The Chromatic Number of the Plane

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Contents

1 Problem Statement

The Chromatic Number of the Plane, also called the Hadwiger-Nelson problem, is an unsolved geometric graph theory problem first formulated by Edward Nelson in 1950. The problem asks for the minimum number of colours needed to colour the plane such that any two points of unit distance (i.e. the distance between them is exactly one) from each other have a different colour, the value is often referred to as CNP. In 1951, de Brujin and Erdős proved that when all finite subgraphs of an infinite graph G can be coloured with c colours, G could be coloured with c colours as well. This reduced the Hadwiger-Nelson problem to that of finding the largest possible chromatic number for a finite unit-distance graph.[1] For any readers unfamiliar with the term, a unit-distance graph is a graph formed by distinct points, with an edge between two points if the distance between them is exactly one.

2 Early work on bounds

2.1 An Upper Bound

A construction of a colouring of the plane utilizing 7-colours can be seen as follows. Consider a tiling of the plane of hexagons with diameter slightly less than 1. In this case, all points within the same hexagon are less than unit distance, and a hexagon can be surrounded with 6 hexagons of different colours. Under this tiling, any 2 points of the same colour are either less than unit distance, in which case they are in the same hexagon, or more than unit distance, in which case they are separated by at least one hexagon of a different colour. This tiling gives $CNP \leq 7$ as an upper bound. See fig. 1 for a visualization of such a tiling.

It's certainly worth noting that no improvements have been made on this upper bound since the discovery of this hexagonal tessellation by J. Isbell in the 1950s.

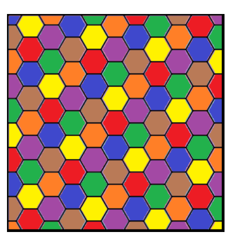

Figure 1: An example of a 7-colourable tiling of the plane, where each hexagon has diameter slightly less than one.

2.2 Trivial Lower Bounds

While proving $CNP > 3$ also proves that $CNP > 2$, a proof that $CNP > 2$ is presented anyway to show its simplicity.
Suppose the plane was 2-colourable. Now consider an equilateral triangle with side length equal to one embedded in the plane as a unit-distance graph. Then, two of the three points of the triangle must be the same colour, and since their corresponding side length is one, it contradicts that the plane has a valid 2-colouring.

Similarly, suppose the plane was 3-colourable. Consider an embedding of the Moser Spindle (fig. 2), a unit distance, 4-colourable graph, in the plane. The Moser Spindle graph is not 3-colourable, so two adjacent vertices contained within it must share the same colour, contradicting that the plane has a valid 3 colouring. The same result can instead be shown using an embedding of the Golomb graph (fig. 2), a separate unit-distance, 4-colourable graph. Thus, $CNP \geq 4$.

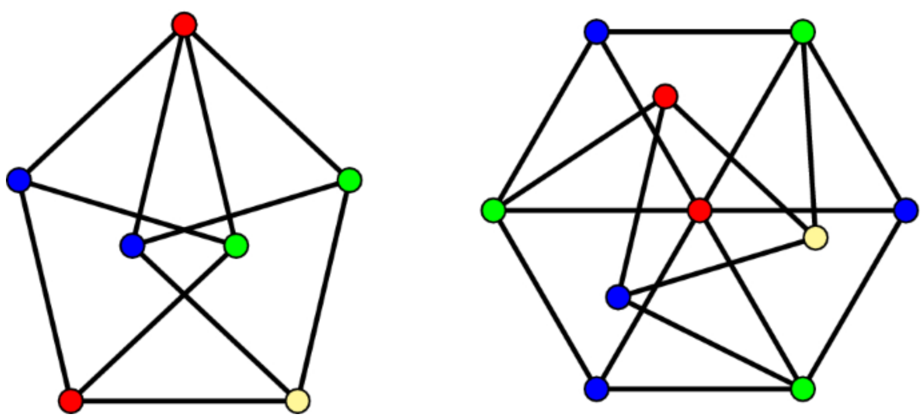

Figure 2: The Moser Spindle (left) and the Golomb Graph, both 4-coloured.

