Analysis I
Lecture Notes

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This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

The textbook of the class is Understanding Analysis, by Stephen Abbott. I am additionally using Principles of mathematical analysis, by Walter Rudin. Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 What are the real numbers?

Lecture 1

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

1.1 Algebraic Structure

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

Definition 1.1 (Field Axioms)

A set F is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

i. (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

ii. (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

iii. (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.

iv. (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = (-x) + x = 0$. If $x \neq 0$, there exits an element x^{-1} such that $xx^{-1} = x^{-1}x = 1$.

v. (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$.

1.2 Order Structure

1.2.1 Ordering

Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

Definition 1.2 (Ordering)

An *ordering* on a set F is a relation, represented by $\leq$, with the following properties:

i. $x \leq y$ or $y \leq x$, for all $x, y \in F$.

ii. If $x \leq y$ and $y \leq x$, then $x = y$.
iii. If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.

Additionally, a field F is called an *ordered field* if F is endowed with an ordering $\leq$ that satisfies

iv. If $y \leq z$, then $x + y \leq x + z$.
v. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Lecture 2

1.2.2 Bounds

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.

Definition 1.3 (Upper bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is an *upper bound of* A if $a \leq b$ for all $a \in A$.

If a set A has an upper bound, we say that A is *bounded above*.

Definition 1.4 (Supremum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *least upper bound of* A, or *supremum of* A, denoted by $\sup(A)$, if:
i. s is an upper bound of A, and
ii. if b is any upper bound of A, then $s \leq b$.

Proposition 1.5 (The supremum, if it exists, is unique)

If s and s' are both supremum of A, then $s = s'$.

Example

Let $F = \mathbb{Q}$ and $A = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\}$. Then, $\sup(A) = 0$.

Analogously, we can define lower bounds and least upper bounds.

Definition 1.6 (Lower bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is a *lower bound of A* if $a \geq b$ for all $a \in A$.

If a set A has a lower bound, we say that A is *bounded below*.

Definition 1.7 (Infimum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *greatest lower bound of A*, or *infimum of A*, denoted by $\inf(A)$, if:

i. s is a lower bound of A, and
ii. if b is any lower bound of A, then $s \geq b$.

1.2.3 Completeness

Definition 1.8 (Completeness)

Given F an ordered field, we say F is *complete* if, for any subset $A \subset F$ bounded above and nonempty, the supremum of A exists[a].

[a] and is an element of F, as the definition requires.

Theorem 1.9 (Unique complete ordered field)

There exists a unique complete ordered field, up to isomorphism.

The proof of this theorem is beyond the scope of this course. One can show the existence of such a field by creating a field of Dedekind cuts. A Dedekind cut is a subset $C \subset \mathbb{Q}$ such that, if $c \in C$, then all rational numbers $x < c$ also are in C; and that C has no supremum in $\mathbb{Q}$. Addition can be defined using set addition. Multiplication is harder to define, since it is needed a separation between "non-negative" and "negative" numbers. Ordering can be defined using subsets. Finally, one has to prove all the axioms (field axioms, ordering axioms, and the axiom of completeness).

Definition 1.10 (Real numbers)

The set of real numbers, denoted by $\mathbb{R}$, is the complete ordered field.

Question. If $\mathbb{R}$ is defined in such a axiomatic way, how can we say that $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$?

Recall that the statement $\mathbb{Z} \subset \mathbb{Q}$ is also a strange statement. A rational number is actually a pair of integers; how can a single integer be also a pair of integers? To be fair, in a set-theoretical sense, it is in fact untrue that $\mathbb{Z} \subset \mathbb{Q}$. However, the set of rational numbers of the form $\frac{n}{1}$ has the same structure (with respect to multiplication and addition) as the set of integers. Therefore, when we say "$\mathbb{Z} \subset \mathbb{Q}$", we actually mean that there exists a subset of $\mathbb{Q}$ that is isomorphic to $\mathbb{Z}$. This difference is usually not interesting for us, when studying Analysis. In the rare cases where this kind of difference is relevant, we say sentences like "there is a copy of $\mathbb{Z}$ in $\mathbb{Q}$."

In a similar fashion, the additive group generated by the identity of $\mathbb{R}$* is isomorphic to $\mathbb{Z}$; as well as the field generated by the identity of $\mathbb{R}$† is isomorphic to $\mathbb{Q}$. Therefore, there are copies of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$, or, more informally, $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$.

1.3 Absolute value

Before we dig more deeply into the idea of a supremum, consider this definition that comes just from the structure of an ordered field.

Definition 1.11 (Absolute value)

If F is an ordered field, and $x \in F$, let

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

Theorem 1.12

If F is an ordered field, and $x \in F$, then $|x| \geq 0$.

Proof. If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x = |x|$. ∎

Theorem 1.13

If F is an ordered field, and $x \in F$, then $|-x| = |x|$.

*In other words, the smallest additive group that contains the identity of $\mathbb{R}$.

†In other words, the smallest field that contains the identity of $\mathbb{R}$.

Proof. If $x \geq 0$, then $0 = x + (-x) \geq 0 + (-x) = -x$, therefore $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$, therefore $|-x| = -x = |x|$. ∎

Theorem 1.14

If F is an ordered field, and $x, y \in F$, then $|xy| = |x||y|$.

Proof. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.

If $x \geq 0$ and $y \leq 0$, then $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the previous case with x and $-y$ and also Theorem 1.13 to obtain $|xy| = |-xy| = |x(-y)| = |x||-y| = |x||y|$.

If $x \leq 0$ and $y \geq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$. So we apply the first case with $-x$ and y and also Theorem 1.13 to obtain $|xy| = |-xy| = |(-x)y| = |-x||y| = |x||y|$.

If $x \leq 0$ and $y \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$ and $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the first case with $-x$ and $-y$ and also Theorem 1.13 to obtain $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. ∎

Theorem 1.15 (Triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x + y| \leq |x| + |y|.$$

Proof. If $x \geq 0$, then $|x| = x$. If $x \leq 0$, then $x \leq 0 = x + (-x) \leq 0 + (-x) = x$, so $|x| = -x \geq x$. In either case, $|x| \geq x$.

Thus, $|x| + |y| \geq x + y$ and $|x| + |y| = |-x| + |-y| \geq -x - y = -(x + y)$. Since $|x + y| = x + y$ or $|x + y| = -(x + y)$, in either case, $|x| + |y| \geq |x + y|$. ∎

Theorem 1.16 (Reverse triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x - y| \geq ||x| - |y||.$$

Proof. Triangle inequality implies that $|x| = |(x-y)+y| \leq |x-y|+|y|$ and $|y| = |(y-x)+x| \leq |y-x|+|x|$. Equivalently, we have $|x-y| \geq |x|-|y|$ and $|x-y| \geq |y|-|x|$; consequently, $|x-y| \geq ||x|-|y||$. ∎

1.4 Supremum again

Lecture 3

Theorem 1.17 (ϵ-sup Theorem)

Given $A \subset \mathbb{R}$ nonempty and bounded above, and given s an upper bound of A, then $s = \sup(A)$ if, and only if, for all $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

Proof. Supppose $s = \sup(A)$. Then, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that $a > s - \epsilon$.

Suppose $s \neq \sup(A)$. Then, there exists an upper bound of A that is smaller than s, say $s - \delta$. Then, it follows that, for $\epsilon = \delta/2$, there is no $a \in A$ such that $a > s - \delta/2$, because $a \leq s - \delta < s - \delta/2$ for all $a \in A$. ∎

Definition 1.18 (Sum of Sets)

Given $A, B \subset \mathbb{R}$, we define their sum as

$$A + B = \{a + b : a \in A, b \in B\}$$

Theorem 1.19 (Supremum of Sum of Sets)

If $A, B \subset \mathbb{R}$ are both nonempty and bounded above, then

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Since $\sup(A)$ is an upper bound of A, it holds that $a \leq \sup(A)$ for all $a \in A$. Since $\sup(B)$ is an upper bound of B, it holds that $b \leq \sup(B)$ for all $b \in B$. Therefore, $a+b \leq \sup(A)+\sup(B)$ for all $a \in A$ and $b \in B$, i.e., $x \leq \sup(A)+\sup(B)$ for all $x \in A + B$; thus, $\sup(A) + \sup(B)$ is an upper bound of $A + B$.

Let $\epsilon > 0$ be any positive real number. ϵ-sup Theorem implies that there exists $a \in A$ such that $a > \sup(A) - \epsilon/2$. ϵ-sup Theorem also implies that there exists $b \in B$ such that $b > \sup(B) - \epsilon/2$. Therefore, there exist $a \in A$ and $b \in B$ such that $a + b >$

$\sup(A) + \sup(B) - \epsilon$; thus, there exists $x \in X$ such that $x > \sup(A) + \sup(B) - \epsilon$. Finally, by ϵ-sup Theorem, $\sup(A + B) = \sup(A) + \sup(B)$. ∎

Lecture 4

1.5 Archimedean Properties

Theorem 1.20 (Archimedean Properties)

i. Given any $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_{>0}$ with $n > x$.
ii. Given any $y > \mathbb{R}_{>0}$, there exists some $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < y$.

Proof. The first statement is equivalent to $\mathbb{Z}_{>0}$ is not bounded above.

Suppose $\mathbb{Z}_{>0}$ is bounded above. Then, there exists $s = \sup(\mathbb{Z}_{>0})$. Therefore, $s - 1$ is not an upper bound of $\mathbb{Z}_{>0}$, i.e., there exists $n \in \mathbb{Z}_{>0}$ such that $s - 1 < n$. However, this implies $s < n + 1 \in \mathbb{Z}_{>0}$, implies s is not an upper bound of $\mathbb{Z}_{>0}$, which is a contradiction.

The second statement follows from the first one by setting $x = \frac{1}{n}$. ∎

Theorem 1.21 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For all $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ with $a < q < b$.

Proof. By Archimedean Properties with $y = b - a > 0$, there exists $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < b - a$.

Let m be the smallest natural number greater than na. Then,

$$m - 1 \leq na < m$$
$$\frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n}.$$

The first inequality implies that $\frac{m}{n} \leq a + \frac{1}{n} < b$, so finally, we conclude that

$$a < \frac{m}{n} < b.$$

∎

Corolary 1.22

For all $a, b \in \mathbb{R}$, with $a < b$, there exist infinitely many $q \in \mathbb{Q}$ with $a < q < b$.

Lecture 5

1.6 Nested Interval Property

Theorem 1.23 (Nested Interval Property)

Suppose we have a sequence of closed intervals $I_n = [a_n, b_n]$, with $a_n \leq b_n$, that are nested decreasing, i.e.,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Proof. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$.

