^{n} \frac{1}{j} \\ &< c\sqrt{n} + \log(2n) \end{aligned}$$

where the last equality uses the fact that $n \geq 5$. And this finishes our proof. $\square$

References

[1] P. Erdös, R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*. L'Enseignement Math., Volume 28, Geneva, 1980. Also available here.