3 Improvement by De Grey

The lower bound of four remained for another 57 years, lasting until 2018 when Aubrey De Grey found a family of unit distance graphs that were not 4 colourable, with the smallest of which known to De Grey at the time of publication being one with 1581 vertices. The graph (fig. 3) is far too dense to make sense of its structure visually, but De Gray lies out his process for its construction in detail in [2]. A short summary of the construction is as follows. Begin by considering the 7-vertex, 12-edge unit distance graph H depicted in fig. 3. This graph can be coloured in four essentially distinct ways using at most 4 colours, with "essentially distinct" referring to colourings that are distinct up to rotation, reflection, and colour transposition. Note that two of these colourings consist of a monochromatic triple, a set of 3 vertices of the same colour. De Grey iteratively constructed graphs using copies of H to build larger graphs that are 4-colourable. From there, he exploits the presence or absence of monochromatic triples in the H subgraphs in proper colourings of these larger graphs to force a resulting graph to not 4-colourable. The 4-colourability check is done with computer assistance due to the size of the graphs. Surprisingly, this computation, based off a modified depth-first search, is efficient due to the forced monochromatic triples in the final resulting graph allowing many colour options to be fixed early in the search. Even for De Grey's initial 20,000 vertex graph, the verification was completed in mere minutes.

Since De Gray's discovery, a set of mathematicians began working to exploit the structural properties of his graph to find smaller examples of unit distance graphs that were not 4-colourable. This was largely done through the collaborative mathematics project Polymath. Since then, smaller 5-colourable unit-distance graphs have been found with the current smallest being a 510-vertex example found by Jaan Partsin 2019. See the polymath thread ([3]) for some interesting discussion and visualizations regarding these size improvements.

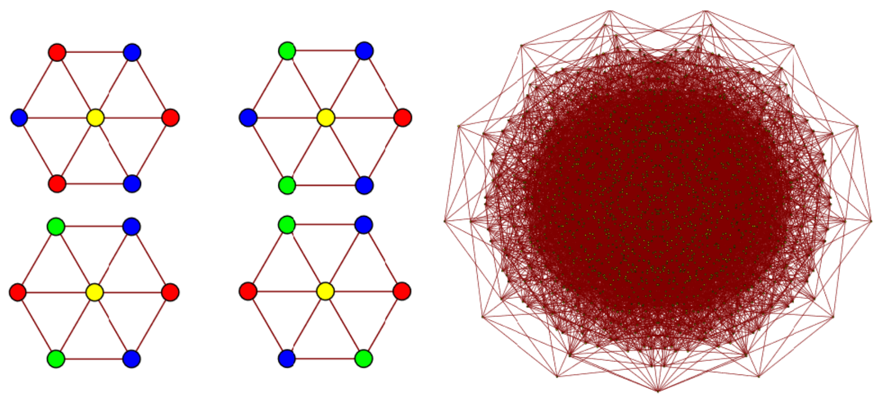

Figure 3: Left: H, the starting graph, and its 4 essentially distinct colourings. Right: A 1581-vertex unit distance graph that is not 4-colourable

4 Axiomatic Issues

4.1 The Axiom of Choice

All proofs of the De Brujin-Erdős Theorem discovered so far are dependent on the assumption that the Axiom of Choice holds. The Axiom of choice, hereafter referred to as AC, says that given any collection of nonempty sets it is possible to make a selection of exactly one element from each set, regardless of if there are infinite sets in the collection. Therefore, in certain models of mathematics where AC is not assumed true, the result by De Brujin and Erdős can not be assumed and the Hadwiger-Nelson problem cannot be reduced to the problem of finding the maximal chromatic number of a finite unit distance graph. The main controversy regarding the validity of AC is that it can be used to show the existence of objects that seem paradoxical or, even stranger, objects where it is consistent (impossible to contradict) that the object is not definable. A notable example of one of these illogical sounding results is what is called the Banach-Tarski paradox. This paradox states that, supposing AC holds, it can be proved that a 3-dimensional ball can be broken down into a finite number of point sets, and reassembled into two 3-dimensional balls with the same volume as the original[4]. While this makes no sense intuitively, the result is not actually a paradox, largely following from it being impossible to measure the volume of a point set.

4.2 Effects on the Hadwiger-Nelson problem

If AC were to not hold, and the Hadwiger-Nelson problem were not reducable by the De Brujin-Erdős Theorem, then it is entirely possible that the chromatic number of the plane (an infinite unit-distance graph) could be greater than the chromatic number of all finite unit-distance graphs. The only way to properly bound CNP by above would be to prove the result for infinite unit-distance graphs, which is certainly a difficult task. Especially since computer assistance is a large factor in recent results, and would likely be much harder to apply to the infinite case.

