ALGEBRAIC DEPENDENCE AND MILNOR K-THEORY

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ABSTRACT. This paper shows that algebraic (in)dependence is encoded in Milnor K-theory of fields. As an application, we show that the isomorphism type of a field is determined by its Milnor K-theory, up to purely inseparable extensions, in most situations.

CONTENTS

1. INTRODUCTION

Let K be a field. The Milnor K-theory of K has a very simple definition:

$$\mathrm{K}_*^{\mathrm{M}}(K) := \frac{\mathrm{T}_*(K^\times)}{\langle x \otimes y \mid x + y = 1 \rangle}$$

where $\mathrm{T}_*(K^\times)$ denotes the tensor algebra of the $\mathbb{Z}$-module $K^\times$, and the two-sided ideal $\langle x \otimes y \mid x + y = 1 \rangle$ consists of the so-called *Steinberg relations*.

In degree one, we have the multiplicative group, $\mathrm{K}_1^{\mathrm{M}}(K) = K^\times$, while the ring structure of $\mathrm{K}_*^{\mathrm{M}}(K)$ involves the additive structure of K as well. It is natural to ask whether the *field K* itself is determined (up-to isomorphism) by $\mathrm{K}_*^{\mathrm{M}}(K)$. This question was considered in [2, 4], focusing mostly on finitely-generated field extensions and eventually relying on the so-called *fundamental theorem of projective geometry* to reconstruct the fields in question.

In this paper, we investigate this question for fields which do not necessarily satisfy any finiteness conditions, and we obtain the following main result.

Main Theorem. *Let K be any field whose absolute transcendence degree is at least* 5*. Then the isomorphism type of K is determined, up to purely inseparable extensions, by the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{K}_*^{\mathrm{M}}(K)$.*

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See Theorem 5.4 for the precise statement. We also prove a similar *relative* result for relatively algebraically closed field extensions of sufficiently large transcendence degree in Theorem 5.3. Note that the theorem above imposes no additional restrictions on K besides the bound on the absolute transcendence degree. For example, this theorem applies to any sufficiently large *algebraically closed* field.

Since we wish to work with fields whose multiplicative group may even be *divisible*, it is important to work with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{K}_*^{\mathrm{M}}(K)$ as opposed to $\mathrm{K}_*^{\mathrm{M}}(K)$ itself. More precisely, if K is radically closed, then the quotient $K^\times/\text{torsion}$ is already a $\mathbb{Q}$-vector space, and thus $\mathbb{Q}^\times$ provides a source of indeterminacy for $\mathrm{K}_*^{\mathrm{M}}(K)/\langle\text{torsion}\rangle$. Namely, any $r \in \mathbb{Q}^\times$ yields an automorphism of $\mathrm{K}_*^{\mathrm{M}}(K)/\langle\text{torsion}\rangle$ defined by $t \mapsto t^r$ in degree one, and such automorphisms only arise from field theory if r is a power of the characteristic of K. By tensoring $\mathrm{K}_*^{\mathrm{M}}(K)$ with $\mathbb{Q}$, we can control for such indeterminacies in our main result.

Furthermore, if K is any field and $K \to K^i$ denotes the perfection of K, then the corresponding map

$$\mathrm{K}_*^{\mathrm{M}}(K) \to \mathrm{K}_*^{\mathrm{M}}(K^i)$$

induces an *isomorphism* after tensoring with $\mathbb{Q}$. Thus, inseparability is an additional source of indeterminacy which must be accounted for, hence we can only expect to recover the isomorphism type of K up to purely inseparable extensions when working with $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{K}_*^{\mathrm{M}}(K)$.

The technical core of this work is in recovering all the information about *algebraic dependence* from Milnor K-theory, see Theorem 4.6. Once we obtain all information about algebraic dependence, our reconstruction results will follow by applying a distant cousin of the fundamental theorem of projective geometry, due to Evans-Hrushovski [8, 9] and Gismatullin [10], based on the *group-configuration theorem*. In the case where the fields (or field extensions) in question are finitely-generated, we can instead use one of the main theorems from [4] to obtain better reconstruction results. For example, Theorem 5.7 (which uses [4, Theorem 4] in an essential way) shows that the isomorphism type of a finitely-generated field K of absolute transcendence degree ≥ 2 is determined by $\mathrm{K}_*^{\mathrm{M}}(K)$ with no need to pass to inseparable extensions.

2. Notation and preliminaries

We will primarily work with a fixed field denoted by K. In some cases we will also consider subfields of K, usually denoted k.

2.1. Quotients of Milnor K-theory. For a subgroup T of $K^\times$, we write

$$\mathrm{K}_*^{\mathrm{M}}(K|T) := \frac{\mathrm{K}_*^{\mathrm{M}}(K)}{\langle T\rangle}$$

where $\langle T\rangle$ refers to the (two-sided) ideal of $\mathrm{K}_*^{\mathrm{M}}(K)$ generated by $T \subset K^\times = \mathrm{K}_1^{\mathrm{M}}(K)$. If T is trivial, then we omit it from the notation to match the standard notation for Milnor K-theory: $\mathrm{K}_*^{\mathrm{M}}(K) := \mathrm{K}_*^{\mathrm{M}}(K|\{1\})$. In the case where $T = k^\times$ for a subfield k of K, we write $\mathrm{K}_*^{\mathrm{M}}(K|k)$ instead of $\mathrm{K}_*^{\mathrm{M}}(K|k^\times)$.

As usual, we will use the notation $\{f_1, \ldots, f_n\}$ to denote the product of $f_1, \ldots, f_n \in K^\times = \mathrm{K}_1^{\mathrm{M}}(K)$ in $\mathrm{K}_n^{\mathrm{M}}(K)$, and such elements of $\mathrm{K}_n^{\mathrm{M}}(K)$ will be called *symbols*. A similar notation and terminology will also be used for the variants of $\mathrm{K}_*^{\mathrm{M}}(K)$ we consider in this paper, while ensuring that the variant being considered is clear from context.

2.2. **Duality.** For a subgroup T of $K^\times$, we write

$$\mathcal{K}_{K|T} := \mathbb{Q} \otimes_{\mathbb{Z}} (K^\times/T).$$

If $T = \{1\}$, then we write $\mathcal{K}_K$ instead of $\mathcal{K}_{K|T}$, and if $T = k^\times$ for a subfield k of K, we will write $\mathcal{K}_{K|k}$ instead of $\mathcal{K}_{K|k^\times}$. The operation of $\mathcal{K}_{K|T}$ will always be written *additively.*

We will consider the dual

$$\mathcal{G}_K := \mathrm{Hom}_{\mathbb{Z}}(K^\times, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Q}}(\mathcal{K}_K, \mathbb{Q})$$

as a topological vector space with respect to the weak topology, where $\mathbb{Q}$ is given the discrete topology. We have an obvious perfect pairing

$$K^\times \times \mathcal{G}_K \to \mathbb{Q}.$$

For a subspace $\mathcal{H}$ of $\mathcal{G}_K$, we write $\mathcal{H}^\perp \subset K^\times$ for the orthogonal of $\mathcal{H}$ with respect to this pairing. For a subgroup T of $K^\times$, we will use the notation $\mathcal{G}_{K|T} \subset \mathcal{G}_K$ for the orthogonal of T with respect to this pairing. As always, if $T = k^\times$ for a subfield k of K, we write $\mathcal{G}_{K|k}$ instead of $\mathcal{G}_{K|k^\times}$. When $T = \{1\}$ is trivial, one has $\mathcal{G}_{K|T} = \mathcal{G}_K$, so our convention of omitting T from the notation in this case still works.

A subgroup T of an abelian group A will be called *saturated* if A/T is torsion-free. For any subgroup T of $K^\times$, the subspace $\mathcal{G}_{K|T}$ is closed, and for a subspace $\mathcal{H} \subset \mathcal{G}_K$, the subgroup $\mathcal{H}^\perp$ is saturated. In fact, if T is any subgroup of $K^\times$, then $\mathcal{G}_{K|T}^\perp$ is the saturation of T (i.e. the smallest saturated subgroup containing T) and if $\mathcal{H} \subset \mathcal{G}_K$ is any subspace then $\mathcal{G}_{K|\mathcal{H}^\perp}$ is the closure of $\mathcal{H}$.

The maps $\mathcal{H} \mapsto \mathcal{H}^\perp$ and $T \mapsto \mathcal{G}_{K|T}$ provide a one-to-one order-reversing correspondence between the closed subspaces of $\mathcal{G}_K$ and the saturated subgroups of $K^\times$. We also have canonical perfect pairing

$$\mathcal{K}_{K|T} \times \mathcal{G}_{K|T} \to \mathbb{Q}$$

associated to any subgroup T of $K^\times$. We will say so explicitly when considering orthogonals with respect to this pairings to avoid any potential confusion with the notation $(-)^\perp$ introduced above.

The base-change $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{K}_*^{\mathrm{M}}(K|T)$ will be denoted by $\mathcal{K}_*(K|T)$. As usual, if $T = \{1\}$ then we write $\mathcal{K}_*(K)$ instead of $\mathcal{K}_*(K|\{1\})$ and if $T = k^\times$ for a subfield k of K, then we write $\mathcal{K}_*(K|k)$ instead of $\mathcal{K}_*(K|k^\times)$. Note that for any subgroup T of $K^\times$, one has $\mathcal{K}_1(K|T) = \mathcal{K}_{K|T}$.

2.3. **Alternating pairs.** Elements of $\mathcal{G}_K$ will be considered both as $\mathbb{Z}$-linear maps $K^\times \to \mathbb{Q}$ and as $\mathbb{Q}$-linear maps $\mathcal{K}_K \to \mathbb{Q}$. If $f \in \mathcal{G}_K$ with $T = \ker(f) \subset K^\times$, then we may also consider f as a $\mathbb{Z}$-linear map $K^\times/T \to \mathbb{Q}$ and as a $\mathbb{Q}$-linear map $\mathcal{K}_{K|T} \to \mathbb{Q}$.

A pair of elements $f, g \in \mathcal{G}_K$ will be called an *alternating pair* provided that

$$f(x) \cdot g(y) = f(y) \cdot g(x)$$

whenever $x, y \in K^\times$ satisfy $x + y = 1$ in K. We denote the associated binary relation on $\mathcal{G}_K$ by $\mathcal{R}_K$:

$$\mathcal{R}_K(f, g) \iff f, g \text{ are an alternating pair.}$$

In fact, for the majority of this paper we will be working with the structure consisting of the following data (associated to various subgroups T of $K^\times$), which we abbreviate as $\mathscr{A}(K|T)$:

1. The $\mathbb{Q}$-vector space $\mathcal{K}_{K|T}$.
2. The topological $\mathbb{Q}$-vector space $\mathcal{G}_{K|T}$.

(3) The canonical pairing $\mathcal{K}_{K|T} \times \mathcal{G}_{K|T} \to \mathbb{Q}$.
(4) The restriction of the relation $\mathcal{R}_K$ to $\mathcal{G}_{K|T}$.