The "nested" condition implies that, if $i < j$, then $[a_i, b_i] \supset [a_j, b_j]$. Therefore, $a_j, b_j \in [a_i, b_i]$, which implies that $a_i \leq a_j \leq b_j \leq b_i$ for all $i < j$. Note that this implies that

$$a_i \leq b_j \text{ and } a_j \leq b_i, \text{ for all } i < j.$$

We can rewrite it as

$$a_i \leq b_j, \text{ for all } i \text{ and } j.$$

This implies that a_i is a lower bound of B for any i, and also implies that b_j is an upper bound of A for any j. Since A is bounded above, we can define $x = \sup(A)$. Clearly, x is an upper bound of A.

Suppose x is not a lower bound of B. Then, there exists n such that $x > b_n$. The ϵ-sup Theorem, with $\epsilon = x - b_n > 0$, implies that there exists m such that $a_m > x - (x - b_n) = b_n$, which contradicts the previous displayed equation. Therefore, x is a lower bound of B.

Finally, x is both an upper bound of A and a lower bound of B, thus, for all n, $a_n \leq x \leq b_n$, i.e., $x \in [a_n, b_n]$. Therefore, x is in such intersection. ∎

1.7 Cardinality

Question. Are all sets with an infinite number of elements the same size?

Definition 1.24 (Cardinality)

Given two sets A and B, we say that A *and* B *have the same cardinality* if there exists a bijection $f: A \to B$. We will write $A \sim B$ to say that A and B have the same cardinality.

Example

The sets $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ have the same cardinality.

Definition 1.25 (Countability)

We say a set S is *countable* if it has the same cardinality as $\mathbb{N}$. If a set is not a finite set and not countable, then we say it is *uncountable*.

Proposition 1.26 ($\mathbb{N}^2$ is countable)

$\mathbb{N}^2$ is countable.

Proof. The function $f: \mathbb{N}^2 \to \mathbb{N}$, defined by

$$f(i, j) = \frac{(i + j - 1)(i + j - 2)}{2} + i$$

is a bijection. ∎

Lecture 6

Theorem 1.27

If A is countable and B is countable, then $A \times B$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable, then $A_1 \times \cdots \times A_n$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable or finite, then $A_1 \times \cdots \times A_n$ is countable or finite.

Theorem 1.28

If $S_1, S_2, \dots$ are each countable, then their union is countable.

Similarly, if $\{S_i\}_{i\in I}$ is a countable or finite collection of sets, which are each countable or finite; then their union is countable.

Example

Let $\mathcal{T}$ be the collection of finite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, let A_i be the collection of subsets of $\{1, 2, \ldots, i\}$. Note that $|A_i| = 2^i$, thus A_i is finite. Then, note that $\varnothing \in A_1$, and, if $S \in \mathcal{T}$ is non-empty, it holds that $S \in A_{\max(S)}$; so $\mathcal{T} = \bigcup_{i=1}^{\infty} A_i$.

Therefore, by Theorem 1.28, we conclude that $\mathcal{T}$ is countable or finite. Since $\mathcal{T}$ is not finite, then it is countable.

Theorem 1.29

If A is countable, and $f : A \to B$ is surjective, then B is countable or finite.

Similarly, if A is countable, and $f : B \to A$ is injective, then B is countable or finite.

In particular, if A is countable, and $A \supseteq B$, then B is countable or finite.

Proposition 1.30 ($\mathbb{Q}$ is countable)

$\mathbb{Q}$ is countable.

Proof. Consider the function $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ defined by $f(a, b) = \frac{a}{b}$. Clearly, $f(p, q) = \frac{p}{q}$ for any $\frac{p}{q} \in \mathbb{Q}$. ∎

Proposition 1.31

$\mathbb{R}$ is not countable.

Proof (using nested intervals). Assume $\mathbb{R}$ is countable. So, there exists a bijection $f : \mathbb{N} \to \mathbb{R}$.

Let $I_1 = [f(1) + 1, f(2) + 2]$. Note that $f(1) \notin I_1$. We will define I_{n+1} recursively. Suppose $I_n = [a, b]$, then, define I_{n+1} as either $[a, \frac{2a+b}{3}]$ or $[\frac{a+2b}{3}, b]$ such that $f(i + 1) \notin I_{n+1}$; that is possible since $f(i + 1)$ cannot be in both sets.

By the Nested Interval Property, there exists a real number $r \in \bigcap_{i=1}^{\infty} I_n$. However, since f is a bijection, there exists $m \in \mathbb{N}$ such that $f(m) = r$. Therefore, $r \notin I_m$, a contradiction. ∎

Lecture 7

Proof (using Cantor's diagonalization). We'll prove $(0, 1)$ is uncountable, which implies $\mathbb{R}$ is uncountable.

Assume $(0, 1)$ is countable, therefore, there exists a bijective function $f : \mathbb{N} \to (0, 1)$.

Let's write out decimal expansions[a] of $f(1), f(2), \ldots$. If there's doubt between a recurrent 9 or a recurrent 0 in the end, we choose the latter form. We write

$$f(i) = 0.a_{i1}a_{i2}a_{i3}\ldots,$$

with $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $b_k = 1$, if a_{kk} is odd, and $b_k = 2$, if a_{kk} is even. Note that $c_k \neq b_{kk}$ and $c_k \notin \{0, 9\}$ for all k. Therefore, $x = 0.b_1b_2b_3\ldots$ cannot be on the image of f; a contradiction. ∎

[a] What are decimal expansions? We only need to know that decimal expansions are unique except for some duplication, like $0.09999 = 0.1$.

Another perspective on the Cantor's proof arises by using the binary base, instead of the decimal base. For each real number $x = 0.x_1x_2x_3\ldots$, we can define a $f(x) = \{n \in \mathbb{N} : a_n = 1\}$. This is almost[‡] a bijection because, but nevertheless, we can conclude that, in some sense,

$$|\mathbb{R}| = 2^{|\mathbb{N}|}.$$

[‡] The same number with two expansions yields a problem.

2 Normed Vector Spaces and Metric Spaces

2.1 Complex Numbers

Definition 2.1 (Complex number)

The set of complex numbers, denoted by $\mathbb{C}$, is the set of pairs (a, b) of real numbers. On top of that, we define addition and multiplication of complex numbers by

- $(a, b) + (c, d) = (a + c, b + d)$.
- $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

Proposition 2.2 ($\mathbb{C}$ is a field)

$(\mathbb{C}, +, \cdot)$ is a field.

Consider the complex numbers of the form $(a, 0)$. Note that $(a, 0) + (a', 0) = (a + a', 0)$ and $(a, 0) \cdot (a', 0) = (aa', 0)$. Therefore, this subset of the complex numbers is isomorphic to $\mathbb{R}$. Similarly as we've seen in previous chapters, we can say that $\mathbb{R} \subset \mathbb{C}$, referring to this natural homomorphism; i.e., if a is a real number, then we'll also use a to talk about the complex number $(a, 0)$.

Definition 2.3 (Imaginary unit)

Let $i = (0, 1) \in \mathbb{C}$.

Proposition 2.4

Let $(a, b) \in \mathbb{C}$. Then,

$$(a, b) = a + bi.$$

Proof.

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) \\ &= (a, b). \end{aligned}$$

■

Definition 2.5 (Conjugate)

Given real numbers a, b and a complex number $z = a + bi$, we define *the conjugate of* z as $a - bi$, denoted by $\overline{z}$.

Definition 2.6 (Real and imaginary part)

Given real numbers a, b and a complex number $z = a + bi$, the numbers a and b are called *the real part* and *the imaginary part of* z, respectively. Symbolically, we can write

$$\mathrm{Re}(z) = a, \qquad \mathrm{Im}(z) = b.$$

Proposition 2.7

Given $z, w \in \mathbb{C}$, we have

- $\overline{z + w} = \overline{z} + \overline{w}$.
- $\overline{zw} = \overline{z} \cdot \overline{w}$.
- $z + \overline{z} = 2\mathrm{Re}(z)$ and $z - \overline{z} = 2i\mathrm{Im}(z)$.
- $z\overline{z}$ is a non-negative real number.

Definition 2.8 (Absolute value of a complex number)

If $z = a + bi$ is a complex number, then we define $|z| = (z\overline{z})^{1/2} = (a^2 + b^2)^{1/2}$.

2.2 Normed Vector Space

Definition 2.9 (Normed Vector Space)

Let F be either $\mathbb{R}$ or $\mathbb{C}$.

A *normed vector space* is a *vector space* W over F, equipped with a norm $||\bullet|| \colon W \to \mathbb{R}$, satistying the following conditions:

i. $||v|| \geq 0$ for all $v \in W$.
ii. $||v|| = 0$ if, and only if, $v = 0$.
iii. $||cv|| = |c| \cdot ||v||$ for all $v \in W$ and for all $c \in F$.
iv. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in W$.

Example

$\mathbb{R}$ is a normed vector space, considering the norm $|x|$. $\mathbb{C}$ is a normed vector space, considering the norm $|z|$.

2.2.1 Euclidean Spaces

Definition 2.10 (Euclidean Space)

Given an positive integer n, consider the vector space $\mathbb{R}^n$. If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, we define the *inner product* by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$||\mathbf{x}|| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

This structure is called the Euclidean n-dimensional space.

Proposition 2.11

The Euclidean n-dimensional space is a normed vector space.

2.3 Metric Space

Definition 2.12 (Metric Space)

A *metric space* is a pair (X, d), where X is a set and $d \colon X \times X \to \mathbb{R}$ is a function, called *metric*, that satisfies:

- $d(x, y) \geq 0$ for all $x, y \in X$; with equality if, and only if, $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example

Consider $d(x, y) = |x - y|$, in all following examples (with the appropriate domain).

- $(\mathbb{Z}, d)$ is a metric space.
- $(\mathbb{Q}, d)$ is a metric space.

- $(\mathbb{R}, d)$ is a metric space.
- $(\mathbb{C}, d)$ is a metric space.

Every normed vector space W is naturally also a metric space, by considering the metric $d : W \times W \to \mathbb{R}$ defined by $d(v, u) = ||v - u||$.

Example ($\mathbb{R}^n$ is a metric space)

Define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$. $(\mathbb{R}^n, d)$ is a metric space.

There are metric spaces which are not normed vector spaces, but they are out of the scope of this course.

From now on in these notes, whenever you read "X is a metric space" or "X is a normed vector space," it is useful to think about the prototypical examples of $X = \mathbb{R}$, $X = \mathbb{C}$, or $X = \mathbb{R}^2$.

3 Limits

3.1 Sequences

Definition 3.1 (Limit of a sequence)

Let X be a metric space. We say a sequence $(a_n) = a_1, a_2, a_3, \ldots$, where $a_i \in X$, *converges to* $a \in X$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$.

