Algebraic Geometry
Lecture Notes

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This is Haverford College's undergraduate MATH H334, officially named Algebra II, instructed by Tarik Aougab. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Zoom.

Contents

1 Introduction: It's all connected

The Markov equation is

$$x^2 + y^2 + z^2 = 3xyz.$$

Let's understand the integer solutions for the Markov equation.

Definition 1.1 (Markov number)
A Markov number $n \in \mathbb{N}$ is any number such that there exists y_0, z_0 such that (n, y_0, z_0) is a solution to the Markov equation. Let m_n be the n-th positive integer Markov number.

Example 1.1 (Markov number)
$(1, 2, 5)$ is a solution to the Markov equation. Thus, $1, 2, 5$ are Markov numbers.

Theorem 1.2 (Caracterization of irrational numbers)
Let $\alpha \in \mathbb{R}$. Then, α is irrational $\iff$ there are infinitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Theorem 1.3 ($\sqrt{5}$ is the best constant)
If $\alpha = \phi = \frac{1+\sqrt{5}}{2}$ and $\beta > \sqrt{5}$, there are only finitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\beta q^2}.$$

If we disregard ϕ and its derivatives, then we can change $\sqrt{5}$ to $2\sqrt{2}$.

Theorem 1.4 (Caracterization of irrational numbers not related to ϕ)
Let $\alpha \in \mathbb{R}$. Then, $\alpha \notin \mathbb{Q}[\phi] \iff$ there are infinitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2\sqrt{2}q^2}.$$

Theorem 1.5 ($2\sqrt{2}$ is the best constant)
If $\alpha = \sqrt{2}$ and $\beta > 2\sqrt{2}$, there are only finitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\beta q^2}.$$

We can disregard $\sqrt{2}$ and its derivatives, and change $2\sqrt{2}$ to $\frac{\sqrt{221}}{5}$; and so on.

This naturally creates a sequence of real numbers, called *Lagrange numbers*, which starts as $\sqrt{5}$, $2\sqrt{2}$, $\frac{\sqrt{221}}{5}$, $\ldots$, L_n, $\ldots$.

Surprisingly, there is a conection between the Markov and Lagrange numbers.

Theorem 1.6 (Markov)

$$L_n = \sqrt{9 - \frac{4}{m_n^2}}$$

Theorem 1.7

Let $\rho_1, \rho_2 \in \mathrm{Hom}(F_2, SL(2,\mathbb{R}))$. If $\mathrm{Tr}(\rho_1(a)) = \mathrm{Tr}(\rho_2(a))$, $\mathrm{Tr}(\rho_1(b)) = \mathrm{Tr}(\rho_2(b))$ and $\mathrm{Tr}(\rho_1(ab^{-1})) = \mathrm{Tr}(\rho_2(ab^{-1}))$, then there exists $A \in SL(2,\mathbb{R})$ such that $\rho_1(w) = A\rho_2 A^{-1}$ for all $w \in F_2$.

The upshot of this theorem is that $\mathrm{Hom}(F_2, SL(2,\mathbb{R}))/\text{conjugation}$ is, in some sense, a subset inside $\mathbb{R}^3$.

For certain homomorphisms $\rho : F_2 \to SL(2,\mathbb{R})$, there exists a magical machine, which we will call *hyperbolic geometry machine*, that sends ρ to the following figure.

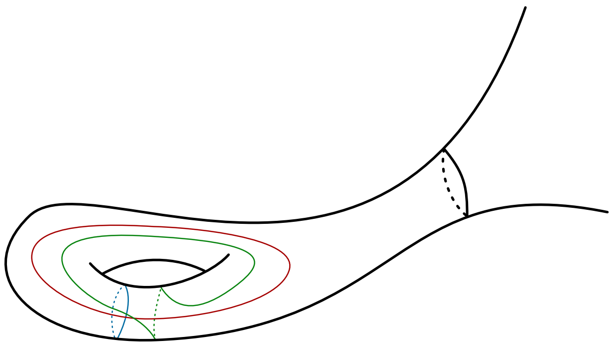


Figure 1: Result of the hyperbolic geometry machine on certain homomorphisms

The length of the blue, green and red loops are replated to $\mathrm{Tr}(\rho(a))$, $\mathrm{Tr}(\rho(b))$ and $\mathrm{Tr}(\rho(ab^{-1}))$.

For certain super special homomorphisms $\rho : F_2 \to SL(2,\mathbb{R})$, this machine sends ρ to this other figure.

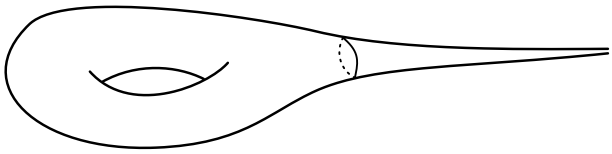


Figure 2: Result of the hyperbolic geometry machine on super special homomorphisms

Theorem 1.8

ρ is super special if, and only if,

$$\mathrm{Tr}(\rho(a))^2 + \mathrm{Tr}(\rho(b))^2 + \mathrm{Tr}(\rho(ab^{-1}) = \mathrm{Tr}(\rho(a))\,\mathrm{Tr}(\rho(b))\,\mathrm{Tr}(\rho(ab^{-1})),$$

i.e., $\left(\frac{\mathrm{Tr}(\rho(a))}{3}, \frac{\mathrm{Tr}(\rho(b))}{3}, \frac{\mathrm{Tr}(\rho(ab^{-1}))}{3}\right)$ is a solution to the Markov equation.