As before, we write $\mathscr{A}(K)$ instead of $\mathscr{A}(K|T)$ if T is trivial and $\mathscr{A}(K|k)$ instead of $\mathscr{A}(K|k^\times)$ when k is a subfield of K.

Recall that the Steinberg relations in Milnor K-theory are generated by basic tensors of the form $x \otimes y$ for $x, y \in K^\times$ satisfying $x + y = 1$. Thus, the alternating condition for pairs of elements of $\mathcal{G}_K$ can be tested using the product in Milnor K-theory, as the following fact summarizes.

Fact 2.1. *Let $f, g \in \mathcal{G}_K$ be given and let $T \subset (\mathbb{Q} \cdot f + \mathbb{Q} \cdot g)^\perp$ be any subgroup. The following are equivalent:*

(1) *For all $x, y \in K^\times/T = \mathrm{K}_1^{\mathrm{M}}(K|T)$ satisfying $\{x, y\} = 0$ in $\mathrm{K}_2^{\mathrm{M}}(K|T)$, one has*
$$f(x) \cdot g(y) = f(y) \cdot g(x).$$
(2) *For all $x, y \in \mathcal{K}_{K|T} = \mathcal{K}_1(K|T)$ satisfying $\{x, y\} = 0$ in $\mathcal{K}_2(K|T)$, one has*
$$f(x) \cdot g(y) = f(y) \cdot g(x).$$
(3) *f, g are an alternating pair.*

In particular, this shows that the data $\mathscr{A}(K|T)$ is *completely determined* (in a functorial manner) by the algebra $\mathcal{K}_*(K|T)$. Indeed, $\mathcal{G}_{K|T}$ is the (weak) dual of $\mathcal{K}_{K|T} = \mathcal{K}_1(K|T)$, and the fact above shows that for $f, g \in \mathcal{G}_{K|T}$ one has $\mathcal{R}_K(f, g)$ if and only if for all $x, y \in \mathcal{K}_1(K|T)$ such that $\{x, y\} = 0$ in $\mathcal{K}_2(K|T)$, one has $f(x) \cdot g(y) = f(y) \cdot g(x)$.

We will borrow some notation and terminology from group theory by considering $\mathcal{R}_K(-,-)$ as being analogous to the condition that two elements of a group commute. Namely, for a closed subspace $\mathcal{H}$ of $\mathcal{G}_K$ we consider the following (closed) subspaces of $\mathcal{G}_K$:

(1) $\mathcal{C}_K(\mathcal{H}) := \{f \in \mathcal{G}_K \mid \forall g \in \mathcal{H},\ \mathcal{R}_K(f, g)\}$, the $\mathcal{R}_K$-*centralizer* of $\mathcal{H}$.
(2) $\mathcal{Z}_K(\mathcal{H}) := \{f \in \mathcal{H} \mid \forall g \in \mathcal{H},\ \mathcal{R}_K(f, g)\}$, the $\mathcal{R}_K$-*centre* of $\mathcal{H}$.

2.4. **Valuations.** Valuations on K will only be considered up-to equivalence. Let v, w be two valuations. Our convention is that $v \le w$ means v is a *coarsening* of w.

Let v be a valuation of K. We shall write $\mathcal{O}_v$ for the valuation ring of v, $\mathfrak{m}_v$ for the maximal ideal of $\mathcal{O}_v$, Kv for the residue field of v and vK for the value group of v. The unit group $\mathcal{O}_v^\times$ will be denoted by U_v and the principal unit group $1 + \mathfrak{m}_v$ will be denoted by U_v^1. If k is a subfield of K, then we write kv and vk for the residue field and value group of the restriction of v to k.

We define:
$$\mathcal{I}_v := \mathcal{G}_{K|\mathrm{U}_v}, \quad \mathcal{D}_v := \mathcal{G}_{K|\mathrm{U}_v^1}.$$
Note that $\mathcal{I}_v \subset \mathcal{D}_v$ and that the exact sequence
$$1 \to Kv^\times \to K^\times/\mathrm{U}_v^1 \to vK \to 1$$
dualizes to an exact sequence
$$0 \to \mathcal{I}_v \to \mathcal{D}_v \to \mathcal{G}_{Kv} \to 0.$$

For a subgroup T of $K^\times$, write Tv for the image of $T \cap \mathrm{U}_v$ in $Kv^\times$ and vT for the image of T in vK. We have an exact sequence
$$1 \to Kv^\times/Tv \to K^\times/(T \cdot \mathrm{U}_v^1) \to vK/vT \to 1$$

which dualizes to an exact sequence of the form

$$0 \to \mathcal{G}_{K|T} \cap \mathcal{I}_v \to \mathcal{G}_{K|T} \cap \mathcal{D}_v \to \mathcal{G}_{Kv|Tv} \to 0.$$

In the special case where $T = k^\times$ for a subfield k of K, our notational conventions are compatible. Namely, the natural exact sequence

$$1 \to Kv^\times/kv^\times \to K^\times/(k^\times \cdot \mathrm{U}_v^1) \to vK/vk \to 1$$

dualizes to an exact sequence

$$0 \to \mathcal{G}_{K|k} \cap \mathcal{I}_v \to \mathcal{G}_{K|k} \cap \mathcal{D}_v \to \mathcal{G}_{Kv|kv} \to 0.$$

The subspace of $\mathcal{K}_K$ generated by the image of U_v^1 will be denoted by $\mathcal{U}_v^1$ and the subspace generated by the image of U_v will be denoted by $\mathcal{U}_v$. Similarly, if T is a subgroup of $K^\times$, we write $\mathcal{U}_{v|T}^1$ for the image of $\mathcal{U}_v^1$ in $\mathcal{K}_{K|T}$ and $\mathcal{U}_{v|T}$ for the image of $\mathcal{U}_v$. As always, when $T = k^\times$, we write $\mathcal{U}_{v|k}$ and $\mathcal{U}_{v|k}^1$ instead of $\mathcal{U}_{v|k^\times}$ and $\mathcal{U}_{v|k^\times}^1$.

Note that $\mathcal{U}_v$ resp. $\mathcal{U}_v^1$ is the orthogonal of $\mathcal{I}_v$ resp. $\mathcal{D}_v$ with respect to the pairing $\mathcal{K}_K \times \mathcal{G}_K \to \mathbb{Q}$. Similarly, $\mathcal{U}_{v|T}$ resp. $\mathcal{U}_{v|T}^1$ is the orthogonal of $\mathcal{G}_{K|T} \cap \mathcal{I}_v$ resp. $\mathcal{G}_{K|T} \cap \mathcal{D}_v$ with respect to the pairing $\mathcal{K}_{K|T} \times \mathcal{G}_{K|T} \to \mathbb{Q}$.

3. The local theory

Our starting point is the following fundamental result.

Theorem 3.1. *Let K be any field, and let $\mathcal{D}$ be a closed subspace of $\mathcal{G}_K$. The following are equivalent:*

1. *For all $f, g \in \mathcal{D}$, one has $\mathcal{R}_K(f, g)$.*
2. *There exists a valuation v of K and a closed subspace $\mathcal{I} \subset \mathcal{D}$ of codimension ≤ 1, such that $\mathcal{D} \subset \mathcal{D}_v$ and $\mathcal{I} \subset \mathcal{I}_v$.*

Variants of this theorem have appeared in the works of Bogomolov [3], Bogomolov-Tschinkel [1], Efrat [6], Koenigsmann [11], Engler-Koenigsmann [7], the author [13, 14], and others, albeit primarily the Galois-theoretic context. The proof of Theorem 3.1 has now been completely formally verified using the `Lean3` interactive theorem prover [5] and its formally verified mathematics library `mathlib` [12], see [15]. We thus omit the proof, referring instead to the references above for the key ideas and to [15] for the computer-verified proof.

The power of this theorem is in the implication (1) $\Rightarrow$ (2), while the converse is a simple consequence of the ultrametric inequality. We will need a slightly stronger variant of the "easy" direction (2) $\Rightarrow$ (1), formulated as follows.

Lemma 3.2. *Suppose that v is a valuation of K and $f, g \in \mathcal{D}_v$ are given. Let f_v and g_v denote the images of f and g in $\mathcal{G}_{Kv}$ under the canonical map $\mathcal{D}_v \to \mathcal{G}_{Kv}$. Then $\mathcal{R}_K(f, g)$ holds if and only if $\mathcal{R}_{Kv}(f_v, g_v)$ holds.*

Proof. Suppose $\mathcal{R}_K(f, g)$ holds, and let $x, y \in Kv^\times$ satisfy $x + y = 1$. We may choose lifts $\widetilde{x}, \widetilde{y} \in \mathrm{U}_v$ of x, y such that $\widetilde{x} + \widetilde{y} = 1$. Thus

$$f_v(x) \cdot g_v(y) = f(\widetilde{x}) \cdot g(\widetilde{y}) = f(\widetilde{y}) \cdot g(\widetilde{x}) = f_v(y) \cdot g_v(y).$$

Conversely, suppose that $\mathcal{R}_{Kv}(f_v, g_v)$ holds and let $x, y \in K^\times$ be such that $x + y = 1$. We must show that

$$f(x) \cdot g(y) = f(y) \cdot g(x).$$

If $v(x) > 0$ then $f(y) = g(y) = 0$ since $f, g \in \mathcal{D}_v$, so the equation in question trivially holds true. The equation similarly holds true if $v(y) > 0$. If $v(x) < 0$ then $y = x \cdot (x^{-1} - 1)$ while $v(x^{-1}) > 0$. Since $f(-1) = 0$ and $f \in \mathcal{D}_v$, it follows that

$$f(y) = f(x) + f(x^{-1} - 1) = f(x) + f(1 - x^{-1}) = f(x).$$

We similarly have $g(y) = g(x)$, so the equation again holds true. The equation similarly holds true if $v(y) < 0$.

The last case to consider is where $v(x) = v(y) = 0$, in which case $x, y \in \mathrm{U}_v$ and the values of f and g at x and y can be computed in the residue field. In other words, letting $\overline{x}$ and $\overline{y}$ denote the images of x and y in $Kv^\times$, we have $\overline{x} + \overline{y} = 1$ so that

$$f(x) \cdot g(y) = f_v(\overline{x}) \cdot g_v(\overline{y}) = f_v(\overline{y}) \cdot g_v(\overline{x}) = f(y) \cdot g(x).$$

In any case, we see that the necessary equation does indeed hold. □

3.1. **Valuative subspaces.** A closed subspace $\mathcal{I}$ of $\mathcal{G}_K$ will be called *valuative* provided that $\mathcal{I} \subset \mathcal{I}_v$ for some valuation v of K.