If this definition holds for some a, we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

If this definition does not hold for any a, we say $\lim_{n\to\infty} a_n$ does not exist, or that the sequence diverges.

Proposition 3.2 (The limit, if it exists, is unique)

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a'$, then $a = a'$.

Proof. For all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N_\epsilon$. For all $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that $d(a_n - a') < \epsilon$ for all $n \geq M_\epsilon$.

Therefore, for all $\epsilon > 0$, there exists $L_\epsilon \in \mathbb{N}$, namely $\max\{N_\epsilon, M_\epsilon\}$, such that $d(a_n, a) < \epsilon$ and $d(a_n, a') < 0$ for all $n \geq L_\epsilon$. Triangle inequality implies that $d(a, a') < 2\epsilon$ for all $\epsilon > 0$; thus $d(a, a') = 0$, and consequently $a = a'$. ∎

Example

We claim that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

This is true because, given $\epsilon > 0$, we can choose N be a natural number larger than $\sqrt{\frac{1}{\epsilon}}$. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N^2} > \frac{1}{n^2} = \left|\frac{1}{n^2} - 0\right|.$$

Example (The limit does not exist)

We claim that $\lim_{n\to\infty}(-1)^n$ does not exist.

Suppose it does exist, namely a. Then, consider $\epsilon = \frac{1}{2}\max\{|a-1|, |a+1|\}$. Not both $|a-1|$ and $|a+1|$ can be zero, so $\epsilon > 0$. However, since $\lim_{n\to\infty} = a$, for that ϵ, it must hold that there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $|a - (-1)^n| < \epsilon$.

In particular, note that plugging in $n \mapsto N$ and $n \mapsto N+1$ imply that $|a-1| < \epsilon$ and $|a+1| < \epsilon$; which is a contradiction given our choice of ϵ.

Example

We claim that $\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$. Note that we can rewrite $\frac{2n+1}{n+3} = 2 - \frac{5}{n+3}$. For any ϵ, there exists $N \in \mathbb{N}$ such that $N > \frac{5}{\epsilon}$. Therefore, for all $n \geq N$, it holds that

$$\left|\left(2 - \frac{5}{n+3}\right) - 2\right| = \frac{5}{n+3} < \frac{5}{N} < \epsilon,$$

and our claim follows.

Lecture 8

Example

We claim that $\lim_{n\to\infty} \frac{2n^2}{5n^3-7} = 0$.

This is true because, given $\epsilon > 0$, we can choose N to be a natural number larger than $\frac{1}{\epsilon}$ and larger than 2. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N} > \frac{1}{n} > \frac{2n^2}{4n^3} > \frac{2n^2}{4n^3 + (n^3 - 7)} = \left|\frac{2n^2}{5n^3 - 7} - 0\right|$$

Theorem 3.3

Given a sequence (v_n) in a metric space X,

$$\lim_{n\to\infty} v_n = L \text{ if, and only if, } \lim_{n\to\infty} d(v_n, L) = 0.$$

Proposition 3.4

Given a sequence $\vec{v}_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n})$ in $\mathbb{R}^k$,

$$\lim_{n\to\infty} \vec{v}_n = (L_1, L_2, \ldots, L_k) \text{ if, and only if, } \lim_{n\to\infty} x_{i,n} = L_i \text{ for all } 1 \leq i \leq k.$$

Theorem 3.5 (Algebraic Manipulation of Limits)

Let W be a normed vector space over F.

Suppose that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ are elements of W and $c, d \in F$. Then,

i. $\lim_{n\to\infty}(ca_n + db_n) = ca + db$

If W is a field, then,

ii. $\lim_{n\to\infty} a_n b_n = ab$
iii. $\lim_{n\to\infty}(1/a_n) = 1/a$ if the $a_n \neq 0$ for all n and $a \neq 0$.
iv. $\lim_{n\to\infty}(a_n/b_n) = a/b$ if the $b_n \neq 0$ for all n and $b \neq 0$.

Proof.

i. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < \frac{\epsilon}{2|c|}$$

for all $n \geq N$. Similarly, there exists M such that

$$|b_n - b| < \frac{\epsilon}{2|d|}$$

for all $n \geq M$. Therefore, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned}|(ca_n + db_n) - (ca + db)| &= |(ca_n - ca) + (db_n - db)| \\ &\leq |ca_n - ca| + |db_n - db| \\ &\leq |c||a_n - a| + |d||b_n - b| \\ &< \epsilon,\end{aligned}$$

thus, $\lim_{n\to\infty} ca_n + db_n$.

ii. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < 1$$

for all $n \geq N$; therefore, $|a_n| < |a| + 1$ for all $n \geq N$.
Since $\lim_{n\to\infty} a_n = a$, there exist M such that

$$|a_n - a| < \frac{\epsilon}{|b|}$$

for all $n \geq M$. Similarly, there exist O such that

$$|b_n - b| < \frac{\epsilon}{2(|a| + 1)}$$

for all $n \geq O$. Therefore, for all $n \geq \max\{N, M, O\}$, it holds that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n b_n = ab$.

iii. Triangle inequality implies that $|a| \leq |a - a_n| + |a_n|$; thus $|a_n| \geq |a| - |a - a_n|$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < \frac{|a|}{2}.$$

for all $n \geq N$. Therefore, $|a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$, and consequently, $\frac{2}{|a|} > \left|\frac{1}{a_n}\right|$ for all $n \geq N$.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists M so that

$$|a_n - a| < \frac{\epsilon |a|^2}{2}.$$

Then, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned} \left|\frac{1}{a_n} - \frac{1}{a}\right| &= |a - a_n| \cdot \left|\frac{1}{a}\right| \cdot \left|\frac{1}{a_n}\right| \\ &< \frac{\epsilon |a|^2}{2} \cdot \frac{1}{|a|} \cdot \frac{2}{|a|} \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

iv. Using **ii** and **iii**, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(a_n \frac{1}{b_n}\right) \\ &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} \frac{1}{b_n}\right) \\ &= a \cdot \frac{1}{b} = \frac{a}{b}. \end{aligned}$$

■

Example

Since $\lim_{n\to\infty}(1+1/n) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n) = 1+0 = 1$ and $\lim_{n\to\infty}(1+1/n^2) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n^2) = 1+0 = 1$, we can conlude that

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2+n}{n^2+1} &= \lim_{n\to\infty}\frac{1+1/n}{1+1/n^2}\\ &= \frac{\lim_{n\to\infty} 1+1/n}{\lim_{n\to\infty} 1+1/n^2}\\ &= \frac{1}{1} = 1.\end{aligned}$$

Lecture 9

Definition 3.6 (Boundness)

Let W be a normed vector space. A sequence $(a_n)_{n\in\mathbb{N}}$, where $a_i \in W$, is bounded if there exists $M \in \mathbb{R}$ so that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Theorem 3.7 (A convergent sequence is bounded)

If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then (a_n) is bounded.

Proof. Let L be the limit of such sequence. Let $\epsilon = 1$. Then, there exists $N \in \mathbb{N}$ so that $|a_n - L| < 1$ for all $n \geq N$. Triangle inequality implies that $|a_n| < |L| + 1$ for all $n \geq N$. Define

$$M = \max\{|a_1|+1, |a_2|+1, \ldots, |a_{N-1}|+1, |L|+1\}.$$

Then, for this choice of M, it holds that $|a_n| < M$ for all $n \in \mathbb{N}$. Therefore, (a_n) is bounded. ∎

Lecture 10

Definition 3.8 (Monotone sequences)

Let (a_n) be a sequence of elements of an ordered set (for example, the real numbers).

We say (a_n) is *monotone increasing* if $a_{n+1} \geq a_n$ for all n.

We say (a_n) is *strictly monotone increasing* if $a_{n+1} < a_n$ for all n.

We say (a_n) is *monotone decreasing* if $a_{n+1} \leq a_n$ for all n.

We say (a_n) is *strictly monotone decreasing* if $a_{n+1} < a_n$ for all n.

Theorem 3.9 (Monotone Convergence Theorem)

Let (a_n) be a sequence of real numbers. If (a_n) is monotone increasing and bounded above, then it converges.

Similarly, if (a_n) is monotone decreasing and bounded below, then it converges.

Proof. We will only prove the first statement. Let $\epsilon > 0$. Let $a = \sup\{a_1, a_2, a_3, \dots\}$. ϵ-sup Theorem implies that there exists N so that $a - a_N < \epsilon$. Since the sequence is monotone increasing, for all $n \geq N$, we have that

$$|a - a_n| = a - a_n < \epsilon;$$

thus, $\lim_{n\to\infty} a_n = n$. ∎

Example

What in the world is $\sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}$? If it exists, it would be plausible to be the limit of the sequence

$$\sqrt{6}, \quad \sqrt{6+\sqrt{6}}, \quad \sqrt{6+\sqrt{6+\sqrt{6}}}, \dots$$

The easier way to make sense of this sequence is using recursion. We will define it as

$$a_1 = \sqrt{6}, \quad \text{and} \quad a_n = \sqrt{6 + a_{n-1}} \text{ for } n \geq 2.$$

We know that $a_1 = \sqrt{6} < \sqrt{6+\sqrt{6}} = a_2$. Suppose that $a_{n-1} < a_n$. Then, $a_n = \sqrt{6 + a_{n-1}} < \sqrt{6 + a_n} = a_{n+1}$. Therefore, by induction, $a_{n+1} > a_n$ for all $n \geq 1$, i.e., the sequence a_n is monotone increasing.

We also know that $a_1 < 10$. Suppose that $a_{n-1} < 10$. Then, $a_n = \sqrt{6 + a_n} < \sqrt{16} < 10$. Therefore, by induction, $a_n < 10$ for all $n \geq 1$, i.e., 10 is an upper bound of a_n.

By the Monotone Convergence Theorem, we conclude that a_n has a limit. Finally,

$$\begin{aligned}\left(\lim_{n\to\infty} a_n\right)^2 &= \lim_{n\to\infty} a_n^2 \\ &= \lim_{n\to\infty} (6 + a_{n-1}) \\ &= 6 + \lim_{n\to\infty} a_n;\end{aligned}$$

therefore, $\lim_{n\to\infty} a_n = 3$ or $\lim_{n\to\infty} a_n = -2$. Since a_n evaluates to positive real numbers, the latter proposition yields a contradiction when plugging $\epsilon \mapsto 1$. Therefore, the former proposition must be true, i.e.,

$$\lim_{n\to\infty a_n} = 3.$$

Theorem 3.10 (Limits preserve $\leq$)

Let (a_n) and (b_n) be sequences of real numbers. Suppose $a_n \leq b_n$ for all n, and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then, $a \leq b$.