2 Introducing Algebraic Varieties

2.1 Definition

Definition 2.1 (Affine hypersurface)
Let K be a field. $K[x_1, \ldots, x_n]$ is the ring of polynomials with coefficients in K, and

Suppose $p \in K[x_1, \ldots x_n]$ and p is not constant. Then,

$$V(p) := \{(k_1, \ldots, k_n) \in K^n \mid p(k_1, \ldots, k_n) = 0\}$$

Example 2.1
Let $K = \mathbb{R}$ and $n = 2$. Consider $p(x_1, x_2) = x_1^2 + x_2^2 - 1$. In this case,

$$V(p) = \{(r_1, r_2) \in \mathbb{R}^2 \mid r_1^2 + r_2^2 - 1 = 0\}.$$

In this case, $V(p)$ represents a circle.

More generally, ellipses, hyperbolas, parabolas are all $V(p)$, for the right choice of p.

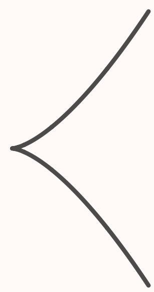
Definition 2.2 (Algebraic variety)
More generally, if $\mathcal{P}$ is a collection of polynomials in $K[X]$, not constants. Define

$$V(\mathcal{P}) = \{(k_1, \ldots, k_n) \in K^n \mid p(k_1, \ldots, k_n) = 0, \forall p \in \mathcal{P}\}.$$

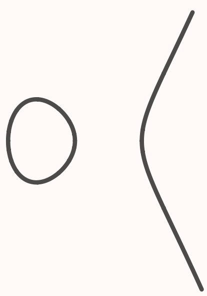
2.2 Examples with $K = \mathbb{R}$

Question 2.1
What sorts of geometric properties can algebraic varieties have?

Example 2.2
Consider $p(x, y) = y^2 - x^3$. Then, $V(p)$ looks like:

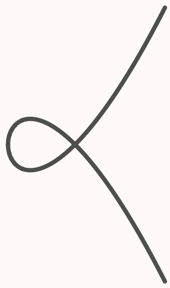


Example 2.3
Consider $q(x, y) = y^2 - x(x^2 - 1)$. Then, $V(q)$ looks like:



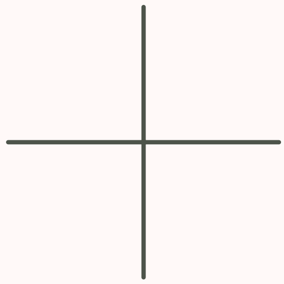
Example 2.4

Consider $r(x, y) = y^2 - x^2(x + 1)$. Then, $V(r)$ looks like:



Example 2.5

Consider $s(x, y) = xy$. Then, $V(s)$ looks like:



Definition 2.3

We say a variety has dimension d if a subset of it "looks like $\mathbb{R}^d$" and if it is the disjoint union of finitely many pieces that each "look like $\mathbb{R}^i$" with $0 \leq i \leq d$

Example 2.6

The dimension of $V(x^2 + y^2 - 1)$ is 1.

Example 2.7

The dimension of $V(\{x, y\}) = \{(0, 0)\}$ is 0.

In Linear Algebra, the number of linearly independent always equals the codimension of the solution set.

Question 2.2

Does this hold for varieties?

Answer. No. $V(x^2 + y^2) = \{(0, 0)\}$, which has dimension 0 (as opposed to the expected $2 - 1 = 1$). Another example is $V(y, y - 1) = \varnothing$, which has dimension -1 (as opposed to the expected $2 - 2 - 2$).

So, linear algebraic dimension count fail for varieties for at least two reasons:

- non-existence of solutions to certain types of algebraic equations (e.g., $x^2 = -1$).
- non-extistence of intersections between parallel lines.

February 17, 2021

To solve the first problem, we'll use the complex numbers instead of the real numbers. To solve the second problem, we'll need to develop *projective spaces*.

3 Introducing Projective Spaces

Basic idea Start with a initial space and add new point to it which keep track of the different "ways" of goign off to infinity in a straight line.

Notation $\mathbb{P}^{\text{dimension}}(\text{field})$.

Example 3.1 (Real projective line)

Consider the projective space $\mathbb{P}^1(\mathbb{R})$: this is just $\mathbb{R}$ plus one additional point "at infinity".

Example 3.2 (Real projective plane)

Consider the projective space $\mathbb{P}^2(\mathbb{R})$: this is $\mathbb{R}^2$ plus an additional point for each line in $\mathbb{R}^2$ through the origin.

Any two parallel lines in $\mathbb{R}^2$ intersect in the $\mathbb{P}^2(\mathbb{R})$.

So, any two lines in $\mathbb{P}^2(\mathbb{R})$ intersect at a point in $\mathbb{P}^2(\mathbb{R})$

Example 3.3 (Complex projective line)

Consider the projective space $\mathbb{P}(\mathbb{C})$: this is just $\mathbb{C}$ plus one additional point "at infinity".

Definition 3.1

In $\mathbb{C}^2$, a *complex line through the origin* is a subvector space of $\mathbb{C}^2$ over $\mathbb{C}$ with dimension 1.

February 19, 2021

Example 3.4

Consider the projective space $\mathbb{P}^2(\mathbb{C})$: this is $\mathbb{C}^2$ plus an additional point "at infinity" for each complex line through the origin.

"How many" new points are there? There is one for each complex line. In fact, if we look to the "slope" of each complex line, it lives inside $\mathbb{P}(\mathbb{C})$ and uniquely identifies each complex line.