Lemma 3.3. *Suppose that $\mathcal{I}$ is valuative. Then there exists a unique minimal valuation $v_\mathcal{I}$ such that $\mathcal{I} \subset \mathcal{I}_{v_\mathcal{I}}$. The valuation $v = v_\mathcal{I}$ is characterized by the following two properties:*

(1) *One has $\mathcal{I} \subset \mathcal{I}_v$, or equivalently $\mathrm{U}_v \subset \mathcal{I}^\perp$.*
(2) *The subgroup $v(\mathcal{I}^\perp)$ contains no nontrivial convex subgroups.*

Proof. The collection of all valuations w such that $\mathcal{I} \subset \mathcal{I}_w$ is nonempty by assumption. Also, if w_i is a chain of such valuations, then the infimum w of the w_i satisfies

$$\mathrm{U}_w = \bigcup_i \mathrm{U}_{w_i}$$

and as $\mathrm{U}_{w_i} \subset \mathcal{I}^\perp$ for all i, it follows that $\mathrm{U}_w \subset \mathcal{I}^\perp$ hence w is also in the collection. This collection is also closed under binary infimums: If w_1 and w_2 are two such valuations and w is their infimum, then

$$\mathrm{U}_{w_1} \cdot \mathrm{U}_{w_2} = \mathrm{U}_w,$$

and since $\mathrm{U}_{w_i} \subset \mathcal{I}^\perp$, we also have $\mathrm{U}_w \subset \mathcal{I}^\perp$, hence w is again in the collection. It follows that this collection has a unique minimal element $v_\mathcal{I}$.

Put $v = v_\mathcal{I}$. Note that $v(\mathcal{I}^\perp)$ contains no nontrivial convex subgroups for otherwise the coarsening associated to such a subgroup would contradict the minimality of v. Conversely, if v satisfies (1) and (2) and w is a coarsening of v satisfying (1), then the convex subgroup of vK associated to w must be contained in $v(\mathcal{I}^\perp)$. This subgroup must be trivial by condition (2) and thus $w = v$. It follows that v is minimal with respect to condition (1), hence $v = v_\mathcal{I}$. This concludes the proof. □

Lemma 3.4. *Suppose that v is a valuation of K and that $\mathcal{H}$ is a closed subspace of $\mathcal{G}_K$. Then $v((\mathcal{I}_v \cap \mathcal{H})^\perp)$ is the saturation of $v(\mathcal{H}^\perp)$ in vK. In particular, if $\mathcal{H} \subset \mathcal{I}_v$ then $v(\mathcal{H}^\perp)$ is saturated.*

Proof. Put $H := (\mathcal{I}_v \cap \mathcal{H})^\perp$. First, note that $v(H)$ is indeed saturated in vK. Indeed, if $n \cdot v(t) \in v(H)$ for some $t \in K^\times$ and some positive integer n, then $t^n \in \mathrm{U}_v \cdot H$ while $\mathrm{U}_v \subset H$ so that $t^n \in H$. Since H is itself saturated, it follows that $t \in H$ hence $v(t) \in v(H)$.

Put $T := \mathrm{U}_v \cdot \mathcal{H}^\perp$. Since $\mathcal{I}_v^\perp = \mathrm{U}_v$, it follows that $\mathcal{G}_{K|T} = \mathcal{I}_v \cap \mathcal{H}$ and thus H is the saturation of T. This means that H/T is torsion, hence $v(H)/v(T)$ is torsion as well, while $v(T) = v(\mathcal{H}^\perp)$. It follows that $v(H)$ is indeed the saturation of $v(\mathcal{H}^\perp)$. □

Lemma 3.5. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{G}_K$, and let v be a valuation of K. Then $v = v_\mathcal{I}$ for $\mathcal{I} = \mathcal{I}_v \cap \mathcal{H}$ if and only if the saturation of $v(\mathcal{H}^\perp)$ in vK contains no nontrivial convex subgroups.*

Proof. Combine Lemmas 3.3 and 3.4. □

3.2. **Detecting valuative subspaces.**

Lemma 3.6. *Let $\mathcal{I}$ be a valuative subspace of $\mathcal{G}_K$ with associated valuation $v := v_\mathcal{I}$. Then one has $\mathcal{D}_v = \mathcal{C}_K(\mathcal{I})$.*

Proof. The inclusion $\mathcal{D}_v \subset \mathcal{C}_K(\mathcal{I})$ follows from Lemma 3.2. Conversely, suppose that $f \in \mathcal{G}_K$ satisfies $\mathcal{R}_K(f, g)$ for all $g \in \mathcal{I}$. Let $x \in K^\times$ be an element satisfying $v(x) > 0$.

Note that for all $g \in \mathcal{I}$, one has $g(1-x) = 0$ since $1 - x \in \mathrm{U}_v \subset \mathcal{I}^\perp$. If there exists some $g \in \mathcal{I}$ such that $g(x) \neq 0$, then one has $f(1-x) = 0$ since

$$f(1-x) \cdot g(x) = f(x) \cdot g(1-x) = 0.$$

Otherwise, there must exists some $y \in K^\times$ such that $0 < v(y) < v(x)$ and some $g \in \mathcal{I}$ such that $g(y) \neq 0$. Indeed, if this does not hold then $[0, v(x)] \subset vK$ would be contained in $v(\mathcal{I}^\perp)$, which cannot happen since $v = v_\mathcal{I}$. With such a y, the argument above shows that $f(1-y) = 0$ while

$$f(1-x) = f((1-x)\cdot(1-y)) = f(1-(y + x\cdot(1-y))).$$

But $v(y + x \cdot (1-y)) = v(y)$, so, again, the argument above shows that $f(1-x) = 0$. In other words, $\mathrm{U}_v^1 \subset f^\perp$, hence $f \in \mathcal{D}_v$. □

Proposition 3.7. *Suppose that $\mathcal{D}$ is a closed subspace of $\mathcal{G}_K$ such that $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$. Then $\mathcal{I} := \mathcal{Z}_K(\mathcal{D})$ is valuative and $\mathcal{D} \subset \mathcal{D}_v$ for $v = v_\mathcal{I}$.*

Proof. By Lemma 3.6 it suffices to show that $\mathcal{I}$ is valuative. Since $\mathcal{I} \neq \mathcal{D}$, there exists $f_1, f_2 \in \mathcal{D}$ such that $\mathcal{R}_K(f_1, f_2)$ does not hold. Put $\mathcal{D}_i = \mathcal{I} + \mathbb{Q} \cdot f_i$. Then $\mathcal{D}_i$ are both closed subspaces of $\mathcal{G}_K$, and the following hold:

(1) $\mathcal{I}$ has codimension 1 in $\mathcal{D}_i$.
(2) $\mathcal{I}$ has codimension 2 in $\mathcal{D}_1 + \mathcal{D}_2$.
(3) $\mathcal{I} = \mathcal{D}_1 \cap \mathcal{D}_2$.
(4) For all $f, g \in \mathcal{D}_i$, one has $\mathcal{R}_K(f, g)$.

By Theorem 3.1, there exist valuations v_i and closed subspaces $\mathcal{I}_i$ of $\mathcal{D}_i$ of codimension ≤ 1 such that $\mathcal{D}_i \subset \mathcal{D}_{v_i}$ and $\mathcal{I}_i \subset \mathcal{I}_{v_i}$.

If v_1 and v_2 are *not* comparable, then, letting v denote their infimum, one has $\mathrm{U}_{v_1}^1 \cdot \mathrm{U}_{v_2}^1 = \mathrm{U}_v$ by the approximation theorem for independent valuations. It follows that $\mathcal{D}_{v_1} \cap \mathcal{D}_{v_2} = \mathcal{I}_v$, hence $\mathcal{I} \subset \mathcal{I}_v$, thereby concluding the proof.

So, assume without loss of generality that $v_1 \leq v_2$ hence

$$\mathcal{I}_{v_1} \subset \mathcal{I}_{v_2} \subset \mathcal{D}_{v_2} \subset \mathcal{D}_{v_1}.$$

Assume for a contradiction that $\mathcal{I}$ is *not valuative*. Then $\mathcal{I}$ is not contained in $\mathcal{I}_{v_i} \cap \mathcal{D}_i$, so we may find $g_i \in \mathcal{I}_{v_i} \cap \mathcal{D}_i$ such that $\mathcal{D}_i = \mathcal{I} + \mathbb{Q} \cdot g_i$. In particular, we have

$$\mathcal{D}_1 + \mathcal{D}_2 = \mathcal{I} + \mathbb{Q} \cdot g_1 + \mathbb{Q} \cdot g_2$$

while $g_1, g_2 \in \mathcal{I}_{v_2}$. It follows that $\mathcal{D}_1 + \mathcal{D}_2 \subset \mathcal{D}_{v_2}$ and that the image of $\mathcal{D}_1 + \mathcal{D}_2$ in $\mathcal{D}_{v_2}/\mathcal{I}_{v_2}$ agrees with the image of $\mathcal{D}_2$ in this quotient, which has dimension ≤ 1. This together with Lemma 3.2 would imply that $\mathcal{R}_K(f, g)$ holds for all $f, g \in \mathcal{D}_1 + \mathcal{D}_2$, which is impossible since $f_1, f_2 \in \mathcal{D}_1 + \mathcal{D}_2$. □

3.3. **Visible valuations.** Let T be a subgroup of $K^\times$ and v a valuation of K. We shall say that v is T-*visible* provided that the following hold:

1. One has $\mathcal{I}_v \cap \mathcal{G}_{K|T} = \mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T}) \neq \mathcal{D}_v \cap \mathcal{G}_{K|T}$.
2. One has $v = v_{\mathcal{I}}$ for $\mathcal{I} = \mathcal{I}_v \cap \mathcal{G}_{K|T}$.

These are precisely the valuations v for which we will be able to characterize $\mathcal{I}_v \cap \mathcal{G}_{K|T}$ and $\mathcal{D}_v \cap \mathcal{G}_{K|T}$ using the relation $\mathcal{R}_K$ restricted to $\mathcal{G}_{K|T}$, as we show in the following theorem. If H is the saturation of T, then $\mathcal{G}_{K|T} = \mathcal{G}_{K|H}$ hence a valuation is T-visible if and only if it is H-visible. When $T = k^\times$ for a subfield k of K, then we shall say "k-visible" instead of "$k^\times$-visible." When $T = \{1\}$ is trivial, we shall say "visible" instead of "$\{1\}$-visible."