3.2 Subsequences

Lecture 11

Definition 3.11 (Subsequence)

Given a sequence (a_n) and a strictly monotone increasing sequence of natural numbers (n_i), the sequence (a_{n_i}) is called a *subsequence* of (a_n).

In other words, we can say that (b_k) is a subsequence of (a_n) if there exists a strictly monotone increasing $f : \mathbb{N} \to \mathbb{N}$ so that $b_k = a_{f(k)}$ for all k.

Theorem 3.12

Let X be a metric space. A sequence of elements in X converges to $L \in X$ if, and only if, every of its subsequences converges to $L \in X$.

Proof. The inverse implication is straightforward, since the sequence is a subsequence of itself. Let's prove the direct implication. Let (a_n) be a sequence so that $a_n \to L$. Let (a_{n_i}) be a subsequence of (a_n). Let $\epsilon > 0$. Since $a_n \to L$, there exists N so that

$$d(L, a_n) < \epsilon,$$

for all $n \geq N$. Note that $n_i \geq i$. Therefore, for the same choice of N, it holds that

$$d(L, a_{n_i}) < \epsilon$$

for all $i \geq N$. Therefore, $a_{n_i} \to L$. ∎

Lecture 12

Theorem 3.13 (Squeeze Theorem)

Let (x_n), (y_n), and (z_n) be sequences of real numbers. If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} y_n = L$.

Proof. For all $n \in \mathbb{N}$, since $x_n \leq y_n \leq z_n$, $|z_n - x_n| = |z_n - y_n| + |y_n - x_n|$, which implies

$$|z_n - x_n| \geq |y_n - x_n|. \tag{3.1}$$

Theorem 3.5 implies that $\lim_{n\to\infty}(z_n - x_n) = \lim_{n\to\infty} z_n - \lim_{n\to\infty} x_n = 0$.

Let $\epsilon > 0$. Therefore, since $(z_n - x_n) \to 0$, there exists N such that $|z_n - x_n| < \epsilon$ for all $n \geq N$. Equation (3.1) implies that, for the same choice of N, it holds that $|y_n - x_n| < \epsilon$ for all $n \geq N$. Therefore, $(y_n - x_n) \to 0$. Since $(x_n) \to L$ and $(y_n - x_n) \to 0$, theorem 3.5 implies $(y_n) \to L$. ∎

Example

We claim that $\lim_{n\to\infty} \sqrt{n^2 + 4n} - n = 2$.

A good intuition for that to be true is that $\sqrt{n^2 + 4n} - n \approx \sqrt{n^2 + 4n + 4} - n = 2$.

Formally,

$$\begin{aligned}\sqrt{n^2 + 4n} - n &= \frac{(n^2 + 4n) - n^2}{\sqrt{n^2 + 4n} + n^2} \\ &= \frac{4}{\sqrt{1 + 4/n} + 1} \to 2.\end{aligned}$$

Theorem 3.14 (Bolzano-Weierstrass Theorem)

Every bounded sequence of real numbers has a convergent subsequence.

Proof. Since (a_n) is bounded, there exists M such that $a_n \leq M$ for all n. Let $I_1 = [-M, M]$. Note that infinitely many terms of (a_n) are in I_1.

Suppose $I_k = [a_k, b_k]$ contains infinitely many terms of (a_n). Define I_{k+1} as either $[a_k, \frac{a_k+b_k}{2}]$ or $[\frac{a_k+b_k}{2}, b_k]$ such that I_{k+1} contais infinitely many terms of (a_n).

Nested Interval Property implies that there exists $x \in I_j$ for all j.

Let $n_1 = 1$, so that $a_{n_1} \in I_1$. Define $n_{i+1} > n_i$, so that $a_{n_{i+1}} \in I_{i+1}$; which is possible since I_{n+1} has infinitely many terms.

For each j, both a_{n_j} and x are in I_j. Since the width of I_j is $2M/2^{j-1}$, we conclude

$$-\frac{2M}{2^{j-1}} + x \leq a_{n_j} \leq \frac{2M}{2^{j-1}} + x,$$

thus the Squeeze Theorem implies $(a_{n_j}) \to x$. ∎

Definition 3.15 (Cauchy sequence)

Let X be a metric space. A sequence of elements in X is *Cauchy* if, for all $\epsilon > 0$, there exists N so that $d(a_m, a_n) < \epsilon$ for all $m, n \geq N$.

Example

We claim that the sequence $a_n = \frac{(-1)^n}{n}$ is Cauchy.

Let $\epsilon > 0$. Choose N larger than $\frac{1}{2\epsilon}$.

Then, for all $n, m \geq N$, it holds that

$$\begin{aligned} \left| \frac{(-1)^n}{n} - \frac{(-1)^m}{m} \right| &= \left| \frac{1}{n} \pm \frac{1}{m} \right| \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{2}{N} \\ &< \epsilon. \end{aligned}$$

Proposition 3.16

Every convergent sequence is Cauchy.

Proof. Let $\epsilon > 0$. Since $(a_n) \to L$, there exists N so that

$$d(a_n, L) < \frac{\epsilon}{2}$$

for all $n \geq N$. Therefore, using the triangle inequality,

$$d(a_n, a_m) \leq d(a_n, L) + d(L, a_m) < \epsilon$$

for all $n, m \geq N$; thus the sequence is Cauchy. ∎

Lecture 13

Proposition 3.17

Let W be a normed vector space. Every Cauchy sequence of elements in W is a bounded sequence.

Proof. Let $\epsilon = 1$. There exist N so that $|a_m - a_n| < 1$ for all $m, n \geq N$. This implies that $|a_m - a_N| < 1$ for all $m \geq N$, and consequently, by triangle inequality, $|a_m| = |a_m - 0| \leq |a_m - a_N| + |a_N - 0| < 1 + |a_N|$ for all $m \geq N$.

Therefore, if we set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\},$$

we conclude $|a_m| < M$ for all m. ■

Proposition 3.18

Let X be a metric space. Let (a_n) be a sequence of elements of X. If (a_n) is Cauchy, and if some subsequence of (a_n) converges to some limit $a \in X$, then the whole sequence (a_n) converges to $a \in X$.

Proof. Let $\epsilon > 0$. Let (a_{k_i}) be such sequence that converges to a. Thus, there exists N so that

$$d(a_{k_n}, a) < \epsilon/2$$

for all $n > N$.

Also, since (a_n) is Cauchy, there exists M so that

$$d(a_m, a_n) < \epsilon/2$$

for all $m, n \geq M$. In particular, by setting $m = k_n \geq n$, we conclude

$$d(a_{k_n}, a_n) < \epsilon/2$$

for all $n \geq M$.

Therefore, for all $n \geq \max\{N, M\}$, it holds that

$$d(a_n, a) \leq d(a_n, a_{k_n}) + d(a_{k_n}, a) < \epsilon;$$

in other words, (a_n) converges to a. ■

Theorem 3.19

Every Cauchy sequence of real numbers is convergent.

Proof. Let (a_n) be a Cauchy sequence o

To be finished. ∎

3.3 Series

Lecture 14

Definition 3.20 (Series)

Given a sequence (a_n), we associate it with a sequence (s_n), defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

As an abuse of notation[a], we denote (s_n) using the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or

$$\sum_{n=1}^{\infty} a_n.$$

We call those expressions *(infinite) series*. Each s_n is called a *partial sum* of this series. If (s_n) converges to s, we say that the series *converges*, which we denote symbolically[b] by

$$\sum_{n=1}^{\infty} a_n = s,$$

which we call the sum of the series; though it is actually the limit of a sequence of partial sums.

If (s_n) diverges, we say that the series diverges.

[a] In my honest opinion, this is a really bad notation.
[b] Using the same symbolic arrangement as before! Who did this?

Note that theorems about sequences can be stated in terms of series and vice versa, by defining $a_1 = s_1$ and $a_n = s_n - s_{n-1}$.

Example

Suppose $a_n = (-1)^n$. Consider the infinite series $-1+1-1+1-1+1-\cdots$. Then, a formula for the partial sums is $s_n = \begin{cases} -1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$ Therefore, the sum of the infinite series does not converge, since $\lim_{n\to\infty} s_n$ does not exist.

Example

Suppose $a_n = \frac{1}{2^n}$. Consider the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. Then, a formula for the partial sums is $s_n = 1 - \frac{1}{2^n}$. Therefore, the sum of the infinite series is 1, since $\lim_{n\to\infty} s_n = 1$.

Proposition 3.21 (Geometric Series)

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r}, & \text{if } -1<r<1 \\ \text{does not converge}, & \text{otherwise.} \end{cases}$$

Proof. Note that

$$s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}.$$

If $-1<r<1$, then $(r_{n+1}) \to 0$, which implies $(s_n) \to \frac{1}{1-r}$. Otherwise, then (r_{n+1}) does not converge, which implies (s_n) does not converge. ∎

Proposition 3.22

Suppose (a_n) is a sequence and $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if, and only if, the partial sums $\sum_{k=1}^{n} a_n$ are bounded.

This proposition 3.22 is a direct corollary of Monotone Convergence Theorem.

Theorem 3.23 (Condensation Test)

Suppose (a_n) is monotone decreasing and $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if, and only if, $\sum_{n=1}^{\infty} 2^n a_{2^n}$.

Proof. Proposition 3.22 implies that it suffices to show that

$$\left(\sum_{k=1}^{n} a_k\right)_{n\in\mathbb{N}} \quad \text{is bounded} \tag{3.2}$$

if, and only if,

$$\left(\sum_{k=1}^{m} 2^k a_{2^k}\right)_{m\in\mathbb{N}} \quad \text{is bounded.} \tag{3.3}$$

Suppose (3.2) is true. Therefore, there exists a constant N so that $\sum_{k=1}^{n} a_k < N$ for all n. Given any $m \in \mathbb{N}$, we will plug $n = 2^m - 1$ in the previous statement. This implies that

$$\sum_{k=1}^{2^m} a_k < N,$$

which implies,

$$\sum$$

∎

Theorem 3.24 (p-series converges)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if, and only if, $p > 1$.

Proof. Condensation Test implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if, and only if,

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} \left(2^{1-p}\right)^n$$

converges. Geometric Series implies that the series above converges if, and only if, $-1 < 2^{1-p} < 1$, which is equivalent to $p < 1$. ∎

Lecture 15

Theorem 3.25 (Algebraic Manipulation of Series)

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Then, for any $c, d \in \mathbb{R}$,

$$\sum_{n=1}^{\infty}(ca_n + db_n)$$

converges to

$$c \cdot \sum_{n=1}^{\infty} a_n + d \cdot \sum_{n=1}^{\infty} b_n.$$

This theorem is a corollary of Algebraic Manipulation of Limits.