4 Exploring some varieties

February 26, 2021

Example 4.1

$$y^2 = \ldots$$

yields a sphere on $\mathbb{P}^2(\mathbb{C})$.

Example 4.2

$$y^2 = x(x^2 - 1)$$

yields a torus on $\mathbb{P}^2(\mathbb{C})$.

Theorem 4.1 (WRONG! Dream Theorem)

Example 4.3

The polynomial $p(x, y) = xy$ yeilds to two spheres that touch at one point in $\mathbb{P}^(\mathbb{C}^2)$; which is not on the list.

Theorem 4.2 (Correct Theorem)

If $p \in \mathbb{C}[x, y]$ ...

5 Projective Spaces

March 01, 2021

Let K be a field (in practice, for this class, $K = \mathbb{R}$ or $\mathbb{C}$).

Theorem 5.1 (n-dimension Projective Space)
Given $n \in \mathbb{Z}_{\geq 0}$, the projective n-dimension space over K, denoted by $\mathbb{P}^n(K)$, is defined as the set

$$\mathbb{P}^n(K) = \{A \subset K^{n+1} : A \text{ is a subspace of } K^{n+1} \text{ with dimension 1over } K\}$$

Small, informal aside: $\mathbb{P}^n(K)$ is more than just a set. It is a topological space — more on this soon.

Example 5.1
$\mathbb{P}^1(\mathbb{R})$ is the set of lines in $\mathbb{R}^2$ that go through the origin.

We can try to use the blue circle to "keep track" of the lines.

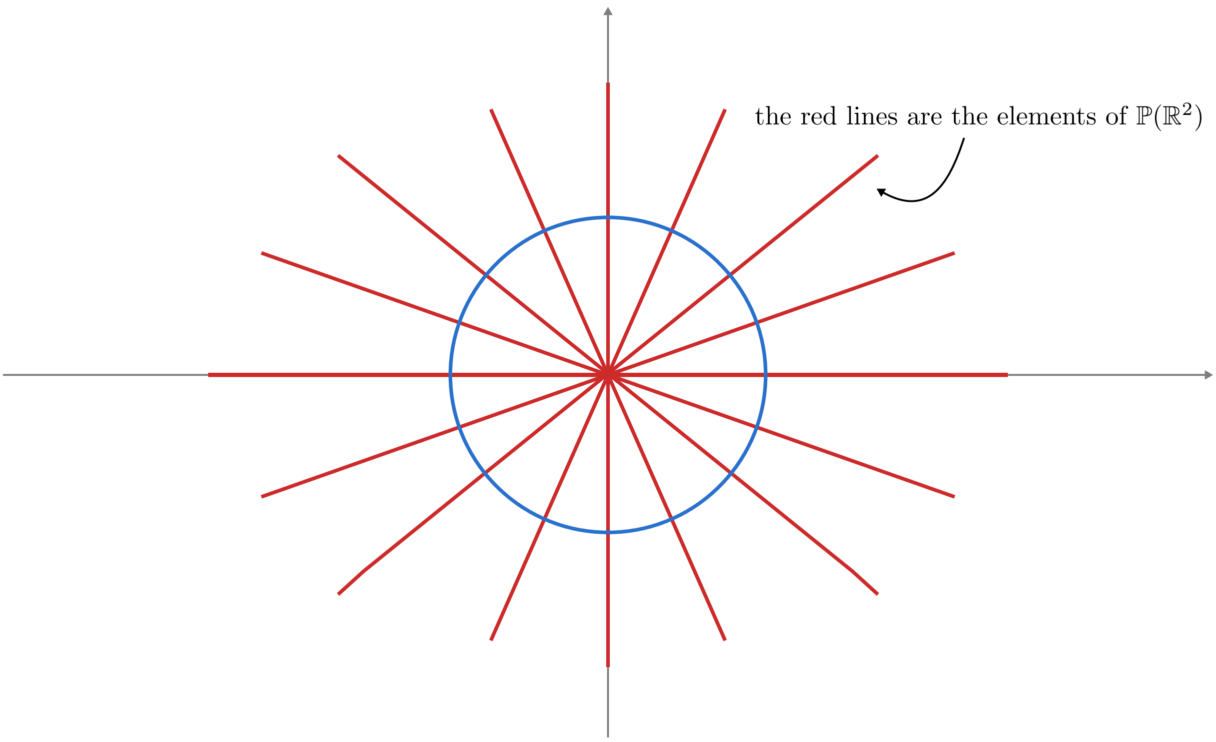


Figure 3: Projective real line

Example 5.2
$\mathbb{P}^2(\mathbb{R})$ is the set of lines in $\mathbb{R}^3$ that go through the origin.

We can try to use the unit sphere to "keep track" of the lines.

Definition 5.2
If U is a $(r+1)$-dimensional subspace of K^{n+1}, then the 1-subspaces of U yield a subset of $\mathbb{P}^n(K)$, and called a r-dimension projective subspace.

Proposition 5.3
Any r-dimension projective subspace is naturally a copy of $\mathbb{P}^r(K)$ inside of $\mathbb{P}^n(k)$.

Proof. Any two r-dimension subspaces of K^{n+1} are related by an isomorphism of $K^{n+1} \to K^{n+1}$ (change of basis). Notice also that $K^{r+1} \subset K^{n+1}$, corr. to zero-ing out the last $n-r$ coordinates is an $(r+1)$-subspace of K^{n+1}. And its 1-subspaces are the elements of $\mathbb{P}^r(K)$, by definition.