Theorem 3.8. *Let T be a subgroup of $K^\times$ and $\mathcal{D} \subset \mathcal{G}_{K|T}$ be a closed subspace. There exists a T-visible valuation of K such that $\mathcal{Z}_K(\mathcal{D}) = \mathcal{I}_v \cap \mathcal{G}_{K|T}$ and $\mathcal{D} = \mathcal{D}_v \cap \mathcal{G}_{K|T}$ if and only if the following conditions hold:*

1. *One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$.*
2. *One has $\mathcal{C}_K(\mathcal{Z}_K(\mathcal{D})) \cap \mathcal{G}_{K|T} = \mathcal{D}$.*

Proof. First suppose that v is indeed T-visible. Put $\mathcal{D} := \mathcal{D}_v \cap \mathcal{G}_{K|T}$ and $\mathcal{I} := \mathcal{I}_v \cap \mathcal{G}_{K|T}$. By assumption, we have $\mathcal{I} = \mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$ and by Lemma 3.6 we have $\mathcal{C}_K(\mathcal{I}) = \mathcal{D}_v$ since $v = v_{\mathcal{I}}$, hence both conditions (1) and (2) hold true.

Conversely, suppose that $\mathcal{D}$ satisfies conditions (1) and (2) and put $\mathcal{I} := \mathcal{Z}_K(\mathcal{D})$. By Proposition 3.7 and condition (1), $\mathcal{I}$ is valuative and, setting $v = v_{\mathcal{I}}$, one has $\mathcal{D} \subset \mathcal{D}_v$. Lemma 3.6 shows that $\mathcal{C}_K(\mathcal{I}) = \mathcal{D}_v$ hence condition (2) implies that $\mathcal{D} = \mathcal{D}_v \cap \mathcal{G}_{K|T}$. We also know that $\mathcal{I}_v \cap \mathcal{G}_{K|T} \subset \mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T}) = \mathcal{I}$ by Lemma 3.2, while $\mathcal{I} \subset \mathcal{I}_v \cap \mathcal{G}_{K|T}$ because $v = v_{\mathcal{I}}$. Thus $\mathcal{I} = \mathcal{I}_v \cap \mathcal{G}_{K|T}$. The fact that v is T-visible follows directly from the definition and the observations above. □

3.4. **Abundant visibility.** This section shows that fields of higher transcendence degree have an abundance of visible valuations.

Lemma 3.9. *Suppose that k is a subfield of K, and let T be any subgroup of $K^\times$ which is contained in $k^\times$. Assume that* $\operatorname{trdeg}(K|k) \geq 1$, *and that v is a valuation of K such that $\mathcal{G}_{K|T} \subset \mathcal{D}_v$. Then v is trivial.*

Proof. If not, then there exists some $t \in K$ which is transcendental over k such that $v(t) > 0$. Thus $1 + t \in \mathrm{U}_v^1 \subset \mathcal{D}_v^\perp \subset \mathcal{G}_{K|T}^\perp \subset \mathcal{G}_{K|k}^\perp$. But $\mathcal{G}_{K|k}^\perp$ is the saturation of $k^\times$ in $K^\times$. This implies that $1 + t$ is algebraic over k, which is impossible. □

Proposition 3.10. *Suppose that k is a subfield of K, and let T be any subgroup of $K^\times$ which is contained in $k^\times$. Let v be a valuation of K such that the saturation of vk in vK contains no nontrivial convex subgroups and such that* $\operatorname{trdeg}(Kv|kv) \geq 1$. *Then v is visible over T.*

Proof. Since the saturation of T in $K^\times$ is contained in the radical closure of k in K, we may assume without loss of generality that T is saturated and that k is radically closed in K. In particular, $k^\times$ is also saturated in $K^\times$ and hence $k^\times = \mathcal{G}_{K|k}^\perp$ while $T = \mathcal{G}_{K|T}^\perp$. By Lemma 3.4, we see that the saturation of vk in vK is $v(\mathcal{I}_0^\perp)$ where $\mathcal{I}_0 = \mathcal{I}_v \cap \mathcal{G}_{K|k}$. On the other hand, our assumption ensures that $\mathcal{G}_{K|k} \subset \mathcal{G}_{K|T}$ hence $\mathcal{I}_0 \subset \mathcal{I}_1 := \mathcal{I}_v \cap \mathcal{G}_{K|T}$, thus $\mathcal{I}_1^\perp \subset \mathcal{I}_0^\perp$. Thus $v(\mathcal{I}_1^\perp)$ contains no nontrivial convex subgroups, so that v is indeed the valuation associated to $\mathcal{I}_1$ due to the characterization from Lemma 3.3.

We must show that

$$\mathcal{I}_v \cap \mathcal{G}_{K|T} = \mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T}) \neq \mathcal{D}_v \cap \mathcal{G}_{K|T}.$$

Recall that $\mathcal{D}_v \cap \mathcal{G}_{K|T} / \mathcal{I}_v \cap \mathcal{G}_{K|T} \cong \mathcal{G}_{Kv|Tv}$. Since $Tv \subset kv^\times$ we find that $\mathcal{G}_{Kv|kv} \subset \mathcal{G}_{Kv|Tv}$, while $\operatorname{trdeg}(Kv|kv) \geq 1$ ensures that $\mathcal{G}_{Kv|kv}$ is infinite dimensional. Thus $\mathcal{I}_v \cap \mathcal{G}_{K|T} \neq \mathcal{D}_v \cap \mathcal{G}_{K|T}$.

Thus all that remains to show is that $\mathcal{I}_v \cap \mathcal{G}_{K|T} = \mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T})$. Note that $\mathcal{I}_v \cap \mathcal{G}_{K|T} \subset \mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T})$ by Lemma 3.2, so we only need to show the other inclusion. As noted above, the image of the composition

$$\mathcal{D}_v \cap \mathcal{G}_{K|T} \hookrightarrow \mathcal{D}_v \twoheadrightarrow \mathcal{D}_v/\mathcal{I}_v \cong \mathcal{G}_{Kv}$$

is precisely $\mathcal{G}_{Kv|Tv}$. By Lemma 3.2, it follows that the image of $\mathcal{Z}_K(\mathcal{D}_v \cap \mathcal{G}_{K|T})$ in $\mathcal{G}_{Kv|Tv}$ is precisely $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv})$. It suffices to show that this image, or equivalently $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv})$, is trivial.

We now have two cases to consider. If $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv}) = \mathcal{G}_{Kv|Tv}$ then by Theorem 3.1, there exists a valuation w of Kv such that $\mathcal{G}_{Kv|Tv} \subset \mathcal{D}_w$ and such that $\mathcal{I}_w \cap \mathcal{G}_{Kv|Tv}$ has codimension ≤ 1 in $\mathcal{G}_{Kv|Tv}$. Lemma 3.9 shows that w is trivial and thus $\mathcal{I}_w$ is trivial, which is impossible since $\mathcal{G}_{Kv|kv} \subset \mathcal{G}_{Kv|Tv}$ is infinite dimensional.

Thus we have $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv}) \neq \mathcal{G}_{Kv|Tv}$. In this case, Proposition 3.7 shows that $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv})$ is valuative and, letting w denote the associated valuation, one has $\mathcal{G}_{Kv|Tv} \subset \mathcal{D}_w$. Again, Lemma 3.9 shows that w is trivial, hence $\mathcal{I}_w$ is trivial, so $\mathcal{Z}_{Kv}(\mathcal{G}_{Kv|Tv})$ is trivial as well. This concludes the proof of the proposition. □

4. Algebraic dependence

Let k be a relatively algebraically closed subfield of K. Our goal in this section is to provide a characterization of algebraic dependence (over k) in K using the algebra $\mathcal{K}_*(K|k)$, or equivalently, using the structure $\mathscr{A}(K|k)$.

4.1. **Milnor-closed subspaces.** Let T be a subgroup of $K^\times$ and $\mathcal{H}$ a subspace of $\mathcal{K}_{K|T}$. We say that $\mathcal{H}$ is *Milnor-closed* provided that for all nontrivial $s \in \mathcal{H}$ and $t \in \mathcal{K}_{K|T}$ such that $\{s,t\} = 0$ in $\mathcal{K}_2(K|T)$, one has $t \in \mathcal{H}$ as well. Any subset S of $\mathcal{K}_{K|T}$ has a *Milnor-closure* which is the smallest Milnor-closed subspace $\mathcal{H}$ of $\mathcal{K}_{K|T}$ that contains S. Explicitly, the Milnor-closure of S can be computed as a union

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_n$$

where $\mathcal{H}_0$ is the subspace generated by S, and $\mathcal{H}_{n+1}$ is the subspace generated by $\mathcal{H}_n$ and all $t \in \mathcal{K}_{K|T}$ such that there exists some nontrivial $s \in \mathcal{H}_n$ where $\{s,t\} = 0$ in $\mathcal{K}_2(K|T)$.

Lemma 4.1. *Let $\mathcal{H}$ be a Milnor-closed subspace of $\mathcal{K}_{K|T}$. Let H be the preimage of $\mathcal{H}$ with respect to the canonical map $K^\times \to \mathcal{K}_{K|T}$. Then H contains the following:*

(1) *The saturation of T.*
(2) *Elements of $K^\times$ of the form $a+b\cdot h$ for any $h\in H$ whose image in $\mathcal{K}_{K|T}$ is nontrivial and any a,b in the saturation of T.*

Proof. Let R denote the saturation of T. It is clear that H contains R. Recall that we have a natural morphism of graded rings

$$\mathrm{K}_*^{\mathrm{M}}(K|R) \to \mathcal{K}_*(K|R) = \mathcal{K}_*(K|T).$$

Consider elements of $K^\times$ the form $a+b\cdot h$ as described in (3). We calculate some symbols in $\mathrm{K}_2^{\mathrm{M}}(K|R)$:

$$\begin{aligned}\{h, a+b\cdot h\} &= \{h, 1-(-b\cdot a^{-1})\cdot h\}\\ &= \{(-b\cdot a^{-1})\cdot h, 1-(-b\cdot a^{-1})\cdot h\}\\ &= 0.\end{aligned}$$

The first equality follows from the fact that $a^{-1}\in R$, the second from the fact that $-b\cdot a^{-1}\in R$, and the last from the Steinberg relations in Milnor K-theory. Since $\mathcal{H}$ is Milnor-closed and $h\in H$ it follows that $a+b\cdot h\in H$ as well. □

4.2. **Detecting algebraic dependence.**

Lemma 4.2. *Let v be a valuation on K. The canonical map*

$$\wedge^*(\mathcal{K}_{K|\mathrm{U}_v}) \to \mathcal{K}_*(K|\mathrm{U}_v)$$

given by the identity in degree one is an isomorphism.