Theorem 3.26 (Comparison Test)

Suppose $0 \leq a_n \leq b_n$ for all n. Then,

i. if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. If $\sum_{n=1}^{\infty} b_n$ converges, then, by Proposition 3.22, the partial sums $\sum_{k=1}^{n} b_k$ are bounded. Since $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$, we conclude the partial sums $\sum_{k=1}^{n} a_k$ are also bounded. Therefore, by Proposition 3.22, $\sum_{n=1}^{\infty} a_n$ converges. Therefore, **i.** is true.

ii. follows from **i.** by contraposition. ∎

Theorem 3.27 (Cauchy Criterion for Series)

A series $\sum_{n=1}^{\infty} a_n$ converges if, and only if, for all $\epsilon > 0$, there exists N so that

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$$

for all $n > m \geq N$.

This theorem is a corollary of Theorem 3.19.

With this theorem, we can provide another proof for **i.** of Comparison Test.

Proof (of **i.** of Comparison Test). If $\sum_{n=1}^{\infty} b_n$ converges, then, by the Cauchy Criterion for Series, for all $\epsilon > 0$, there exists N, so that

$$\left|\sum_{k=m+1}^{n} b_k\right| < \epsilon$$

for all $n > m \geq N$.

For any $\epsilon > 0$, with the choice of N given above, we have that

$$\left|\sum_{k=m+1}^{n} a_k\right| \leq \left|\sum_{k=m+1}^{n} b_k\right| < \epsilon$$

for all $n > m \geq N$. Therefore, by the Cauchy Criterion for Series, $\sum_{n=1}^{\infty} a_n$ converges. ∎

Lecture 16

Theorem 3.28 (Ratio Test)

Given $\sum_{n=1}^{\infty} a_n$, suppose that the limit

$$R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

If $R < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges[a]. If $R > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] In fact, it converges absolutely.

Theorem 3.29 (Divergence Test)

If a series $\sum n = 1^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

3.3.1 Mixed-sign series

Now, we seek to determine if $\sum_{n=1}^{\infty} a_n$ converges when the a_n are a mix of non-negative and negative terms.

Some previous test/tools can be applied to the mixed-sign case:

- Definition of infinite series convergence;
- Cauchy Criterion;
- Geometric Series Test;
- Ratio Test;

- Divergence Test;

but one key test cannot (at least not immediately):

- Comparison Test.

Definition 3.30 (Absolute Convergence)

If $\sum_{n=1}^{\infty} |a_n|$ converges, we say $\sum_{n=1}^{\infty} a_n$ *converges absolutely*.

Definition 3.31 (Conditional Convergence)

If $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges, we say $\sum_{n=1}^{\infty}$ *converges conditionally*.

Theorem 3.32 (Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 3.33 (Alternating Series Test)

Consider (a_n) monotone decreasing with $a_n \geq 0$ for all n and $\lim_{n\to\infty} a_n = 0$. Then, the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_1 = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. Consider the partial sums $s_n = \sum_{i=1}^{n} a_i$.

Define $I_{2k} = [s_{2k}, s_{2k-1}]$ and $I_{2k+1} = [s_{2k}, s_{2k+1}]$.

Note that $s_{2k+2} - s_{2k} = a_{2k+1} - a_{2k+2} \geq 0$ and $s_{2k+2} - s_{2k+1} = -a_{2k+2} \leq 0$. Therefore

$$s_{2k+2} \in [s_{2k}, s_{2k+1}].$$

and consequently,

$$I_{2k+2} \subset I_{2k+1}.$$

Similarly, since $s_{2k+1} - s_{2k-1} = -a_{2k} + a_{2k+1} \leq 0$ and $s_{2k+1} - s_{2k} = a_{2k+1} \geq 0$. Therefore

$$s_{2k+1} \in [s_{2k}, s_{2k-1}].$$

and consequently,

$$I_{2k+1} \subset I_{2k}.$$

Given any $\epsilon > 0$, since $(a_n) \to 0$, there exists ∎

Lecture 17

Theorem 3.34 (Limit Comparison Test)

Suppose $a_n \geq 0$ and $b_n > 0$ for all n, $\sum_{n=1}^{\infty} b_n$ converges and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists. Then, $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, there exists N so that

$$\frac{a_n}{b_n} < L + 1$$

for all $n \geq N$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \\ &< \sum_{n=1}^{N-1} a_n + (L+1)\sum_{n=1}^{\infty} b_n.\end{aligned}$$

Since the first sum is finite and the second sum converges, we conclude that $\sum_{n=1}^{\infty} a_n$ converges. ∎

3.3.2 Reordering a series

Theorem 3.35

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any reordering of $\sum_{n=1}^{\infty}$ will converge to the same value as $\sum_{n=1}^{\infty} a_n$.

Theorem 3.36

If $\sum_{n=1}^{\infty} a_n$ converges conditionally and α is any real number, then there exists an reordering of $\sum_{n=1}^{\alpha}$ that converges to α.

Proof. Define

$$\begin{aligned}a_n^+ &= \max\{a_n, 0\} \\ a_n^- &= \max\{-a_n, 0\}.\end{aligned}$$

Notice that

$$\begin{aligned} a_n &= a_n^+ - a_n^- \\ |a_n| &= a_n^+ + a_n^-. \end{aligned}$$

Suppose, by contradiction, one of the sequences $\sum a_n^+$ or $\sum a_n^-$ converges. Without loss of generality, $\sum a_n^+$ converges. Since $\sum a_n$ converges, we conclude $\sum a_n^- = \sum (a_n^+ - a_n)$ converges. Therefore, $\sum |a_n| = \sum (a_n^+ + a_n^-)$ converges; a contradiction of the conditional convergence.

Therefore, both series $\sum a_n^+$ and $\sum a_n^-$ diverge.

To be finished. ∎

Lecture 18

4 Basic Topology

Definition 4.1 (Neighborhood)

Let X be a metric space. Given any $a \in X$ and $\epsilon > 0$, we define the *ϵ-neighborhood centered at a* as

$$V_\epsilon(a) = \{x \in \mathbb{R} : d(a, x) < \epsilon\}.$$

One can see that, if $X = \mathbb{R}$,

$$V_\epsilon(a) = (a - \epsilon, a + \epsilon).$$

Definition 4.2 (Open Set)

Let X be a metric space. We say $O \subset X$ is open with respect to X if, for all $a \in O$, there exists $\epsilon > 0$ so that

$$V_\epsilon(a) \subset O.$$

We'll usually omit "with respect to X" when the metric space is clear by context. Usually, in the examples, we'll consider $X = \mathbb{R}$.

Example

With respect to $\mathbb{R}$, $(1, 4)$ is an open set; $[1, 4)$ is not open; $(0, \infty)$ is open; $\mathbb{Q}$ is not open; $(1, 3) \cup (4, 6)$ is open; the empty set is open; $\mathbb{R}$ is open.

With respect to $\mathbb{R}^2$,

$$\{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < 1\}$$

is open;

$$\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$$

is open;

$$\{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } 1 < x < 4\}$$

is not open.

Proposition 4.3 (Union of open sets)

Let $\mathcal{C}$ be a collection of open sets. Then,

$$\bigcup_{O \in \mathcal{C}} O$$

is an open set.

Proof. Let $x \in \bigcup_{O \in \mathcal{C}} O$. By definition of union, there exists a set $O_x \in \mathcal{C}$ so that $x \in O_x$. Since O_x is open, there exists $\epsilon > 0$ so that $V_\epsilon(x) \subset O_x$. Since $O_x \subset \bigcup O \in \mathcal{C}O$, we conclude $V_\epsilon(x) \subset \bigcup_{O \in \mathcal{C}} O$.

Since this argument was done for arbitrary x, we conclude $\bigcup_{O \in \mathcal{C}} O$ is open. ∎

Proposition 4.4 (Finite intersection of open sets)

Let $\mathcal{C}$ be a finite collection of open sets. Then,

$$\bigcap_{O \in \mathcal{C}} O$$

is an open set.

Proposition 4.5

There exists a collection $\mathcal{C}$ of open sets such that $\bigcap_{O \in \mathcal{C}} O$ is not open.

Proof. Let $\mathcal{C} = \{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{Z}_{>0}\}$. Then,

$$\bigcap_{O \in \mathcal{C}} O = \{0\},$$

which is not open. ∎

Proposition 4.6 (ϵ-neighborhoods are open)

Given any $a \in X$, and any $\epsilon > 0$, the set $V_r(a)$ is open.

Proof. Let $b \in V_\epsilon(a)$. Therefore, $d(a, b) < \epsilon$. Let $\delta = \epsilon - d(a, b) > 0$.

Let $c \in V_\delta(b)$. Therefore, $d(b, c) < \delta = \epsilon - d(a, b)$. Therefore, by the triangle inequality,

$$d(a, c) \leq d(a, b) + d(b, c) < \epsilon,$$

i.e., $c \in V_\epsilon(a)$. Since this was done for arbitrary c, we conclude $V_\delta(b) \subset V_\epsilon(a)$.

Since this was done for arbitrary b, we conclude $V_\epsilon(a)$ is open. ■

Lecture 19

Definition 4.7 (Limit point of a set)

Let X be a metric space. Given a set $A \subset X$, we say that seme point $x \in X$ is a *limit point* A if, for all $\epsilon > 0$, the neighborhood $V_\epsilon(x)$ intersects A at some point other than x.

Definition 4.8 (Closed set)

Let X be a metric space. We say that $A \subset X$ is closed if every limit point of F is an element of F.

Lecture 20

Definition 4.9 (Closure)

The closure of A, denoted by $\overline{A}$, is the union of A with the set of its limit points.

Definition 4.10 (Interior)

The interior of A, denoted by $\mathring{A}$, is the set

$$\mathring{A} = \{x \in A : \exists \epsilon > 0, V\epsilon(x) \subset A\}$$

Definition 4.11 (Boundary)

The boundary of A, denoted by ∂A, is $\overline{A} - \mathring{A}$.

Proposition 4.12

$\overline{A}$ is closed and, for any closed C containing A, it also holds that C contains $\overline{A}$.

In this sense, $\overline{A}$ is the smallest closed set that contains A.

Proposition 4.13

$\mathring{A}$ is open and, for any open set $O \subset A$, it also holds taht $O\mathring{A}$.

In this sense, $\mathring{A}$ is the largest open set contained in A.

Theorem 4.14 (Sequence Interpretation of Limit Point)

Given a set A in a metric space X, $w \in W$ is a limit point of A if, and only if, there exists a sequence (v_n) in A so that $v_n \neq w$ for all n and $\lim_{n\to\infty} v_n = w$.