Definition 5.4

If a vector space V has dimension n, and $U \subset V$ is a subspace, the *co-dimension of* U, denoted $\operatorname{cod}(U)$, is

$$\operatorname{cod}(U) = n - \dim(U).$$

Lemma 5.5

Let S_1, S_2 be any two projective subspaces of $\mathbb{P}^n(K)$. Then,

$$\operatorname{cod}(S_1 \cap S_2) \leq \operatorname{cod}(S_1) + \operatorname{cod}(S_2).$$

Equivalentely,

$$\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - n.$$

March 03, 2021

Sketch. Using tools from Linear Algebra, we can conclude that given two subspaces $V_1, V_2 \subset V$,

$$\dim(V_1 \cap V_2) \geq \dim(V_1) + \dim(V_2) - \dim(V).$$

An $(r+1)$-dimensional subspace $\tilde{S}_1$ of K^{n+1} has codimension $(n+1)-(r+1) = n-r$. And, the associated projective subspace S_1 of $\mathbb{P}^n(K)$ has same codimension. So, the inequality for vector spaces implies the inequality for projective spaces.

We will use a lot the connection between vector spaces and projective spaces.

Example 5.3

Any two projective 2-spaces in $\mathbb{P}^3(\mathbb{R})$ intersect in at least a (projective) line.

Definition 5.6

Let $p \in \mathbb{P}^n(K)$ — we may call p a *projective point*, or simply a *point* — and let L_p be the corresponding line throught the origin in K^{n+1}. (*Technically, those are the same, but it is useful to separate them.*)

Then, if $\vec{a} \in K^{n+1}$, $\vec{a} \in L_p$, $\vec{a} \neq \vec{0}$, we call $\vec{a}$ a *coordinate set for* p — or simply *coordinates for* p.

An unforunate fact is that a single projective point p doesn't have a unique coordinate set.

Proposition 5.7

Given two non-zero $\vec{a}, \vec{b} \in K^{n+1}$, they are coordinate for the same point in $\mathbb{P}^n(K)$ if, and only if, there exists $\lambda \in K$ such that

$$\vec{a} = \lambda \cdot \vec{b},$$

i.e., if 0, $\vec{a}$ and $\vec{b}$ are collinear.

Example 5.4

Let's think about $\mathbb{P}^2(\mathbb{R})$.

For any point $(x, y, z) \in \mathbb{R}^3$ such that $z \neq 0$, we can divide by z and get $\left(\frac{x}{z}, \frac{y}{z}, 1\right)$ — *which represents the same projective point in $\mathbb{P}^2(\mathbb{R})$ as (x, y, z).*

Therefore, except for the projective points (lines) in the xy-plane, we can handle the problem of non-unique representation of projective points by referring to a projective point by the *unique* point in $\mathbb{R}^3$ with a 1 in the last coordinate. See fig. 4.

So, the plane $z = 1$ (a copy of $\mathbb{R}^2$) can be naturally identified with the subset of $\mathbb{P}^2(\mathbb{R})$ consisting of projective points that represent lines *not* in the xy-plane. The remaining projective points can be identified with a copy of $\mathbb{P}^1(\mathbb{R})$ — which we usually call *the line at infinity*.

In general, one can always imagine $\mathbb{P}^n(K)$ as a copy of K^n together with a copy of $\mathbb{P}^{n-1}(K)$ *"at infinity"* — the latter we call *the hyperplane at infinity*.

Remark. There is no preferred hyperplane at infinity. In our example, the choice of the plane $z = 1$ was completely arbitrary.

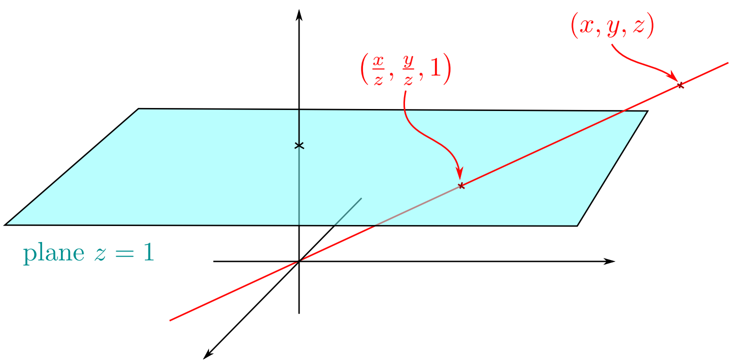


Figure 4: Real Projective Plane

Lemma 5.8

Any $(n-1)$-dimensional projective subspace W in $\mathbb{P}^n(K)$ can be chosen as the hyperplane at infinity.

March 05, 2021

Once we choose a hyperplane at infinty, we denote it by $\mathbb{P}^{n-1}_{\infty}(K)$.

Lemma 5.9

Let $\mathbb{P}^{n-1}_i(K)$ denote the projective subspace of $\mathbb{P}^n(K)$ whose projective points lie on the hyperplane $x_i = 0$. Then

$$\mathbb{P}^n(K) = \bigcup_{i=1}^{n+1} \left(\mathbb{P}^n(K) \setminus \mathbb{P}^{n-1}_i(K)\right).$$

In other words, the affine parts of $\mathbb{P}^n(K)$ associated to the choices of $x_i = 0$ $(i = 1, \ldots, n+1)$ as the hyperplane at infinity, jointly cover all of $\mathbb{P}^n(K)$.

Proof. $\mathbb{P}^n(K) \setminus \mathbb{P}^{n-1}_i$ counts every line, but those entirely contained in the hyperplane $x_i = 0$. Thus, we only miss the points contained in all hyperplanes $x_i = 0$, $i = 1, \ldots, n+1$; which is no line.