Proof. Note that the map is surjective and that $\mathrm{U}_v = \mathcal{I}_v^\perp$. We will identify $\mathcal{K}_{K|\mathrm{U}_v}$ with $\mathbb{Q}\otimes_\mathbb{Z} vK$ and for $t\in K^\times$, we abuse the notation and write $v(t)$ for the image of t in $\mathbb{Q}\otimes_\mathbb{Z} vK = \mathcal{K}_{K|\mathrm{U}_v}$. The kernel of the map in question is generated by $r_{x,y} := v(x)\wedge v(y)$ where $x,y\in K^\times$ satisfy $x+y=1$. We claim that all such $r_{x,y}$ are already trivial. If $v(x)=0$ or $v(y)=0$ then $r_{x,y}=0$, so there is nothing to show. If $v(x)>0$ then $v(y)=0$ so $r_{x,y}=0$, and similarly if $v(y)>0$. Otherwise, $v(x)<0$ and $v(y)=v(x)$ so again $r_{x,y}=0$. In any case, the kernel is trivial, so the map in question is an isomorphism. □

Lemma 4.3. *Suppose that k is a subfield of K and T is a subgroup of $K^\times$ which is contained in $k^\times$. Let $t_1,\ldots,t_n\in K^\times$ be given, and let $\bar{t}_i$ denote the image of t_i in $\mathcal{K}_{K|T}$. If $\{\bar{t}_1,\ldots,\bar{t}_n\}=0$ in $\mathcal{K}_n(K|T)$, then $t_1,\ldots,t_n$ are algebraically dependent over k.*

Proof. If $t_1,\ldots,t_n$ are algebraically independent, then we may find a k-valuation v on K such that $v(t_1),\ldots,v(t_n)$ are linearly independent in $\mathbb{Q}\otimes_\mathbb{Z} vK = \mathcal{K}_{K|\mathrm{U}_v}$. For example, we can take the discrete rank n valuation on $k(t_1,\ldots,t_n)$ associated to the regular sequence $(t_1,\ldots,t_n)$ and choose v to be some prolongation of this valuation to K. Since $T\subset k^\times\subset \mathrm{U}_v$, the assertion follows from Lemma 4.2. □

4.3. **Geometric lattices.** Suppose that k is a relatively algebraically closed subfield of K. The collection of all relatively algebraically closed subextension of $K|k$ will be denoted by $\mathbb{G}(K|k)$. This is a *complete* lattice, with respect to inclusion of subfields of K, meaning that any set has a greatest lower bound (the infimum) and a smallest upper bound (the supremum). In this case, the infimum is computed by taking intersections in K, and the supremum is computed by taking the relative algebraic closure of the compositum in K. We call $\mathbb{G}(K|k)$ the *geometric lattice* associated to $K|k$.

For a field F, write F^i for the perfect closure of F. Note that restriction along $K \hookrightarrow K^i$ induces an isomorphism of geometric lattices

$$\mathbb{G}(K^i|k^i) \cong \mathbb{G}(K|k),$$

where the inverse is given by $M \mapsto M^i$. We will make use of the following result from [10], which fundamentally relies on the work of Evans-Hrushovski [8, 9] based on the *group configuration theorem*. This theorem should be thought of as an analogue of the fundamental theorem of projective geometry, but for an incidence geometry associated to $\mathbb{G}(K|k)$ as opposed to a projective space.

Theorem 4.4 ([10], Theorem 4.2)**.** *Suppose that $K|k$ and $L|l$ are relatively algebraically closed extensions of fields. Assume that* $\mathrm{trdeg}(K|k) \geq 5$*, and that $\varphi : \mathbb{G}(K|k) \cong \mathbb{G}(L|l)$ is an isomorphism of geometric lattices. Then there exists an isomorphism $\Phi : K^i \cong L^i$ of fields satisfying $\Phi(k^i) = l^i$ such that $\varphi(M) = \Phi(M^i) \cap L$ for all $M \in \mathbb{G}(K|k)$. Furthermore, Φ is unique with these properties up-to composition with some power of the p-power Frobenius $x \mapsto x^p$, where p is the characteristic exponent of K.*

Proof. Since $\mathbb{G}(K|k) \cong \mathbb{G}(L|l)$, we have $\mathrm{trdeg}(L|l) = \mathrm{trdeg}(K|k) \geq 5$, as the transcendence degree of a relatively algebraically closed field extension is the Krull dimension of the associated geometric lattice.

The only part that doesn't follow immediately from [10, Theorem 4.2] is that [8, 9, 10] all write $\mathbb{G}(K|k)$ for the *combinatorial geometry* associated to $K|k$ as opposed to the geometric lattice as we have defined above. But the two approaches are easily seen to be equivalent (this is a very well-known fact of matroid theory).

Indeed, let us write $\mathbb{G}'(K|k)$ for the combinatorial geometry associated to $K|k$. This object refers to the set of all relatively algebraically closed subextensions M of $K|k$ such that $\mathrm{trdeg}(M|k) = 1$, and $\mathbb{G}'(K|k)$ is endowed with a *closure operator* cl which associates to a subset $S \subset \mathbb{G}'(K|k)$ the set $\mathrm{cl}(S)$ of all $M \in \mathbb{G}'(K|k)$ such that $M \subset \overline{k(S)} \cap K$. A subset S of $\mathbb{G}'(K|k)$ is called *closed* provided that $\mathrm{cl}(S) = S$.

One can functorially recover $\mathbb{G}'(K|k)$ from $\mathbb{G}(K|k)$, and vice-versa, as follows. First note that $\mathbb{G}(K|k)$ has a unique minimal element $\perp$ corresponding to the field k. Next, note that $\mathbb{G}'(K|k)$ is the set of *atoms* of $\mathbb{G}(K|k)$, i.e. the set of elements M of $\mathbb{G}(K|k)$ which are different from $\perp$ and minimal with that property. The closure operator cl is obtained using the lattice structure of $\mathbb{G}(K|k)$ as follows:

$$\mathrm{cl}(S) = \{M \in \mathbb{G}'(K|k) \mid M \leq \mathrm{Sup}(S)\},$$

where $\mathrm{Sup}(S)$ denotes the supremum of S in $\mathbb{G}(K|k)$. Conversely, one may identify $\mathbb{G}(K|k)$ with the lattice of closed subsets of $\mathbb{G}'(K|k)$.

Going back to the context of the theorem, we have an isomorphism $\varphi : \mathbb{G}(K|k) \cong \mathbb{G}(L|l)$ of lattices, which induces an isomorphism $\varphi' : \mathbb{G}'(K|k) \cong \mathbb{G}'(L|l)$ of combinatorial geometries. By [10, Theorem 4.2(ii)], there exists an isomorphism $\Phi : K^i \cong L^i$ of fields with $\Phi(k^i) = l^i$ such that for all $M \in \mathbb{G}'(K|k)$, one has $\varphi'(M) = \Phi(M^i) \cap L$.

It remains to show that $\varphi(M) = \Phi(M^i) \cap L$ for *all* $M \in \mathbb{G}(K|k)$, but this follows easily from the fact that $\mathbb{G}(K|k)$ and $\mathbb{G}(L|l)$ are *atomistic* lattices, meaning that every element M of $\mathbb{G}(K|k)$ is the supremum of the atoms it bounds from above (and similarly for $\mathbb{G}(L|l)$). $\square$

4.4. **Geometric subspaces.** Suppose that k is a relatively algebraically closed subfield of K. Let L be any subextension of $K|k$. Then the canonical map

$$\mathcal{K}_{L|k} \to \mathcal{K}_{K|k}$$

is injective, and we will identify $\mathcal{K}_{L|k}$ with its image in $\mathcal{K}_{K|k}$. A subspace of $\mathcal{K}_{K|k}$ will be called *geometric* if it is of the form $\mathcal{K}_{L|k}$ for some *relatively algebraically closed* subextension L of $K|k$. The collection of all geometric subspaces of $\mathcal{K}_{K|k}$ will be denoted by $\mathbb{G}_{\mathcal{K}}(K|k)$, considered as a poset with respect to inclusion in $\mathcal{K}_{K|k}$.

Proposition 4.5. *The canonical map $\mathbb{G}(K|k) \to \mathbb{G}_{\mathcal{K}}(K|k)$ sending $L \in \mathbb{G}(K|k)$ to $\mathcal{K}_{L|k}$ is an order isomorphism $\mathbb{G}(K|k) \cong \mathbb{G}_{\mathcal{K}}(K|k)$. In particular, $\mathbb{G}_{\mathcal{K}}(K|k)$ is also a complete lattice.*

Proof. This map is clearly monotone and surjective. Conversely, suppose that L_1 and L_2 are two elements of $\mathbb{G}(K|k)$, that $\mathcal{K}_{L_1|k} \leq \mathcal{K}_{L_2|k}$ and that $t \in L_1$ is some element. Letting $\bar{t}$ denote the image of t in $\mathcal{K}_{K|k}$, we have $\bar{t} \in \mathcal{K}_{L_1|k}$, which is contained in $\mathcal{K}_{L_2|k}$. Thus there exists some positive integer n and some constant $c \in k^\times$ such that $c \cdot t^n \in L_2^\times$. This implies that t is algebraic over L_2 and thus $t \in L_2$. In other words, we have $L_1 \leq L_2$ in $\mathbb{G}(K|k)$. In particular, the map in question is then injective, hence bijective, and its inverse is also monotone. ☐

4.5. **Milnor closure vs. algebraic closure.** We continue to work with a relatively algebraically closed subfield k of K. Recall that we have a canonical pairing

$$\mathcal{K}_{K|k} \times \mathcal{G}_{K|k} \to \mathbb{Q}.$$

For the rest of this subsection, we will use the notation $\mathcal{E}^\perp$ to denote the orthogonal of a subspace $\mathcal{E} \subset \mathcal{G}_{K|k}$ with respect to the above pairing. Although this overloads the notation $(-)^\perp$ introduced previously, it should be clear from context when $\mathcal{E}^\perp$ refers to a subspace of $\mathcal{K}_{K|k}$ as opposed to a subgroup of $K^\times$.

The following theorem is the technical core of this paper. It provides a characterization of the elements $\mathbb{G}_{\mathcal{K}}(K|k)$ as subspaces of $\mathcal{K}_{K|k}$.