Theorem 4.15 (Sequence Interpretation of a Closed Set)

Given a set A in a metric space X, A is closed if, and only if, for any sequence (v_n) in A that has limit w in W, it holds that $w \in A$.

Lecture 21

Theorem 4.16

A set is open if, and only if, its complement is closed.

Corolary 4.17

The union of a finite collection of closed sets is closed.

Corolary 4.18

The intersection of any collection of closed sets is closed.

Lecture 22

Definition 4.19 (Open Cover)

Given $A \subset X$, and a collection $\{O_\lambda\}$ of open sets in X, we call $\{O_X\}$ an *open cover of* A, if $A \subset \bigcup_\lambda O_\lambda$.

A *open subcover* of a given cover $\{O_\lambda\}$ of A is a subset of $\{O_\lambda\}$ that still covers A.

Definition 4.20 (Compactness)

We say taht $A \subset X$ is *compact* if every open cover of A admits a finite subcover.

Example

I claim that the set $[0, 1)$ is not compact. Consider the open cover

$$(-1/2, 1/2), (-2/3, 2/3), (-3/4, 3/4), \ldots.$$

Their union is $(-1, 1)$, which covers $[0, 1)$. However, any finite subcover will have union of the form $(-(n-1)/n, (n-1)/n)$, which does not cover $[0, 1)$.

Definition 4.21 (Sequentially Compactness)

We say that $A \subset X$ is *sequentially compact* if every sequence in A has a subsequence that converges to an element of A.

Example

I claim that the set $\mathbb{N}$ is not sequentially compact. Consider the sequence

$$1, 2, 3, 4, 5, \dots .$$

No subsequence of this sequence converges.

I also claim that the set $\mathbb{N}$ is not compact. Consider the open cover

$$(1 - \tfrac{1}{2}, 1 + \tfrac{1}{2}), (2 - \tfrac{1}{2}, 2 + \tfrac{1}{2}), (3 - \tfrac{1}{2}, 3 + \tfrac{1}{2}), \dots .$$

No finite subset of this open cover also covers A.

Theorem 4.22 (Heine-Borel)

If $A \subset \mathbb{R}^n$, the following statements are equivalent:

i. A is compact;
ii. A is sequentially compact;
iii. A is closed and bounded.

Proof. To be done. ■

Lecture 23

4.1 The Cantor set

Definition 4.23 (Cantor set)

Let E_0 be the interval $[0, 1]$. Remove the middle third open interval to obtain E_1, which therefore can be written as

$$[0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1].$$

Remove the middle third open intervals to obtain E_2, which therefore can be written as

$$[0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Define $E_3, E_4, \dots$ analogously. The cantor set $\mathcal{C}$ is defined as the intersection of all sets E_n, i.e.,

$$\mathcal{C} = \bigcap_{n=0}^{\infty} E_n.$$

Proposition 4.24

The Cantor set $\mathcal{C}$ is nowhere dense, i.e., $\mathring{\overline{\mathcal{C}}} = \varnothing$.

Theorem 4.25

The Cantor set is uncountable.

Proposition 4.26

The Cantor set is the set of all numbers x that can be written as

$$x = (0.a_1a_2a_3\dots)_3 = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

with $a_i \in \{0, 2\}$.

Lecture 24

4.2 Connectedness

Definition 4.27 (Separatedness)

If A, B are nonempty sets in X, we say that A, B are *separated* if

$$\overline{A} \cap B = \varnothing \quad \text{and} \quad A \cap \overline{B} = \varnothing.$$

Definition 4.28 (Connectedness)

We say that a set E in X is *disconnected* if there exist nonempty sets A, B that are separated, with $E = A \cup B$.

We say that E is *connected* if E is not disconnected.

Example

The cantor set $\mathcal{C}$ is disconnected in $\mathbb{R}$ (consider $A = \mathcal{C} \cap [0, 1/3]$, and $B = \mathcal{C} \cap [2/3, 1]$).

The set of rational numbers is disconnected in $\mathbb{R}$ (consider $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$, and $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$).

Question. What sets are connected in $\mathbb{R}$?

Definition 4.29

We say that $E \subset \mathbb{R}$ is an interval if, given $x < y$, with $x, y \in E$, we have $c \in E$ for all c satisfying $x < c < y$.

In other words, intervals are sets of the form

$$\varnothing, \{a\}, (a, b), [a, b), (a, b], [a, b], [a, \infty), (a, \infty), (-\infty, b), (-\infty, b), \mathbb{R}.$$

(One may say that $\varnothing$ and $\{a\}$ are trivial intervals.)

Theorem 4.30

A set $E \subset \mathbb{R}$ is connected if, and only if, it is an interval.

Proof. Let's first prove the direct implication. Suppose E is not an interval. Then, there exist $a \in E, b \in E, c \notin E$ with $a < c < b$. Let $A = (-\infty, c) \cup E$, and $B = (c, -\infty)$. Such sets are separated, thus E is disconnected.

Let's now prove the converse implication. To be done. ∎

Definition 4.31 (Path)

Given a metric space X, a *continuous path* from p to q in W is a continuous[a] function $f\colon [0, 1] \to X$ with $f(0) = p$ and $f(1) = q$.

[a] In a few weeks, we will know what this means.

Definition 4.32 (Path-connectedness)

We say that a set A is X is *path-connected* if, given any $v_1, v_2 \in A$, there is a continuous path lying in A starting v_1 and ending at v_2.

Theorem 4.33

If A is path-connected, then A is connected.

Example (Topologist's Comb)

The converse of the theorem above is false.

Let $K = \{1/n : n \in \mathbb{Z}_{>0}\}$. Let

$$S = (\{0\} \times [0,1]) \cup (K \times [0,1]) \cup ([0,1] \times \{0\}).$$

S is connected in $\mathbb{R}^2$, but it is not path connected.

5 Calculus

Lecture 25

Definition 5.1 (Limit of a function to a point)

Let X, Y be a metric spaces. Let p be a limit point of $A \subset X$. Given $f\colon A \to Y$, we say that

$$\lim_{x \to p} f(x) = L$$

if, and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(f(x), L) < \epsilon$$

for all $x \in A$ satisfying $0 < d(x, p) < \delta$.

Example

I claim that $\lim_{x \to 4} \sqrt{x} = 2$.

For all $\epsilon > 0$, define $\delta = \epsilon > 0$. Then, if $0 < |x - 4| < \delta$, it holds that

$$|\sqrt{x} - 2| < |(\sqrt{x} - 2)(\sqrt{x} + 2)| = |x - 4| < \delta = \epsilon,$$

as desired.

Example

I claim that $\lim_{(x,y) \to (1,2)} (x + y) = 3$.

For all $\epsilon > 0$, define $\delta = \epsilon/2 > 0$. Then, if $0 < ||(x, y) - (1, 2)|| < \delta$, it holds that

$$\begin{aligned}
|(x + y) - 3| &= |(x - 1) + (y - 2)| \\
&\leq |x - 1| + |y - 2| \\
&\leq \sqrt{(x-1)^2 + (y-2)^2} + \sqrt{(x-1)^2 + (y-2)^2} \\
&= 2||(x, y) - (1, 2)|| \\
&< \epsilon,
\end{aligned}$$

as desired.

Theorem 5.2 (Algebraic Manipulation of Function Limits)

Let X and Y be metric spaces. Let p be a limit point of $A \subset X$. Let $f, g \colon A \to Y$. Suppose $\lim_{x\to p} f(x) = a$, $\lim_{x\to p} g(x) = b \in W$.

If X and Y are vector spaces over a field F, then:

i. $\lim_{x\to p}(cf(x) + dg(x)) = ca + db$

If Y is a field, then:

ii. $\lim_{x\to p}(f(x)g(x)) = ab$
iii. $\lim_{x\to p}(1/f(x)) = 1/a$ if $a \neq 0$.
iv. $\lim_{x\to p}(f(x)/g(x)) = a/b$ if $b \neq 0$.

Proof.

i. Let $\epsilon > 0$. Since $\lim_{x\to p} f(x) = a$, there exists $\delta > 0$ such that

$$|f(x) - a| < \frac{\epsilon}{2|c|}$$

whenever $0 < |x - p| < \delta$. Since $\lim_{x\to p} g(x) = b$, there exists $\gamma > 0$ such that

$$|g(x) - b| < \frac{\epsilon}{2|d|}$$

whenever $0 < |x - p| < \gamma$. Therefore, for all $x \in X$ satisfying $0 < |x - p| < \min\{\delta, \gamma\}$, we conclude that

$$\begin{aligned}
|(cf(x) + dg(x)) - (ca + db)| &= |c(f(x) - a) + d(g(x) - b)| \\
&\leq |c(f(x) - a)| + |d(g(x) - b)| \\
&\leq |c||(f(x) - a)| + |d||(g(x) - b)| \\
&< |c|\frac{|\epsilon|}{2|c|} + |d|\frac{\epsilon}{2|d|} = \epsilon;
\end{aligned}$$

therefore, $\lim_{x\to p}(cf(x) + dg(x)) = ca + db$.

ii. Let $\epsilon > 0$. Since $\lim_{x\to p} f(x) = a$, there exist $\delta > 0$ such that $|f(x) - a| < 1$ whenever $0 < d(x, p) < \delta$; therefore,

$$|f(x)| < |a| + 1$$

whenever $0 < d(x, p) < \delta$.

Since $\lim_{x\to p} f(x) = a$, there exist $\gamma > 0$ such that

$$|f(x) - a| < \frac{\epsilon}{|b|}$$

whenever $0 < d(x,p) < \gamma$. Similarly, there exist $\beta > 0$ such that

$$|g(x) - b| < \frac{\epsilon}{2(|a|+1)}$$

whenever $0 < d(x,p) < \beta$. Therefore, for all $x \in A$ satisfying $0 < d(x,p) < \min\{\delta, \gamma, \beta\}$, it holds that

$$\begin{aligned} |f(x)g(x) - ab| &= |f(x)g(x) - f(x)b + f(x)b - ab| \\ &= |f(x)(g(x) - b) + b(f(x) - a)| \\ &\le |f(x)(g(x) - b)| + |b(f(x) - a)| \\ &\le |f(x)||g(x) - b| + |b||f(x) - a| \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to p} f(x)g(x) = ab$.

iii. Triangle inequality implies that $|a| \le |a - f(x)| + |f(x)|$; thus $|f(x)| \ge |a| - |a - f(x)|$. Since $\lim_{x\to p} f(x) = a$, there exist $\delta > 0$ such that $|f(x) - a| < \frac{|a|}{2}$ whenever $0 < d(x,p) < \delta$. Therefore, $|f(x)| > |a| - \frac{|a|}{2} = \frac{|a|}{2}$, and consequently,

$$\frac{2}{|a|} > \left|\frac{1}{f(x)}\right|$$

whenever $0 < d(x,p) < 0$.