The lower bound for the Hadwiger-Nelson problem would however not be affected, as it would still hold that the chromatic number of the plane cannot be less than the chromatic number of a subset of the plane.

5 A General Statement and Similar Problems

One may ask the question, "Is a similar problem on a different metric space easier? Harder?" This section will introduce some examples to show the varying complexity of that question and how different it can be in approach. Referring to the metric of a set as a distance function, the problem can be stated in its most general form as:

"Given a metric space M, with distance function d. How many colours are necessary to colour all elements of M such that for all $m, n \in M$, if $d(m, n) = k$, m and n have different colours."

Clearly, taking $\mathbb{R}^2$ and the Euclidean distance as a metric space with $k = 1$ gives the standard definition of the Hadwiger-Nelson problem. For the following examples, we define $\chi(M, d, k)$, where M is the set, d is the metric, and k is an element of the image of d. This notation is convenient, and certainly much easier to write. The Hadwiger-Nelson Problem, for example, would be represented as $CNP = \chi(\mathbb{R}^2, || \cdot ||, 1)$, where $|| \cdot ||$ is the Euclidean norm, much shorter than the problem statement given in section 1.

5.1 Higher Dimension

We first will look at problems of the form $(\chi(\mathbb{R}^n), ||\cdot||, 1)$ for $n > 2$. This is the same as the statement of the Hadwiger-Nelson problem for dimension greater than two. For example, the $n = 3$ case (usually referred to as the chromatic number of space) is currently known to be bounded below by 6, and bounded above by 15. The upper bound, shown by D. Coulson, was a result of a lattice colouring scheme[5], while the lower bound, shown by O. Nechustan, results from a three-dimensional, unit-distance graph that is shown to have no 5-colouring[6].

In the n-dimensional case the current best known bounds are

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$$

with $o(1)$ being a function $f(n)$ such that $\lim_{n\to\infty} f(n) = 0$. The lower bound is a result of A. M. Raigorodksii[7] and the upper bound a result of Larman and Rogers[8]. An alternate proof of the upper bound can be exhibited using a multilattice approach, as shown by Prosanov.[9] While still complicated, it is simpler than the original proof my Larman and Rogers.

5.2 Lower Dimension

Here, we consider the problem $\chi(\mathbb{R}, ||\cdot||, 1)$. The simplest way to phrase this problem would be to ask how many colours are needed to colour the real numbers such that if x is one colour, say red, $x \pm 1$ are both not red. The solution is almost as simple as the statement, consider two colours, say blue and green. $\mathbb{R}$ can be coloured by considering an infinite sequence of distinct sub-intervals $[a, b)$, with $0.5 < b - a < 1.0$ such that their union covers $\mathbb{R}$, and colouring them in an alternating fashion to construct a valid 2-colouring. Since a 1-colouring is clearly not possible, this is the minimum, so $\chi(\mathbb{R}, ||\cdot||, 1) = 2$.

In fact, the same approach works for any k, as long as $\frac{k}{2} < b - a < k$
Thus, the real numbers are always 2-colourable under the Euclidean norm.

5.3 An example on the integers

Consider $\chi(\mathbb{Z}^2, ||\cdot||, 1)$. A fun way to visualize this problem is to consider an infinite sheet of grid paper, with each square being of side length 1. How can this be coloured such that no square on the grid has 2 connected vertices of the same colour? In other words, it asks the same question as the Hadwiger-Nelson problem, but only for points with integral values.

A 2-colouring can be shown as follows. First, colour $(0, 0)$, say green. Then, $(1, 0), (-1, 0), (0, 1), (0, -1)$ must be coloured the other colour, say orange. Repeating the process a few times displays a very intuitive pattern depicted in fig(4). From the figure, a clear formulaic approach to the colouring is to consider the sets of points $S_1 = \{(x, x + 2b) : x \in \mathbb{Z}, b \in \mathbb{Z}\}$ and $S_2 = \{(x, y - 1) : (x, y) \in \mathbb{Z}\}$. A valid colouring is formed from colouring all points of S_1 green, and all points of S_2 orange. Its clear to see that $S_1 \cup S_2 = \mathbb{Z}^2$

and that the four neighbours of any point in S_1 are in S_2, and similarly the four neighbours of a point in S_2 are in S_1, where

$$(u, v) \in N((x, y)) \to ||(u, v) \cdot (x, y)|| = 1$$

This shows that the above is a valid 2-colouring, and thus $\chi(\mathbb{Z}^2, ||\cdot||, 1) = 2$.