March 8, 2021

5.1 Projective completion

Definition 5.10 (Homogenieous subset)

A *homogeneous subset* S of K^n us any subset satisfying

$$x \in S \implies cx \in S, \forall c \in K.$$

Another way to think about this: A homogeneous subset is a union of lines through the origin.

Definition 5.11 (Homogeneous variety)

A *homogeneous variety* V in K^n is an algebraic variety that is also homogeneous.

Definition 5.12 (Projective variety)

A *projective variety* V in $\mathbb{P}^n(K)$ corresponding to all 1-subspaces of K^{n+1} lying in a homogeneous variety.

Definition 5.13 (Projective completion)

Let's embed K^n in K^{n+1}, by setting the last variable to 1. Then, in some sense, we are embedding K^n in $\mathbb{P}^n(K)$. Let V be in K^n, and be an algebraic variety. Then, the *projective completion* of V, denoted by $\overline{V}$ is the smallest projective variety in $\mathbb{P}^n(K)$ containing V.

We'll need some theorems and propositons to study the variety $V(z - x^3)$.

Theorem 5.14 (Bézout)

The projective completion of a variety, $V(p)$ in $\mathbb{P}^2(\mathbb{C})$, intersects any complex line n times, counting multiplicity, where $n = \deg(p)$.

Proposition 5.15

Under certain circumstances, we'll be able to conclude that all of those intersections occur within the real part of $\mathbb{C}^2$ (after adding in points at infinity).

If $\mathbb{C}^2 = \{(x, z) = (x_1 + ix_2, z_1 + iz_2) : x_1, x_2, z_1, z_2 \in \mathbb{R}\}$, then the real part of $\mathbb{C}^2$ is the x_1z_1-plane.

Proposition 5.16

Give any (real) line L through origin in the x_1z_1-plane, there exists a unique complex line in $\mathbb{C}^2$ containing L. Futhermore, if L, L' are two distinct lines through the origin, then the corresponding complex lines containing each are not equal.

Therefore, in $\mathbb{P}^2(C)$, the real part of $\mathbb{C}^2$ turns into a copy of $\mathbb{P}^2(\mathbb{R})$.

The circumstances alluded to in Proposition 5.15 arise when $p(x, z) = z - x^3$ and the complex line is the unique one intersecting the real part of $\mathbb{C}^2$ in the z_1-axis. March 10, 2021

March 12, 2021

March 15, 2021

Definition 5.17

Let $p \in K[x_1, \ldots, x_n]$, $\deg(p) = d$. Then, the homogenization of p is a polinomial $H_{x_{n+1}}(p) \in K[x_1, \ldots, x_{n+1}]$ defined by

$$H_{x_{n+1}}(p) = x_{n+1}^d p(X_1/X_{n+1}, \ldots, X_n/X_{n+1}).$$

If $p \in K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, then define $H_{x_i}(p)$ similarly.

Lemma 5.18

If $V(p_1, \ldots, p_r) \subset \mathbb{C}^n$, then $\overline{V} = V(H_{n+1}(p_1), \cdots, H_{n+1}(p_r)) \subset \mathbb{C}^{n+1}$

March 17, 2021

Definition 5.19

The affine part of V is defined to be $V \cap \left(\mathbb{P}^n(K) \setminus \mathbb{P}^{n-1}_\infty(K)\right)$. Sometimes, we call this the *dehomonization of V*.

Definition 5.20

Let $q(x_1, \dots, x_{n+1}) \in K[x_1, \dots, x_{n+1}]$ be homogeneous. Then the *dehomogenization of q at x_i* is

$$D_i(q) = q(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_{n+1}).$$

March 19, 2021

March 22, 2021

6 Multivariable Calculus

March 24, 2021

Definition 6.1 (Differentiable functions in one variable)

Let $U \subset \mathbb{R}$ be an open subset and let $f : U \to \mathbb{R}$. Then, f is called *differentiable* at $a \in U$ if there exists a line through $(a, f(a)) \in \mathbb{R}^2$, given by some equation $y = f(a) + c(x-a)$ for some $c \in \mathbb{R}$ so that

$$\lim_{x \to a} \frac{f(x) - (f(a) + c \cdot (x-a))}{x-a} = 0.$$

If this occurs, then the *derivative* of f at a is c.

Proposition 6.2 (Basic rules of differentiation)

For any f, g differentiable, it holds:

(a) $(\lambda f)'(a) = \lambda \cdot f'(a)$;
(b) $(f+g)'(a) = f'(a) + g'(a)$;
(c) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;
(d) $(f/g)'(a) = \frac{g(x)f'(x) - f(x)g'(x)}{g(a)^2}$;
(e) $(x^n)' = nx^{n-1}$.

Definition 6.3 (Limit of a function)

Given $f : \mathbb{R}^n \to \mathbb{R}$, then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = c$ means that given any $\epsilon > 0$, there exists $\delta > 0$ so that, for any $\vec{y}$ satisfying $d_{\mathbb{R}^n}(\vec{a}, \vec{y}) < \delta$, it holds $|f(\vec{y}) - c| < \epsilon$.

Definition 6.4 (Differentable functions from multiple variables to one variable)

Let $U \subset \mathbb{R}^n$ open, and $f : U \to \mathbb{R}$. Then, f is *differentiable* at $\vec{a} \in U$ if there exists a hyperplane through $(a_1, \dots, a_n f(\vec{a})) \in \mathbb{R}^{n+1}$, given by some equation of the form

$$x_{n+1} = f(a) + c_1(x_1 - a_1) + \cdots + c_n(x_n - a_n)$$

so that

$$\lim_{\vec{x} \to \vec{a}} \frac{f(x) - (f(a) + c_1(x_1 - a_1) + \cdots + c_n(x_n - a_n))}{|x_1 - a_1| + \cdots + |x_n - a_n|} = 0.$$

If this occurs, then the *derivative* of f at $\vec{a}$ is $(c_1, \dots, c_n)$.