Theorem 4.6. *Assume that* $\operatorname{trdeg}(K|k) \geq 2$. *Let k be a relatively algebraically closed subfield of K, and let $\mathcal{H}$ be a Milnor-closed subspace of $\mathcal{K}_{K|k}$. Let H denote the preimage of $\mathcal{H}$ with respect to the map $K^\times \to \mathcal{K}_{K|k}$, and let L denote the relative algebraic closure of $k(H)$ in K. Let $\mathscr{V}_{\mathcal{H}}$ denote the collection of all closed subspaces $\mathcal{D}$ of $\mathcal{G}_{K|k}$ satisfying the following conditions:*

1. *One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$.*
2. *One has $\mathcal{C}_K(\mathcal{Z}_K(\mathcal{D})) \cap \mathcal{G}_{K|k} = \mathcal{D}$.*
3. *One has $\mathcal{D}^\perp \cap \mathcal{H} = 0$.*

Then one has

$$\mathcal{K}_{L|k} = \bigcap_{\mathcal{D} \in \mathscr{V}_{\mathcal{H}}} \mathcal{Z}_K(\mathcal{D})^\perp.$$

Conversely, any geometric subspace of $\mathcal{K}_{K|k}$ arises in this way. More precisely, if $\mathcal{H} = \mathcal{K}_{L|k}$ for $L \in \mathbb{G}(K|k)$, then $\mathcal{H}$ is Milnor-closed and one has

$$\mathcal{H} = \bigcap_{\mathcal{D} \in \mathscr{V}_{\mathcal{H}}} \mathcal{Z}_K(\mathcal{D})^\perp.$$

Proof. Let $\mathscr{V}_{\mathcal{H}}$ be as in the statement of the theorem. By Theorem 3.8, the subspaces $\mathcal{D} \in \mathscr{V}_{\mathcal{H}}$ all have the form $\mathcal{D} = \mathcal{D}_v \cap \mathcal{G}_{K|k}$ where v is a k-visible valuation of K such that $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$. Furthermore, in this case one has $\mathcal{Z}_K(\mathcal{D}) = \mathcal{I}_v \cap \mathcal{G}_{K|k}$ hence also $\mathcal{Z}_K(\mathcal{D})^\perp = \mathcal{U}_{v|k}$. Thus, the intersection in question is precisely

$$\Delta := \bigcap_v \mathcal{U}_{v|k}$$

where v varies over the k-visible valuations of K such that $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$.

Let us first handle the case where $\mathcal{H} = 0$, so $L = k$. By the above discussion, it suffices to show that for every nontrivial $x \in \mathcal{K}_{K|k}$, there exists some k-visible valuation v of K such that $x \notin \mathcal{U}_{v|k}$. Replace x by $n \cdot x$ for some positive integer n to assume without loss of generality that x is the image of a transcendental element $t \in K^\times$. Extend t to a transcendence base

$$\mathcal{B} = \{t\} \cup \mathcal{B}_0$$

of $K|k$, write $M = k(\mathcal{B}_0)$, and let v be a prolongation of the t-adic valuation on $M(t)$ to K. Mote that $Kv|Mv$ is algebraic and $Mv = M$ while $vk = 0$. By Proposition 3.10 and our assumptions on $\operatorname{trdeg}(K|k)$, we see that v is indeed visible, while $v(t) > 0$ by construction. Finally, if $x \in \mathcal{U}_{v|k}$ then $c \cdot t \in \mathrm{U}_v$ for some $c \in k^\times$, but $v(c \cdot t) = v(t) > 0$, so this cannot happen. Thus the assertion of the theorem holds true in the case where $\mathcal{H} = 0$. Assume for the rest of the proof that $\mathcal{H} \neq 0$.

In order to show that $\Delta = \mathcal{K}_{L|k}$, it suffices to show that a k-visible valuation v of K is trivial on L if and only if $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$. Indeed, in this case $\Delta = \bigcap_v \mathcal{U}_{v|k}$ where v varies over the k-visible valuations which are trivial on L. Hence $\mathcal{K}_{L|k} \subset \Delta$. If $x \in \mathcal{K}_{K|k} \smallsetminus \mathcal{K}_{L|k}$, then choose some $t \in K \smallsetminus L$ such that the image of t in $\mathcal{K}_{K|k}$ is $n \cdot x$ for some positive integer n. This t is then transcendental over L. Complete t to a transcendence basis $\mathcal{B} = \{t\} \cup \mathcal{B}_0$ of $K|L$, put $M := L(\mathcal{B}_0)$, and let v be an extension of the t-adic valuation on $M(t)$ to K. Arguing similarly to the above, we see that v is k-visible, trivial on L, and that $x \notin \mathcal{U}_{v|k}$. This shows that indeed, $\Delta = \mathcal{K}_{L|k}$ provided that a k-visible valuation is trivial on L if and only if $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$.

Suppose that v is trivial on L. We have a canonical injective map

$$L^\times / k^\times \to Kv^\times / kv^\times$$

and after tensoring with $\mathbb{Q}$ we obtain

$$\mathcal{K}_{L|k} \to \mathcal{K}_{Kv|kv}$$

which is again injective. This last map is precisely the composition

$$\mathcal{K}_{L|k} \hookrightarrow \mathcal{U}_{v|k} \twoheadrightarrow \mathcal{U}_{v|k}/\mathcal{U}^1_{v|k} = \mathcal{K}_{Kv|kv}$$

and thus $\mathcal{U}^1_{v|k} \cap \mathcal{K}_{L|k} = 0$. Since $\mathcal{H} \subset \mathcal{K}_{L|k}$, we also have $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$.

We must now show that a k-visible valuation v is trivial on L provided that $\mathcal{U}^1_{v|k} \cap \mathcal{H} = 0$. *This is the crux of the proof.* So, assume that v is k-visible and *nontrivial* on L. Since $L|k(H)$ is algebraic, it follows that v is nontrivial on $k(H)$.

We observe that $k(H)^\times = H$ as subgroups of $K^\times$. Indeed, first note that $k^\times \subset H$ and H is multiplicatively closed. Thus, it suffices to show that $H \cup \{0\}$ is additively closed, and since H is a subgroup, for this it suffices to show that $1 + t \in H \cup \{0\}$ whenever $t \in H$. If $t \in k^\times$, then this is obvious, and if not, then its image in $\mathcal{K}_{K|k}$ is nontrivial, so that $1 + t \in H$ by Lemma 4.1, using the assumption that $\mathcal{H}$ is Milnor-closed.

In any case, we have $k(H)^\times = H$. Since $\mathcal{H} \neq 0$, hence $k^\times \neq k(H)^\times = H$ there must exist some element $t \in H \smallsetminus k^\times$ such that $v(t) > 0$. Since $1 + t \in H \smallsetminus k^\times$ as well, the image of $1 + t$ in $\mathcal{K}_{K|k}$ is a nontrivial element of $\mathcal{U}^1_{v|k} \cap \mathcal{H}$, showing that $\mathcal{U}^1_{v|k} \cap \mathcal{H} \neq 0$.

The final assertion is easy. If $\mathcal{H} = \mathcal{K}_{L|k}$ then $\mathcal{H}$ is Milnor-closed by Lemma 4.3, while the preimage of $\mathcal{K}_{L|k}$ in $K^\times$ is $L^\times$ since L is relatively algebraically closed in K. So, the first part of the theorem, which is already proved, gives the desired claim. □

4.6. **Recovering transcendence degree.** We continue with the context above where k is a relatively algebraically closed subfield of K.

Lemma 4.7. *Let d be any positive integer. Assume that there exists a closed subspace $\mathcal{D}$ of $\mathcal{G}_{K|k}$ such that the following conditions hold true:*

(1) *One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D} = \mathcal{C}_K(\mathcal{Z}_K(\mathcal{D})) \cap \mathcal{G}_{K|k}$.*
(2) *One has $d \leq \dim_K(\mathcal{Z}_K(\mathcal{D}))$.*

Then $\operatorname{trdeg}(K|k) \geq d$.

Proof. This is a simple consequence of Theorem 3.8 in conjunction with Abhyankar's inequality. By Theorem 3.8, there exists a k-visible valuation v of K such that $\mathcal{I} := \mathcal{Z}_K(\mathcal{D}) = \mathcal{I}_v \cap \mathcal{G}_{K|k}$. Dualizing $\mathcal{I}$, we obtain $\mathcal{K}_{K|k}/\mathcal{U}_{v|k} = \mathbb{Q} \otimes_{\mathbb{Z}} (vK/vk)$, and our assumption tells us that $\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} (vK/vk)) \geq d$. Choose $t_1, \ldots, t_d \in K$ whose images in $\mathbb{Q} \otimes_{\mathbb{Z}} (vK/vk)$ are linearly independent, and consider $L := k(t_1, \ldots, t_d)$, a finitely-generated subextension of $K|k$, which has the property that

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} (vL/vk)) \geq d.$$

By Abhyankar's inequality, we find

$$d \leq \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} (vL/vk)) \leq \operatorname{trdeg}(L|k) \leq \operatorname{trdeg}(K|k),$$

as claimed in the lemma. □

Lemma 4.8. *Suppose that d is a positive integer and that* $\operatorname{trdeg}(K|k) > d$. *Then there exists a closed subspace $\mathcal{D}$ of $\mathcal{G}_{K|k}$ such that the following conditions hold true:*

(1) *One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D} = \mathcal{C}_K(\mathcal{Z}_K(\mathcal{D})) \cap \mathcal{G}_{K|k}$.*
(2) *One has $d \leq \dim_K(\mathcal{Z}_K(\mathcal{D}))$.*

Proof. Let $t_1, \ldots, t_d \in K$ be algebraically independent over k, and extend $\{t_1, \ldots, t_d\}$ to a transcendence base of $K|k$ of the form

$$\mathcal{B} = \{t_1, \ldots, t_d\} \cup \mathcal{B}_0.$$

By assumption on $\operatorname{trdeg}(K|k)$, the set $\mathcal{B}_0$ is nonempty. Put $L := k(\mathcal{B}_0)$ and $M := k(\mathcal{B}) = L(t_1, \ldots, t_d)$. Consider the valuation associated to the system of regular parameters $(t_1, \ldots, t_d)$ on M. This is a discrete rank d valuation which is trivial on L. Extend it in some way to a valuation v on K. Since $\operatorname{trdeg}(L|k) > 0$, it follows from Proposition 3.10 that v is visible, while $vk = 0$ and the images of $t_1, \ldots, t_d$ in $\mathbb{Q} \otimes_{\mathbb{Z}} vK$ are rationally independent. Thus $\dim_{\mathbb{Q}}(\mathcal{I}_v \cap \mathcal{G}_{K|k}) \geq d$ as well, while condition (1) follows from Theorem 3.8. □

We will use Lemmas 4.7 and 4.8 primarily to provide a *lower bound* on $\operatorname{trdeg}(K|k)$, which will be required in order to apply Theorem 4.6. We summarize this observation in the following lemmas.

Lemma 4.9. *Suppose that $K|k$ and $L|l$ are two relatively algebraically closed field extensions, and that $\mathcal{K}_*(K|k) \cong \mathcal{K}_*(L|l)$ as algebras. If d is a positive integer such that $d < \operatorname{trdeg}(K|k)$, then $d \leq \operatorname{trdeg}(L|l)$.*

Proof. The isomorphism $\mathcal{K}_*(K|k) \cong \mathcal{K}_*(L|l)$ induces an isomorphism of structures $\mathscr{A}(K|k) \cong \mathscr{A}(L|l)$ by Fact 2.1 and the surrounding discussion. By Lemma 4.8, there exists a closed subspace $\mathcal{D}$ of $\mathcal{G}_{K|k}$ such that

(1) One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D} = \mathcal{C}_K(\mathcal{D}) \cap \mathcal{G}_{K|k}$.
(2) One has $d \leq \dim_K(\mathcal{Z}_K(\mathcal{D}))$.