Let $\epsilon > 0$. Since $\lim_{x\to p} f(x) = a$, there exists $\gamma > 0$ so that

$$|f(x) - a| < \frac{\epsilon|a|^2}{2}$$

whenever $0 < d(x,p) < \gamma$. Then, for all $x \in A$ satisfying $0 < d(x,p) < \min\{\delta, \gamma\}$, it holds that

$$\begin{aligned} \left|\frac{1}{f(x)} - \frac{1}{a}\right| &= |a - f(x)| \cdot \left|\frac{1}{a}\right| \cdot \left|\frac{1}{f(x)}\right| \\ &< \frac{\epsilon|a|^2}{2} \cdot \frac{1}{|a|} \cdot \frac{2}{|a|} \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to p} \frac{1}{f(x)} = \frac{1}{a}$.

iv. Using **ii** and **iii**, we have

$$\begin{aligned}\lim_{x\to p}\frac{f(x)}{g(x)} &= \lim_{x\to p}\left(f(x)\frac{1}{g(x)}\right)\\ &= \left(\lim_{x\to p} f(x)\right)\left(\lim_{x\to p}\frac{1}{g(x)}\right)\\ &= a\cdot\frac{1}{b} = \frac{a}{b}.\end{aligned}$$

■

Theorem 5.3 (Sequence Interpretation of Function Limits)

Given a function $f\colon A\subset X\to Y$,

$$\lim_{x\to p} f(x) = L$$

if, and only if, for all sequences (x_n) in A with $\lim_{n\to\infty} x_n = p$ and $x_n \neq c$ for all n, we have $\lim_{n\to\infty} f(x_n) = L$.

Proof. Let's first prove the direct implication. Suppose $\lim_{x\to p} f(x) = L$. Let (x_n) be a sequence in A that such that $x_n \to p$ as $n\to\infty$ and $x_n \neq p$ for all n. Since $x_n \neq p$ for all n, we have that $d(x_n, p) > 0$ for all n. Let $\epsilon > 0$. There exists $\delta > 0$ such that $d(f(x), L) < \epsilon$ whenever $0 < d(x,p) < \delta$. Additionally, there also exists N such that $0 < d(x_n, p) < \delta$ for all $n \geq N$. Therefore, we conclude $d(f(x_n), L) < \epsilon$ for all $n \geq N$. This implies that $\lim_{n\to\infty} f(x_n) = L$, as desired.

Let's now prove the reverse implication. Suppose $\lim_{x\to p} f(x) \neq L$. This implies that there exists $\epsilon > 0$ such that, for all $\delta > 0$, there exists $x \in A$ satisfying $0 < d(x,p) < \delta$ and $d(f(x), L) > \epsilon$. Define $x_n \in A$ such that $0 < d(x_n, p) < \frac{1}{n}$ and $d(f(x_n), L) > \epsilon$, which exists by the previous sentence. Note that, since $0 < d(x_n, p) < \frac{1}{n}$, we conclude that $\lim_{n\to\infty} x_n = p$. However, since $d(f(x_n), L) > \epsilon$ for all n, we conclude that $\lim_{n\to\infty} f(x_n) \neq L$. This concludes the argument. ■

Theorem 5.4 (Squeeze Theorem for Functions)

Suppose $f, g, h\colon A\subset X\to\mathbb{R}$. Suppose $\lim_{x\to p} g(x) = \lim_{x\to p} h(x) = L$ and $g(x) \leq f(x) \leq h(x)$ for all $x\in A$. Then, $\lim_{x\to p} f(x)$ exists, and is L.

Proof. Corolary of Sequence Interpretation of Function Limits and Squeeze Theorem. ■

Example

Let $D\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Then, D does not have a limit at any point.

Lecture 26

Lecture 27

5.1 Continuity

Definition 5.5 (Continuity)

Given a limit point $p \in A \subset X$ and a function $f\colon A \to Y$, we say that f *is continuous at* p if

$$\lim_{x \to p} f(x) = f(p).$$

Additionally, we say that f *is continuous on* B if f is continuous at b for all $b \in B$.

Theorem 5.6 (Sequence Interpretation of Continuity)

Given $f\colon A \subset X \to Y$ and $p \in A$, f is continuous at c if, and only if, we have $\lim_{n\to\infty} f(x_n) = f(c)$ for all sequences (x_n) in A with $\lim_{n\to\infty} x_n = p$.

Note that the sequence is not required to satisfy $x_n \neq c$. The theorem above follows from Definition 5.5 and Theorem 5.3.

Theorem 5.7 (Composition of continuous functions)

Suppose $f\colon A \subset X \to Y$ and $g\colon B \subset Y \to Z$ with the range of f contained in B. Suppose f is continuous at $c \in A$, g is continuous at $f(c)$. Then, $g \circ f$ is continuous at c.

Proof. Let (x_n) be any sequence in A such that $x_n \to c$ as $n \to \infty$. Since f is continuous at c, it follows that $f(x_n) \to f(c)$ as $n \to \infty$. Since g is continous at $f(c)$, it follows that $g(f(x_n)) \to g(f(c))$ as $n \to \infty$.

Since this was done for arbitrary sequence (x_n), it follows that $g \circ f$ is continuous at c. ∎

Lecture 28

Example (Devil's staircase)

If $\mathcal{C}$ is the Cantor set on $[0,1]$, then the Cantor function $c : [0,1] \to [0,1]$ can be defined by

To be finished.

Lecture 29

Theorem 5.8

Given a function $f\colon X \to Y$, f is continuous on X if, and only if, for all open sets $O \subset Y$, the preimage $f^{-1}(O)$ is an open set in X.

Definition 5.9

We say that a set $S \subset A \subset X$ is relatively open with respect to A (or, simply, open in A) if there exists an open set O in X so that $S = O \cap A$.

Theorem 5.10

Given a function $f\colon A \subset X \to Y$, f is continuous on A if, and only if, for all open sets $O \subset Y$, the preimage $f^{-1}(O)$ is relatively open with respect to A.

Lecture 30

Theorem 5.11

Given a function $f\colon X \to Y$, f is continuous on X if, and only if, for all closed sets $V \subset Y$, the inverse image $f^{-1}(V)$ is an closed set.

Theorem 5.12

Given a function $f\colon A \subset X \to Y$, f is continuous on X if, and only if, for all closed sets $V \subset Y$, the inverse image $f^{-1}(V)$ is relatively closed with respect to A.

Theorem 5.13

If $f\colon A \subset X \to Y$ is continuous on some $S \subset A$ and S is a sequentially compact set, then $f(S)$ is sequentially compact.

Proof. Let $y_1, y_2, \dots \in f(S)$. For each n, there exists $x_n \in A$ such that $f(x_n) = y_n$. Since S is sequentially compact, the sequence $x_1, x_2, \dots$ has a converging subsequence, say $x_{n_1}, x_{n_2}, \dots$, that converge to $L \in S$.

Since f is continuous, we can argue that the subsequence $y_{n_1}, y_{n_2}, \ldots$ converges to $f(L) \in f(S)$, as desired to show that $f(S)$ is continuous. ■

If $f\colon S \to \mathbb{R}$ is continuous, and S is sequentially compact, then $f(S)$ is compact; thus it is closed. This means that $\sup(f(S)), \inf(f(S)) \in f(S)$, which implies that f attains a maximum and a minimum. The especial case when $S = [a, b]$ will be used a lot in the future.

Theorem 5.14 (Extreme Value Theorem)

If $f\colon A \subset X \to \mathbb{R}$ is continuous on some $S \subset A$, and S is a sequentially compact set, then there are points $x_0, x_1 \in S$ such that

$$f(x_0) \leq f(x) \leq f(x_1)$$

for all $x \in S$.

Theorem 5.15

If $f\colon A \subset X \to Y$ is continuous on some $C \subset A$ with C a connected set, then $f(C)$ is connected.

Lecture 31

Lecture 32

Corolary 5.16 (Generalized Intermediate Value Theorem)

If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and A is connected; and $f(a) \leq d \leq f(b)$ for some $a, b \in A$ and $d \in \mathbb{R}$; then there exists $c \in A$ such that $f(c) = d$.

Corolary 5.17

There exists a pair of antipodal points on Earth's surface with the same temperature.

5.2 Derivative

Definition 5.18 (Derivative)

Given $f\colon A \subset \mathbb{R} \to \mathbb{R}$, the derivative of f at a interior point $x \in A$ is

$$f'(x) = \lim_{\epsilon \to 0} \frac{g(x+\epsilon) - g(x)}{\epsilon},$$

if the limit exists. If the limit does not exist, we say $f'(x)$ does not exist.

Definition 5.19 (Partial derivative)

Given $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$, the i-th partial derivative of f at a interior point $\mathbf{v} = (v_1, \ldots, v_n) \in A$ is

$$\frac{\partial f}{\partial x_i}\mathbf{v} = \lim_{\epsilon \to 0} \frac{f(v_1, \ldots, v_i + \epsilon, \ldots, v_n) - f(v_1, \ldots, v_i, \ldots, v_n)}{\epsilon},$$

if the limit exists.

Example

Let $f\colon \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $f(x) = 1/x$. I claim that $f'(x) = -1/x^2$, for any $x \in \mathbb{R} - \{0\}$. Indeed,

$$\begin{aligned} f'(x) &= \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\frac{1}{x+\epsilon} - \frac{1}{x}}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{x - (x+\epsilon)}{x(x+\epsilon)\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{-1}{x(x+\epsilon)} \\ &= \frac{-1}{x^2}. \end{aligned}$$

Example

Let $f\colon [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. I claim that $f'(x) = \frac{1}{2\sqrt{x}}$, for any $x \in (0, \infty)$. Indeed,

$$\begin{aligned} f'(x) &= \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\sqrt{x+\epsilon} - \sqrt{x}}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{(\sqrt{x+\epsilon} - \sqrt{x})(\sqrt{x+\epsilon} + \sqrt{x})}{\epsilon(\sqrt{x+\epsilon} + \sqrt{x})} \\ &= \lim_{\epsilon \to 0} \frac{1}{\sqrt{x+\epsilon} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

Theorem 5.20

If $f'(x)$ exists, then f is continuous at x.