Just as before, c_i measures the "instantaneous" rate of change of f, as we move a little bit in the x_i direction. These c_i's are called *partial derivatives* of f at $\vec{a}$ and denoted by

$$c_i = \frac{\partial f}{\partial x_i}(\vec{a}) = f_{x_i}(\vec{a}).$$

Definition 6.5 (Limit of a function with multivariable output)

Given $f : \mathbb{R}^n \to \mathbb{R}^m$, then $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{c}$ means that given any $\epsilon > 0$, there exists $\delta > 0$ so that, for any $\vec{y} \in B_\delta(\vec{a})$, it holds $f(\vec{y}) \in B_\epsilon$.

Definition 6.6 (Differentable functions from multiple variables to one variable)

Let $U \subset \mathbb{R}^n$ open, and $f : U \to \mathbb{R}^m$. We can consider the *coordinate functions* $f^1, \dots, f^m : U \to \mathbb{R}$ so that

$$f(\vec{x}) = (f^1(\vec{x}), \dots, f^m(\vec{x})).$$

Then, f is differentiable at $\vec{a}$ if all $f^1, \dots, f^m$ are differentiable at $\vec{a}$.

March 26, 2021

So, f is differentiable at $\vec{a}$ means that, near $\vec{a}$, f is well approximated by a linear transformation defined by the matrix with partial derivatives $\frac{\partial f^i}{\partial x_j}$, i.e.

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1} & \cdots & \frac{\partial f^m}{\partial x_n} \end{pmatrix}}_{\text{Jac} f(\vec{a}),\text{ the jacobian matrix}} \cdot \vec{h} + \|\vec{h}\|\ \rho(\vec{h}),$$

with $\rho(\vec{h}) \to \vec{0}$ as $\vec{h} \to \vec{0}$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$, we can identity the notion of derivative of f at $\vec{a}$ with $\text{Jac} f(\vec{a})$. March 31, 2021

Definition 6.7 (Differentiable complex functions in one variable)
Let $f : \mathbb{C} \to \mathbb{C}$. Then, f is called *differentiable* or $\mathbb{C}$-*differentiable* or *holomorphic* if there exists a complex line through $(a, f(a)) \in \mathbb{C}^2$, given by some equation $y = f(a) + c(x-a)$ for some $c \in \mathbb{C}$ so that

$$\lim_{x \to a} \frac{f(x) - (f(a) + c \cdot (x-a))}{x-a} = 0,$$

with $x \in \mathbb{C}$. If this occurs, then the *derivative* of f at a is c.

Theorem 6.8 (Cauchy-Riemann equations)
Given $f : \mathbb{C} \to \mathbb{C}$, rewrite f as

$$f(x+iy) = u(x,y) + i \cdot v(x,y),$$

for all $x, y \in \mathbb{R}$ and some $u, v : \mathbb{R}^2 \to \mathbb{R}$.

Then, f is holomorphic at $z_0 = x_0 + iy_0 \in \mathbb{C}$ if, and only if, all of the following happen:

(*i*) $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at (x_0, y_0).

(*ii*)
$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \quad \text{and} \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

April 02, 2021

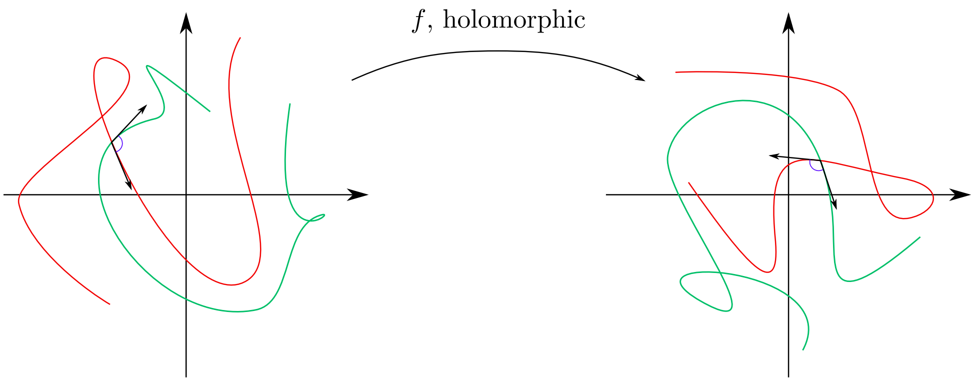


Figure 5: Angles are preserved in holomorphic functions.

Proposition 6.9

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Rewrite f as

$$f(x+iy) = u(x,y) + i \cdot v(x,y).$$

Then, if we view f as a function from $\mathbb{R}^2 \to \mathbb{R}^2$, via

$$f(x,y) = (u(x,y), v(x,y)),$$

its Jacobian matrix $\operatorname{Jac} f$ has orthogonal columns. So, $\operatorname{Jac} f$ is an orthogonal matrix. In especial, if $\vec{a}, \vec{b} \in \mathbb{R}^2$, then $\angle(\vec{a}, \vec{b}) = \angle(\operatorname{Jac} f \cdot \vec{a}, \operatorname{Jac} f \cdot \vec{b})$.