Transferring this subspace across the isomorphism $\mathcal{G}_{K|k} \cong \mathcal{G}_{L|k}$, we obtain a closed subspace of $\mathcal{G}_{L|l}$ satisfying the conditions of Lemma 4.7, which implies that $d \leq \operatorname{trdeg}(L|l)$. □

Lemma 4.10. *Suppose that $K|k$ and $L|l$ are two relatively algebraically closed field extensions, and that $\mathcal{K}_*(K|k) \cong \mathcal{K}_*(L|l)$ as algebras. Assume that $\operatorname{trdeg}(K|k) \geq 3$. Then $\operatorname{trdeg}(K|k) = \operatorname{trdeg}(L|l)$.*

Proof. By Lemma 4.7, we see that $\operatorname{trdeg}(L|l) \geq 2$, hence Theorem 4.6 applies to both $K|k$ and $L|l$. This theorem provides us with an isomorphism of lattices

$$\mathbb{G}_{\mathcal{K}}(K|k) \cong \mathbb{G}_{\mathcal{K}}(L|l)$$

hence also $\mathbb{G}(K|k) \cong \mathbb{G}(L|l)$ by Proposition 4.5. Since the transcendence degree of a relatively algebraically closed field extension is the Krull dimension of the corresponding geometric lattice, the claim follows. □

5. Main results

In this section, we present and prove the main results of this paper. We split up this section into three subsections:

(1) The first regarding *relative results*, dealing with relatively algebraically closed field extensions $K|k$ of sufficiently large transcendence degree, while using $\mathcal{K}_*(K|k)$.
(2) The second regarding *absolute results*, dealing with arbitrary fields of sufficiently large *absolute* transcendence degree (i.e. transcendence degree over the prime subfield), while using $\mathcal{K}_*(K)$.
(3) The third dealing with finitely-generated relatively algebraically closed extensions $K|k$ over perfect fields and finitely-generated fields L, using $\mathrm{K}_*^{\mathrm{M}}(K|k)$, $\mathrm{K}_*^{\mathrm{M}}(L)$.

In cases (1) and (2), our key tool is one of the main results of Evans-Hrushovski and Gismatullin [8, 9, 10], which we have summarized in Theorem 4.4 above. In case (3), our key tools will be the reconstruction result due to Cadoret-Pirutka [4, Theorem 4].

In the first two cases, we will only be able to recover fields *up-to inseparable extensions*, or more precisely, we will be able to recover the perfect closure of the field in question. Recall that we write F^i for its perfect closure of a field F. Of course, if F has characteristic zero, then $F = F^i$. If F has positive characteristic p, then one has

$$F^i = \bigcup_{n \geq 0} F^{1/p^n}.$$

Note that if $K|k$ is a relatively algebraically closed extension of fields, then $K^i|k^i$ is also relatively algebraically closed. The canonical map $K^\times \to (K^i)^\times$ induces an *injective* map $K^\times/k^\times \to (K^i)^\times/(k^i)^\times$, and the induced map of $\mathbb{Q}$-modules $\mathcal{K}_{K|k} \to \mathcal{K}_{K^i|k^i}$ is an *isomorphism.*

We first prove an auxiliary lemma that will be necessary to show certain *uniqueness* properties of the isomorphisms of fields that we obtain. This lemma has appeared before in [9, Theorem 1.1] and in [2, Lemma 13], although the proof of [9] is more complicated as it relies on the more general result [8, Theorem 2.2.2], while [2] has a blanket assumption that the fields in question have characteristic zero and the base field is algebraically closed. The argument we give is an adaptation of the proof from [2] which avoids the restrictions on the base field.

Lemma 5.1. *Let $K|k$ be a relatively algebraically closed extension of fields. Suppose that $x, y \in K$ are algebraically independent over k. Let $a, b \in K \smallsetminus k$ be two elements such that* $\mathrm{trdeg}(k(a,x)|k) = 1$, $\mathrm{trdeg}(k(b,y)|k) = 1$ *and* $\mathrm{trdeg}(k(a\cdot b, x\cdot y)|k) = 1$. *Then there exist nonzero integers m and n such that, modulo $k^\times$, one has $a^n = x^m$, $b^n = y^m$.*

Proof. Embed $K^\times/k^\times$ into $\overline{K}^\times/\overline{k}^\times$ to assume without loss of generality that both k and K are algebraically closed. The elements a, b and $a\cdot b$ are all contained in the compositum $\overline{k(x)}\cdot\overline{k(y)} =: M \subset K$. Since x and y are algebraically independent over k, we may identify

$$\mathrm{Gal}(M|k(x,y)^i) = \mathrm{Gal}(\overline{k(x)}|k(x)^i) \times \mathrm{Gal}(\overline{k(y)}|k(y)^i),$$

where the two projections are the usual restriction maps. This lets us identify $\mathrm{Gal}(\overline{k(x)}|k(x)^i)$ with the subgroup of $\mathrm{Gal}(M|k(x,y)^i)$ which fixes $\overline{k(y)}$ pointwise, and similarly with x and y interchanged.

With this identification, any $\sigma \in \mathrm{Gal}(\overline{k(x)}|k(x)^i)$ acts on $a\cdot b$ as $\sigma(a)\cdot b$, and thus $\sigma(a)/a = \sigma(a\cdot b)/(a\cdot b) =: t_\sigma$. As x and $x\cdot y$ are algebraically independent, we have

$$t_\sigma \in \overline{k(x)} \cap \overline{k(x\cdot y)} = k.$$

But $\sigma(a) = t_\sigma \cdot a$ and $k(x)^i(a)$ is a finite extension of $k(x)^i$, hence t_σ must be a root of unity. By symmetry, for any $\tau \in \mathrm{Gal}(\overline{k(y)}|k(y))$, we also have $\tau(b) = s_\tau \cdot b$ for some root of unity s_τ.

It follows that for all $\gamma \in \mathrm{Gal}(M|k(x,y)^i)$, the elements $\gamma(a)/a$ and $\gamma(b)/b$ are both roots of unity. The action of $\mathrm{Gal}(M|k(x,y)^i)$ on $k(x,y)^i(a,b)$ factors through a finite quotient, and thus we deduce that there exists some positive integer n_1 such that for all $\gamma \in \mathrm{Gal}(M|k(x,y)^i)$, one has $\gamma(a^{n_1}) = a^{n_1}$ and $\gamma(b^{n_1}) = b^{n_1}$. In other words, $\mathrm{Gal}(M|k(x,y)^i)$ acts trivially on a^{n_1} and b^{n_1}, which implies that $a^{n_1} \in k(x)^i$, $b^{n_1} \in k(y)^i$. Arguing similarly with $x\cdot y$ in place of y and x^{-1} in place of x, we see that there exists an integer n_2 such that $a^{n_2} \in k(x)^i$ and $(a\cdot b)^{n_2} \in k(x\cdot y)^i$. Taking $n = n_1 \cdot n_2$, it follows that $a^n \in k(x)^i$, $b^n \in k(y)^i$ and $(a\cdot b)^n \in k(x\cdot y)^i$. Further replacing n by an integer of the form $n\cdot p^k$, where p is the characteristic exponent of k, we may assume that $a^n \in k(x)$, $b^n \in k(y)$ and $(a\cdot b)^n \in k(x\cdot y)$.

In particular, we may write

$$a^n = \prod_{c\in k}(x-c)^{m_c},\ b^n = \prod_{c\in k}(y-c)^{n_c},$$

modulo constants, where all but finitely many of the $m_c, n_c \in \mathbb{Z}$ are zero. In particular, we must also have

$$(a\cdot b)^n = \prod_{c\in k}(x-c)^{m_c}\cdot(y-c)^{n_c},$$

modulo constants. But this element is contained in $k(x\cdot y)$, so the only irreducible polynomials from $k[x,y]$ that may appear in the factorization of $(a\cdot b)^n$ are of the form $(x\cdot y - c)$ for some $c \in k$. Combining these observations, we deduce that $m_c = n_c = 0$ for all $c \in k^\times$, and that

$m_0 = n_0$. In other words, there exists an integer m such that $a^n = x^m$ and $b^n = y^m$ modulo constants, as required. □

Recall that $\mathcal{K}_{K|k}$ is written *additively*. It will be useful to reinterpret the above lemma in the context of $\mathcal{K}_{K|k}$. First of all, if S is *any subset* of $\mathcal{K}_{K|k}$, we write $\mathrm{acl}_{K|k}(S)$ for the *smallest* element of $\mathbb{G}_{\mathcal{K}}(K|k)$ which contains S. Since $\mathbb{G}_{\mathcal{K}}(K|k)$ is a complete lattice, this $\mathrm{acl}_{K|k}(S)$ is well-defined. We say that two elements $x, y \in \mathcal{K}_{K|k}$ are *dependent* provided that there exists a geometric subspace of the form $\mathcal{K}_{L|k}$ where $L \in \mathbb{G}(K|k)$ with $\mathrm{trdeg}(L|k) \leq 1$ such that $x, y \in \mathcal{K}_{L|k}$. If x, y are *not dependent* then we shall say that they are *independent* and in this case $\mathrm{acl}_{K|k}(x, y) = \mathcal{K}_{L|k}$ for some $L \in \mathbb{G}(K|k)$ with $\mathrm{trdeg}(L|k) = 2$. If $x, y \in K^\times$ with images $\overline{x}, \overline{y}$ in $\mathcal{K}_{K|k}$, then $\overline{x}, \overline{y}$ are (in)dependent in the above sense if and only if x, y are algebraically (in)dependent over k.