Proof. Note that

$$\begin{aligned}\lim_{\epsilon\to 0} f(x+\epsilon)-f(x) &= \lim_{\epsilon\to 0}\epsilon\left(\frac{f(x+\epsilon)-f(x)}{\epsilon}\right)\\ &= \left(\lim_{\epsilon\to 0}\epsilon\right)\left(\lim_{\epsilon\to 0}\frac{f(x+\epsilon)-f(x)}{\epsilon}\right)\\ &= 0\cdot f'(x)=0.\end{aligned}$$

Thus, $f(x)=\lim_{\epsilon\to 0} f(x+\epsilon)$, which implies f is continuous at x. ∎

Lecture 33

Theorem 5.21 (Algebraic Manipulation of Derivatives)

If $f\colon A\subset\mathbb{R}\to\mathbb{R}$, $g\colon A\subset\mathbb{R}\to\mathbb{R}$, and $f'(c)$, $g'(c)$ exist, then:

i. $(f+g)'(c)=f'(c)+g'(c)$;
ii. $(kf)'(c)=kf'(c)$;
iii. $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$
iv. $(f/g)'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{g(c)^2}$, if $g(c)\neq 0$.

Proof.

i. To be done.
ii. To be done.
iii. Note that

$$\begin{aligned}(fg)'(c) &= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}\\ &= \lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c}\\ &= \lim_{x\to c}\frac{f(x)(g(x)-g(c))+(f(x)-f(c))g(c)}{x-c}\\ &= \lim_{x\to c}\frac{f(x)(g(x)-g(c))}{x-c}+\lim_{x\to c}\frac{(f(x)-f(c))g(c)}{x-c}\\ &= f(c)g'(c)+f'(c)g(c).\end{aligned}$$

iv. To be done.

∎

Lecture 34

Theorem 5.22 (Chain Rule)

Suppose $f\colon A \subset \mathbb{R} \to \mathbb{R}$ and $g\colon B \subset \mathbb{R} \to \mathbb{R}$, with $f(A) \subset B$. Let $c \in A$. If $f'(c)$ and $g'(f(c))$ exist, then

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. Let

$$u(x) = \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}$$

and let

$$v(y) = \begin{cases} \frac{g(y)-g(f(c))}{y-f(c)} & y \neq f(c) \\ g'(f(c)) & y = f(c). \end{cases}$$

Note that u is continuous at c and v is continuous at $f(c)$, and

$$f(x) - f(c) = u(x)(x-c);$$
$$g(y) - g(f(c)) = v(y)(y - f(c)).$$

Therefore,

$$\begin{aligned} g(f(x)) - g(f(c)) &= v(f(x))(f(x) - f(c)) \\ &= v(f(x))u(x)(x-c); \end{aligned}$$

thus

$$\begin{aligned} (g \circ f)'(c) &= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \\ &= \lim_{x \to c} v(f(x))u(x) \\ &= g'(f(c))f'(c), \end{aligned}$$

as desired. ∎

Theorem 5.23 (Interior Maximum Theorem)

If f' exists at all points in (a, b) and f attains a maximum on (a, b) at a point $c \in (a, b)$, then $f'(c) = 0$.

Proof. Recall that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

By the Sequence Interpretation of Function Limits, we conclude that

$$f'(c) = \lim_{n \to \infty} \frac{f(c + 1/n) - f(c)}{1/n}$$

Since $f(c + 1/n) \leq f(c)$ we conclude that $f'(c) \leq 0$.

Similarly, we also have

$$f'(c) = \lim_{n \to \infty} \frac{f(c - 1/n) - f(c)}{-1/n}$$

Since $f(c + 1/n) \leq f(c)$ we conclude that $f'(c) \geq 0$.

Thus, $f'(c) = 0$, as desired. ∎

Corolary 5.24 (Interior Minimum Theorem)

If f' exists at all points in (a, b) and f attains a minimum on (a, b) at a point $c \in (a, b)$, then $f'(c) = 0$.

Theorem 5.25 (Darboux's Theorem)

Suppose that $f\colon [a, b] \to \mathbb{R}$ is such that the derivative $f'(x)$ exists at every $x \in [a, b]$, and suppose that d is between $f'(a)$ and $f'(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = d$.

Note that this theorem states that the conclusion of the Intermediate Value Theorem holds for f', even if f' is not continuous.

Proof. Without loss of generality, suppose $f'(b) < d < f'(a)$. Let $g(x) = f(x) - dx$. Note that $g'(a) = f'(a) - d > 0$, and $g'(b) = f'(b) - d < 0$.

Let x_n be any sequence converging to a, with $x_n > a$ for all n. Note that

$$0 < g'(a) = \lim_{n \to \infty} \frac{g(x_n) - g(a)}{x_n - a};$$

thus, there exists n such that $\frac{g(x_n) - g(a)}{x_n - a} > 0$, and consequently, since $x_n > a$, we have that $g(n) > g(a)$, so the maximum of g is not attained on a.

Let y_n be any sequence converging to b, with $y_n < b$ for all n. Note that

$$0 > g'(b) = \lim_{n\to\infty} \frac{g(y_n) - g(b)}{y_n - b};$$

thus, there exists n such that $\frac{g(y_n)-g(b)}{y_n-b} < 0$, and consequently, since $y_n < b$, we have that $g(n) > g(b)$, so the maximum of g is not attained on b.

Therefore, the maximum of g is attained at some $c \in (a, b)$. By the Interior Maximum Theorem, we conclude $g'(c) = 0$; thus $f'(c) = d$, as desired. ∎

Example

Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Such function is continuous and differentiable, and its derivative is

$$g'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Although g' is not continuous at 0, Darboux's Theorem implies that, for all $a, b \in \mathbb{R}$ and d between $g'(a)$ and $g'(b)$, there exists $c \in (a, b)$ such that $g'(c) = d$.

Theorem 5.26 (Mean Value Theorem)

Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(a) - f(b)}{a - b}.$$

Lemma 5.27 (Rolle's Theorem)

Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let's divide into three scenarios:

- If there exists x such that $f(x) > f(a) = f(b)$. This implies that the maximum of f is not attained at a nor at b; thus, the maximum of f is attained at some

$c \in (a,b)$. By the Interior Maximum Theorem, we conclude $f'(c) = 0$.
- If there exists x such that $f(x) < f(a) = f(b)$. This implies that the minimum of f is not attained at a nor at b; thus, the minimum of f is attained at some $c \in (a,b)$. By the Interior Minimum Theorem, we conclude $f'(c) = 0$.
- If neither scenario above occur, we have that $f(x) = f(a) = f(b)$ for all $x \in (a,b)$. Therefore, for any $c \in (a,b)$, we have $f'(c) = 0$.

In either case, there exists $c \in (a,b)$ such that $f'(c) = 0$, as desired. ■

Proof (of Mean Value Theorem). Define

$$g(x) = f(x) - (x-a)\left(\frac{f(a)-f(b)}{a-b}\right).$$

Note that

$$\begin{aligned} g(b) = f(b) - (b-a)\left(\frac{f(a)-f(b)}{a-b}\right) &= f(a) \\ = f(a) - (a-a)\left(\frac{f(a)-f(b)}{a-b}\right) & \\ = g(a). & \end{aligned}$$

Thus, we can apply Rolle's Theorem to conclude that there exists $c \in (a,b)$ such that $g'(c) = 0$; therefore,

$$f'(c) = \frac{f(a)-f(b)}{a-b}.$$

■

Theorem 5.28 (Generalized Mean Value Theorem)

Suppose $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Then, there exists $c \in (a,b)$ such that

$$f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$$

Proof (sketch only). Apply Rolle's Theorem on

$$h(t) = f(t)(g(b)-g(a)) - g(t)(f(b)-f(a)).$$

■

Theorem 5.29 (L'Hôpital's Rule)

Suppose that

- f, g are continuous on $[c-\epsilon, c+\epsilon]$,
- f, g are differentiable on $(c-\epsilon, c+\epsilon)$,
- $f(c) = g(c) = 0$, and
- $g(x) \neq 0$ and $g'(x) \neq 0$ on $(c-\epsilon, c+\epsilon) - \{c\}$.

Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

if the limit on the right side exists.

Proof. Let $L = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

Consider an arbitrary sequence (x_n) with $\lim_{n\to\infty} x_n = c$ and $x_n \neq c$ for all n.

By Generalized Mean Value Theorem, there exists d_n between x_n and c with

$$f'(d_n)(g(x_n) - g(c)) = g'(d_n)(f(x_n) - f(c));$$

in other words,

$$\frac{f'(d_n)}{g'(d_n)} = \frac{f(x_n)}{g(x_n)}.$$

Note that, since d_n is between x_n and c, and $x_n \to c$, we conclude $d_n \to c$. Thus,

$$\begin{aligned}\lim_{x\to c} \frac{f'(x)}{g'(x)} &= \lim_{n\to\infty} \frac{f'(d_n)}{g'(d_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)}.\end{aligned}$$

Since this was done for any sequence x_n, we conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}.$$

∎

Lecture 35

Lecture 36

5.3 Riemann Integral

Lecture 37

Definition 5.30 (Upper and Lower Sums)

Given a bounded function $f\colon [a,b] \to \mathbb{R}$ and a collection $P = \{x_0, x_1, \ldots, x_n\}$ — we'll call P a partition – with $a = x_0 < x_1 < \cdots < x_n = b$, let

$$\begin{aligned} M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\}, \\ m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\}, \end{aligned}$$

for $k \in \{1, \ldots, n\}$. Then, define

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M_k (x_k - x_{k-1}), \\ L(f,P) &= \sum_{k=1}^{n} m_k (x_k - x_{k-1}). \end{aligned}$$

Proposition 5.31

If $f\colon [a,b] \to \mathbb{R}$ is bounded and P is any partition of $[a,b]$, then

$$L(f,P) \le U(f,P).$$

Definition 5.32

Given two partitions P, Q of $[a,b]$, we say that Q is a refinement of P if, and only if, $P \subset Q$.

Proposition 5.33

If $f\colon [a,b] \to \mathbb{R}$ is bounded, P is any partition of $[a,b]$, and Q is a refinement of P, then

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$

Theorem 5.34

If $f\colon [a,b] \to \mathbb{R}$ is bounded, and P_1, P_2 are partitions of $[a,b]$, then

$$L(f,P_1) \le U(f,P_2).$$

Proof. Let $Q = P_1 \cup P_2$. Then,

$$L(f,P_1) \le L(f,Q) \le U(f,Q) \le U(f,P_2).$$

∎

Definition 5.35

If $f\colon [a,b] \to \mathbb{R}$ is bounded, define

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\},$$
$$L(f) = \sup\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

Proposition 5.36

If $f\colon [a,b]$ is bounded, then $L(f) \leq U(f)$.

Definition 5.37 (Riemann integral)

If $f\colon [a,b]$ is bounded, and $L(f) = U(f)$, we define

$$\int_a^b f(x)\,dx = L(f) = U(f).$$