Therefore, holomorphic functions are approximated (to better and better accuracy) by orthogonal linear transformations. So long as $f' \neq 0$, this means that angles between curves are preserved, as seen in fig. 5.

April 05, 2021

Lemma 6.10 (Chain rule)

If $f, g : K \to K$ are differentiable, then $f \circ g$ is differentiable and

$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z).$$

The linear map that best approximates $f \circ g$ nearby z is to multiply the number $g'(z)$ and then follow up by multiplying $f'(g(z))$.

Lemma 6.11 (Chain rule)

Given $g : K^n \to K^m, f : K^m \to K^p$ and f, g are both K-differentiable. Then $f \circ g : K^n \to K^p$ is differentiable, and

$$\operatorname{Jac}(f \circ g)(\vec{z}) = \operatorname{Jac} f(g(\vec{z})) \cdot \operatorname{Jac} g(\vec{z}).$$

6.1 Power series

Definition 6.12 (Power series)

A power series is a function $f : U \to \mathbb{C}$, with U being an open set, given by an expression of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

i.e., given any $z_0 \in U$, $\lim_{m \to \infty} \sum_{n=0}^{m} a_n z_0^n$ exists, and we define $f(z_0)$ to be this value.

Definition 6.13 (Absolute convergence)

An expression of the form $\sum_{n=0}^{\infty} a_n z_0^n$ is said to *converge absolutely* if $\sum_{n=0}^{\infty} ||a_n z_0^n|| < \infty$.

Proposition 6.14

For $z_0 \in \mathbb{C}$, if $\sum_{n=0}^{\infty} a_n z_0^n$ converges absolutely, then $\sum_{n=0}^{\infty} a_n z_0^n$.

Definition 6.15 (Radius of Convergenge)

Given $f(z) = \sum_{n=0}^{\infty} a_n z^n$, f is said to have *radius of convergence* $\rho \in [0, \infty]$ if ρ is is the supremum of $\{\sigma : \forall z \in \mathbb{C}, ||z|| < \sigma \implies \sum_{n=0}^{\infty} a_n z^n \text{ converges absolutely}\}$.

Proposition 6.16

The radius of convergence ρ is $\sup\{t \geq 0 \mid \{|a_n| t^n\}_{n=0}^{\infty} \text{ is a bounded sequence}\}$.

April 07, 2021

Lemma 6.17

A power series and its term-by-term differentiation (i.e., given $\sum_{n=0} a_n z^n$, its term-by-term differentiation is $\sum_{n=1}^{\infty} n a_n z^{n-1}$) have the same radius of convergence.

Theorem 6.18

Let ρ be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n 0$. Then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in $B_\rho(0)$. And, the derivative of f is gotten by doing ter-by-term differentiation.

Theorem 6.19

Any holomorphic function on $\mathbb{C}$ can be expressed as a power series. *(Key idea: Taylor series.)*

April 09, 2021

Theorem 6.20

Any holomorphic function on $\mathbb{C}$ can be differentiated any finite number of times.

6.2 Integration

Definition 6.21 (Real-to-Complex Integral)

Given $f : [a, b] \to \mathbb{C}$ continous. Define

$$\int_a^b f(t)\, dt = \int_a^b \mathrm{Re}[f(t)]\, dt + i \int_a^b \mathrm{Im}[f(t)]\, dt$$

Definition 6.22 (Curve)

A *curve* in $\mathbb{C}$ is a continous function $\gamma : [a, b] \to \mathbb{C}$. A curve is called *closed* if $\gamma(a) = \gamma(b)$. A curve is *simple* if $\gamma|_{(a,b)}$ is one-to-one and $\gamma(c) \neq \gamma(a)$ or $\gamma(b)$ for any $c \in (a, b)$. A *Jordan curve* is an simple and closed curve.

We denote $\gamma^* = g([a, b])$, together with an orientation.

Definition 6.23 (Countour Integration)

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve so that $\gamma'(t)$ exists and is continous. Then, if $f : \mathbb{C} \to \mathbb{C}$, the *countour integral* or *path integral* of f along γ by

$$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Some comments:

(i) $\int_\gamma f$ often depends only on γ^*.
(ii) If $-\gamma$ denotes performing γ byt in the opposite direction, then

$$\int_{-\gamma} f = -\int_\gamma f.$$

Theorem 6.24 (Cauchy's integral theorem)

Suppose $f : U \to \mathbb{C}$, $U \subset \mathbb{C}$ open, and f is holomorphic. Let $\gamma : [a, b] \to U$ be simple and closed. Then,

$$\int_\gamma f = 0.$$

Theorem 6.25 (Cauchy's integral formula)

Let $f : U \to \mathbb{C}$ holomorphic, with U open. Let γ be any circle lying in U_1 bounding some disk in D (counterclockwise). Given $w \in D$.

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, dz$$

Theorem 6.26 (Cauchy's Argument Principle)

In the context of above. Assume $f(z) \neq 0, \forall z \in \gamma^*$ and assume there are N zeros in D (counting multiplicity). Then,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = N.$$

7 Topology

April 12, 2021

Proposition 7.1

If $M \subset \mathbb{R}^n$ is an m-manifold, and $x \in M$, then there exists a way to partition the n variables $\{x_1, \ldots, x_n\}$ into two subsets, $\{x_{i_1}, \ldots, x_{i_m}\}$ and $\{x_{j_1}, \ldots, x_{j_{n-m}}\}$, and a continous function f from an open subset V of $\mathbb{R}^m$ to $\mathbb{R}^{n-m}$, so that there exists an open set U around $x \in M$ and with

$$U = \{(x_1, \ldots, x_n) : f(x_{i_1}, \ldots, x_{i_m}) = (x_{j_1}, \ldots, x_{j_{n-m}})\}.$$

Definition 7.2

Also, if on the above definition, if for each $x \in M$, we can choose f to be a smooth function, M is called a smooth manifold. (*Small lie: there shouls also be something said about how the f's change from point to point in order to say that M itself is "smooth" but we'll ignore this for now.*)

Let's apply this perspective to $V(p)$, $p \in \mathbb{C}[x, y]$. We expect a variety to look like a manifold whenever the derivative of the defining polynomial doesn't vanish.