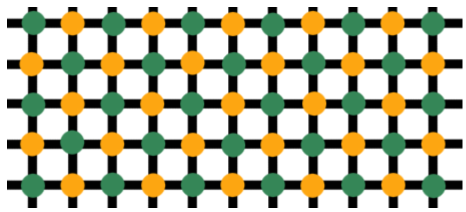

Figure 4: A small portion of $\mathbb{Z}^2$ coloured using 2 colours.

5.4 Something a bit different

So far all examples of metric spaces have used infinite sets of numbers, but that's not a requirement. As long as there exists a suitable metric for the objects, any set of objects can be used. With that in mind, consider a set of strings $S = \{$"abc", "arc", "art", "bat", "cab", "car", "cat", "crc", "crr", "tab", "tar", "trr"$\}$, this set, combined with the Hamming Distance, say d, a measurement of how many characters differ between two strings, is a metric space. Thus, we can consider $\chi(S, d, 1)$. To answer this problem we first construct a graph as follows. Let $G = (V, E)$ with $V = \{x | x \in S\}$, $E = \{(x, y) | x \in V, y \in V, d(x, y) = 1\}$. In English, this is a graph with each "word" in the set S as a vertex, with an edge between two words in the graph if and only if they differ by exactly one character. A visualization of the graph is provided in fig 5, with a valid 3-coluoring. Its clear to see that this is the minimal, as a 2-colouring is not possible due to the K_3 subgraph present in G.

While perhaps not the most mathematically useful example, this exhibits that the general problem can be applied to a variety of different sets and issues, and also shows that in the case of a finite set that the problem can be approached by first constructing the graph and then verifying its colourability.

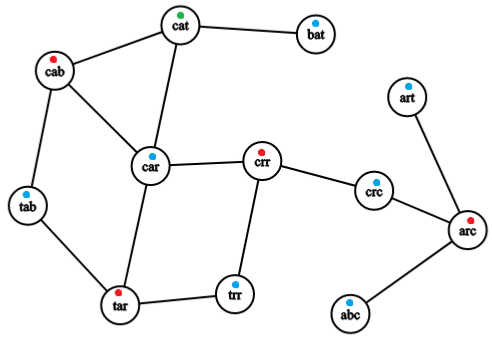

Figure 5: G exhibited with a 3-colouring

6 Conclusion

From this study, its clear that the Hadwiger-Nelson problem is a far deeper problem than its simple statement makes it seem and that only through the multitude of approaches from a multitude of talented mathematicians, has it become as close to solved as it is today. Its tie-ins with the axioms of set theory give even more ambiguity to a solution, though AC is widely regarded to be true, so the problem's reliance on it will likely remain a footnote. De Grey's recent breakthroughs gives hope that a solution is not only possible, but feasible through computer assistance and smart construction, even if it takes another 57 years. Furthermore, generalizing the problem to allow the colouring of arbitrary metric spaces may help provide a new approach to the problem, but even if not, it is a fun exercise on different metric spaces.

7 References

[1] N.G. De Brujin, P. Erdős. (1951), *A colour problem for infinite graphs and a problem in the theory of relations*. Nederl. Akad. Wetensch. Proc. Ser. A, 54: 371–373,

[2] A. De Grey. (2018), *The Chromatic Number of the Plane is at least 5*. Geombinatorics, 28: 5–18, arXiv:1804.02385

[3] Various. (2018), *https://dustingmixon.wordpress.com/2019/07/08/polymath16-thirteenth-thread-bumping-the-deadline/#comment-23999*

[4] T. Tao (2011), *An introduction to Measure Theory*. p.3

[5] Coulson, D. (2002), *A 15-colouring of 3-space omitting distance one", Discrete Math., 256 (1–2): 83–90*. doi:10.1016/S0012-365X(01)00183-2

[6] O. Nechustan. (2002), *On the Space Chromatic Number* Discrete Mathematics 256 (2002) 499-507

[7] A. M. Raigorodskii. (2000). *On the chromatic number of a space* Uspekhi Mat. Nauk, 55(2(332)):147–148

[8] D. G. Larman and C. A. Rogers. (1972) *The realization of distances within sets in Euclidean space* Mathematika, 19:1–24

[9] R. Prosanov. (2018). *A new proof of the Larman-Rogers upper bound for the chromatic number of the Euclidean space* arXiv:1610.02846