Definition 7.3

Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at a. Then, *a is a zero of order n* if there exists some $h : \mathbb{C} \to \mathbb{C}$ holomorphic at a, so that

(*i*) $f(x) = (x - a)^n h(x)$

(*ii*) $h(a) \neq 0$

Theorem 7.4

Suppose $p(x, y) \in \mathbb{C}[x, y]$ such that

(a) $p(0, 0) = 0$;

(b) $\frac{\partial p}{\partial y}(0, 0) \neq 0$.

Then, there exists an open ball B centered at $(0, 0)$ such that $B \cap V(p)$ is the graph of a function $\phi : \mathbb{C}_x \to \mathbb{C}_y$, which is holomorphic at $0 \in \mathbb{C}_x$.

April 14, 2021

Sketch of the proof of Theorem 7.4. This proof will be devided into two steps. First, we'll find the desired $\phi : \mathbb{C}_x \to \mathbb{C}_y$. Secondly, we'll prove it is holomorphic at $0 \in \mathbb{C}_x$.

The polynomial $p(0, y)$ has a zero at $y = 0$ of order 1. Thus, if $\tilde{B}$ is a suficiently small open ball about $0 \in \mathbb{C}_y$, there are no other zeros in $\tilde{B}$, so the argument principle implies that

$$\frac{1}{2\pi i} \int_{\partial \tilde{B}} \frac{\frac{\partial p}{\partial y}(0, y)}{p(0, y)} dy = 1.$$

Also, if $c \in \mathbb{C}_x$ very close to $0 \in \mathbb{C}_x$,

$$\frac{1}{2\pi i} \int_{\partial \tilde{B}} \frac{\frac{\partial p}{\partial y}(c, y)}{p(c, y)} dy$$

has to be very close to 1, by continuity of the output of this contour integral as a function of the inputs involved.

Trick: Choose c so small so that the argument principle still applies, i.e., for c sufficiently tiny, $p(0, y)$ having no zeros on $\partial \tilde{B} \implies p(c, y)$ has no zeros on $\partial \tilde{B}$ either. (Key point: Any continous function achieces a minimum on a compact set, and $\partial \tilde{B}$ is a compact set.)

So,

$$\frac{1}{2\pi i} \int_{\partial \tilde{B}} \frac{\frac{\partial p}{\partial y}(0, y)}{p(0, y)} dy \in \mathbb{N},$$

and it is close to 1. Thus, $p(c, y)$ has exactly 1 zero inside $\tilde{B}$. Define, for such c, $\phi(c) :=$ this zero.

Note that, for $x \in \mathbb{C}_x$ close to 0, $(x, \phi(x)) \in V(p)$, since by plugging $Y \mapsto \phi(x)$ into $p(x, Y)$, we get zero.

From the first part of our argument, we know that there are open balls $U \subset \mathbb{C}_x$, centered at $0 \in \mathbb{C}_x$, and $\Delta \subset \mathbb{C}_y$ so that

$$\forall c \in U, p(c, y) \text{ has a unique root in } \Delta,$$

and it's of order 1.

Consider the function

$$\frac{\frac{\partial p}{\partial y}(c, y)}{p(c, y)}.$$

At least within Δ, we can express this as a rational Taylor series of the form

$$\frac{\frac{\partial p}{\partial y}(c, y)}{p(c, y)} = \frac{a_0 + a_1(y - \phi(c)) + a_2(y - \phi(c))^2 + \cdots}{y - \phi(c)}.$$

Note that $a_0 \neq 0$, because $a_0 = 0 \implies \frac{\partial p}{\partial y}(c, \phi(c)) = 0$, which is a contradiction.

Then,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial\Delta} \frac{y \frac{\partial p}{\partial y}(c, y)}{y - \phi(c)} dy &= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{y a_0 + a_1 y(y - \phi(c)) + a_2 y(y - \phi(c))^2 + \cdots}{y - \phi(c)} dy \\ &= a_0 \phi(c), \end{aligned}$$

by Cauchy's integral formula.

By using the same continuity arguments we used in the first part of our argument, we can conclude that when x is sufficiently close to 0, $p(x, y) \neq 0$ for any $y \in \partial\Delta$.

So, close by to $\partial\Delta$, we can express $\frac{\frac{\partial p}{\partial y}(x,y)}{p(x,y)}$ as a power series of the form $\sum_{n=0}^{\infty} g_n(y) x^n$.

Each of these coefficients, $g_n(y)$, are themselved holomorphic functions of y in the boundary of this ball. And because of the well-behavedness of holomorphic functions, we can integrate term by term. Thus,

$$\begin{aligned} \phi(c) &= \frac{1}{2a_0 \pi i} \int_{\partial\Delta} \frac{y \frac{\partial p}{\partial y}(c, y)}{p(c, y)} dy \\ &= \sum_{n=0}^{\infty} \left(\int_{\partial\Delta} g_n(y) dy \right) c^n \\ &= \sum_{n=0}^{\infty} b_n c^n. \end{aligned}$$