Lemma 5.2. *Let $x, y \in \mathcal{K}_{K|k}$ be two independent elements. Let $a, b \in \mathcal{K}_{K|k}$ be two elements such that $a \in \mathrm{acl}_{K|k}(x)$, $b \in \mathrm{acl}_{K|k}(y)$ and $a + b \in \mathrm{acl}_{K|k}(x + y)$. Then there exists some $r \in \mathbb{Q}^\times$ such that $a = r \cdot x$ and $b = r \cdot y$.*

Proof. The map $\mathcal{K}_{K|k} \to \mathcal{K}_{\overline{K}|\overline{k}}$ is injective and compatible with the notion of (in)dependence introduced above. Also, the map

$$\mathrm{K}_1^{\mathrm{M}}(\overline{K}|\overline{k}) \to \mathcal{K}_{\overline{K}|\overline{k}}$$

is an *isomorphism* since $\overline{K}^\times$ is divisible. By mapping into $\mathcal{K}_{\overline{K}|\overline{k}}$ and identifying this with $\mathrm{K}_1^{\mathrm{M}}(\overline{K}|\overline{k})$, the assertion of this lemma reduces to that of Lemma 5.1. □

5.1. **The relative case.**

Theorem 5.3. *Let $K|k$ be a relatively algebraically closed extension of fields satisfying $\mathrm{trdeg}(K|k) \geq 5$. Then $K^i|k^i$ are determined up-to isomorphism by the algebra $\mathcal{K}_*(K|k)$. More precisely, if $L|l$ is another relatively algebraically closed extension of fields and*

$$\varphi_* : \mathcal{K}_*(K|k) \xrightarrow{\cong} \mathcal{K}_*(L|l)$$

is an isomorphism of $\mathbb{Q}$-algebras, then there exists some $r \in \mathbb{Q}^\times$ and an isomorphism of fields

$$\Phi : K^i \xrightarrow{\cong} L^i$$

such that $\Phi(k^i) = l^i$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_{K^i|k^i} & \xrightarrow{\Phi} & \mathcal{K}_{L^i|l^i} \\ \uparrow & & \uparrow \\ \mathcal{K}_{K|k} & \xrightarrow[r\cdot\varphi_1]{} & \mathcal{K}_{L|l}. \end{array}$$

Here, the vertical maps are those induced by the inclusions $K \subset K^i$ and $L \subset L^i$, and the map labeled Φ is induced by the isomorphism $\Phi : K^i \cong L^i$ of fields. Finally, Φ is unique with these properties up-to composition with some power of the p-power Frobenius $x \mapsto x^p$, where p is the characteristic exponent of K.

Proof. Let φ_* be as in the statement of the theorem and put $\varphi = \varphi_1$. By Lemma 4.10, we see that $\mathrm{trdeg}(L|l) = \mathrm{trdeg}(K|k) \geq 5$. By Theorem 4.6 and the discussion around Fact 2.1, we see that the map $\mathcal{H} \mapsto \varphi\mathcal{H}$ is an *isomorphism* of lattices $\mathbb{G}_{\mathcal{K}}(K|k) \cong \mathbb{G}_{\mathcal{K}}(L|l)$. By Proposition 4.5, we obtain an isomorphism $\psi : \mathbb{G}(K|k) \cong \mathbb{G}(L|l)$ of geometric lattices satisfying

$$\varphi\mathcal{K}_{M|k} = \mathcal{K}_{\psi M|l}$$

for all $M \in \mathbb{G}(K|k)$. Thus, by [10, theorem 4.2] (see Theorem 4.4), there exists an isomorphism $\Phi : K^i \cong L^i$ of fields satisfying the following conditions:

(1) $\Phi(k^i) = l^i$.
(2) One has $\psi(M) = \Phi(M^i) \cap L$ for all $M \in \mathbb{G}(K|k)$.

And furthermore, this Φ is unique up-to composition with some power of the p-power Frobenius $x \mapsto x^p$, where p is the characteristic exponent of K.

To conclude, we must show that there exists some $r \in \mathbb{Q}^\times$ making the diagram from the statement of the theorem commute. Let $\varphi' : \mathcal{K}_{K|k} \cong \mathcal{K}_{L|l}$ be the isomorphism induced by Φ, i.e. φ' is the composition

$$\mathcal{K}_{K|k} \xrightarrow{\cong} \mathcal{K}_{K^i|k^i} \xrightarrow{\Phi} \mathcal{K}_{L^i|l^i} \xleftarrow{\cong} \mathcal{K}_{L|l}.$$

We must show that there exists some $r \in \mathbb{Q}^\times$ such that $r \cdot \varphi = \varphi'$.

Let $x, y \in \mathcal{K}_{K|k}$ be two independent elements. The pair $\varphi(x)$, $\varphi(y)$ is also independent, while the following pairs are all *dependent*:

$$\varphi(x), \varphi'(x);\ \ \varphi(y), \varphi'(y);\ \ \varphi(x) + \varphi(y), \varphi'(x) + \varphi'(y).$$

By Lemma 5.2, there exists some $r(x, y) \in \mathbb{Q}^\times$ such that

$$r(x, y) \cdot \varphi(x) = \varphi'(x),\ \ r(x, y) \cdot \varphi(y) = \varphi'(y).$$

Since $\varphi(x)$ and $\varphi(y)$ are also *linearly independent* over $\mathbb{Q}$, it follows that $r(x, y) = r(y, x)$.

We claim that $r(x, y)$ does not depend on the choice of x, y. First, if z is another element which is independent from x, then $r(x, y) = r(x, z)$ since $r(x, y) \cdot \varphi(x) = r(x, z) \cdot \varphi(x)$. Now, if x', y' is any other pair of independent elements, we show that $r(x, y) = r(x', y')$ as follows:

(1) If x, x' are *dependent*, then x', y are independent, hence

$$r(x, y) = r(y, x) = r(y, x') = r(x', y) = r(x', y').$$

(2) If x, x' are *independent*, then

$$r(x, y) = r(x, x') = r(x', x) = r(x', y').$$

In any case, $r(x, y)$ doesn't depend on the choice of x, y, and we put $r := r(x, y)$ for some choice of an independent pair x, y.

Now, if x is any nontrivial element of $\mathcal{K}_{K|k}$, we may find some $y \in \mathcal{K}_{K|k}$ which is independent from x, and observe that

$$r \cdot \varphi(x) = r(x, y) \cdot \varphi(x) = \varphi'(x).$$

Of course, this equality holds with $x = 0$ as well, so we deduce that indeed $r \cdot \varphi = \varphi'$. This concludes the proof of the theorem. □

5.2. **The absolute case.**

Theorem 5.4. *Let K be a field whose absolute transcendence degree is at least five. Then K^i is determined up-to isomorphism by the algebra $\mathcal{K}_*(K)$. More precisely, suppose that L is another field and*

$$\varphi_* : \mathcal{K}_*(K) \xrightarrow{\cong} \mathcal{K}_*(L)$$

is an isomorphism of $\mathbb{Q}$-algebras. Let k be the relative algebraic closure of the prime subfield of K and l the relative algebraic closure of the prime subfield of L. Then φ_* induces an isomorphism $\overline{\varphi}_* : \mathcal{K}_*(K|k) \cong \mathcal{K}_*(L|l)$. Furthermore, there exists some $r \in \mathbb{Q}^\times$ and an isomorphism of fields

$$\Phi : K^i \xrightarrow{\cong} L^i$$

which necessarily satisfies $\Phi(k^i) = l^i$, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_{K^i|k^i} & \xrightarrow{\Phi} & \mathcal{K}_{L^i|l^i} \\ \uparrow & & \uparrow \\ \mathcal{K}_{K|k} & \xrightarrow[r\cdot\overline{\varphi}_1]{} & \mathcal{K}_{L|l}. \end{array}$$

Here, the vertical maps are those induced by the inclusions $K \subset K^i$ and $L \subset L^i$, and the map labeled Φ is induced by the isomorphism $\Phi : K^i \cong L^i$ of fields. Finally, Φ is unique with these properties up-to composition with some power of the p-power Frobenius $x \mapsto x^p$, where p is the characteristic exponent of K.

By using Theorem 5.3, in order to prove Theorem 5.4 it suffices to give a characterization of the kernel of $\mathcal{K}_K \to \mathcal{K}_{K|k}$ using $\mathcal{K}_*(K)$, where k is the relative algebraic closure of the prime subfield. If this kernel is denoted by Δ, then one has

$$\mathcal{K}_*(K|k) = \frac{\mathcal{K}_*(K)}{\langle\Delta\rangle}$$

so that we can then apply Theorem 5.3. We describe the characterization of this kernel in the following proposition.

For the rest of this subsection, we will use the notation $\mathcal{E}^\perp$ to denote the orthogonal of a subspace $\mathcal{E} \subset \mathcal{G}_K$ with respect to the pairing

$$\mathcal{K}_K \times \mathcal{G}_K \to \mathbb{Q}.$$

As before, it should be clear from context when $\mathcal{E}^\perp$ refers to a subspace of $\mathcal{K}_K$ as opposed to a subgroup of $K^\times$.

Proposition 5.5. *Let K be any field, and let k denote the relative algebraic closure of the prime subfield of K. Assume that* $\mathrm{trdeg}(K|k) \geq 2$. *Put $\Delta := \ker(\mathcal{K}_K \to \mathcal{K}_{K|k})$. For a nonzero $t \in \mathcal{K}_K$, write $\mathcal{H}_t$ for the Milnor closure of the subset $\{t\}$. Let $\mathscr{V}_t$ denote the collection of closed subspaces $\mathcal{D} \subset \mathcal{G}_K$ satisfying the following conditions:*

1. *One has $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$.*
2. *One has $\mathcal{C}_K(\mathcal{Z}_K(\mathcal{D})) = \mathcal{D}$.*
3. *One has $\mathcal{H}_t \cap \mathcal{D}^\perp = 0$.*

Then one has

$$\Delta = \bigcap_{t \in \mathcal{K}_K \smallsetminus \{0\}} \bigcap_{\mathcal{D} \in \mathscr{V}_t} \mathcal{Z}_K(\mathcal{D})^\perp.$$

Proof. Note that $\mathcal{H}_t$ remains unchanged if we replace t by $n \cdot t$ for any nonzero integer n. Thus, we may restrict our attention to those nonzero $t \in \mathcal{K}_K$ which are in the image of $K^\times$. Note also that the kernel of $K^\times \to \mathcal{K}_K$ is the torsion subgroup of $K^\times$.

If K has positive characteristic then $k^\times$ is the torsion of $K^\times$ hence $\mathcal{K}_*(K) = \mathcal{K}_*(K|k)$. Letting $t \in K \smallsetminus k$ be given, we deduce from Theorem 4.6 that the intersection

$$\bigcap_{\mathcal{D} \in \mathscr{V}_t} \mathcal{Z}_K(\mathcal{D})^\perp$$

is precisely the geometric subspace $\mathcal{K}_{L|k}$ where L is the relative algebraic closure of $k(t)$ in K. As t varies, the intersection of all these geometric subspaces is trivial, so the assertion of the proposition follows.

Assume for the rest of the proof that K has characteristic zero. Let $\mathcal{D}$ be a subspace satisfying conditions (1) and (2). By Theorem 3.8, we see that $\mathcal{D} = \mathcal{D}_v$ for some visibile (i.e. $\{1\}$-visible) valuation v of K, and that $\mathcal{I}_v = \mathcal{Z}_K(\mathcal{D})$.

Let t be an element of $K^\times$ which is not a root of unity, and let H_t denote the preimage of $\mathcal{H}_{\bar{t}}$ in $K^\times$, where $\bar{t}$ denotes the image of t in $\mathcal{K}_K$. Assume that such a $\mathcal{D}$ also satisfies condition (3) for this $\bar{t}$. We claim that the v mentioned above must be trivial on $\mathbb{Q}$ hence also on k. For this, it suffices to show that $2 \in H_t$. Indeed, in this case we would find that $\{2, 3, \ldots\} \subset H_t$ since $\mathcal{H}_{\bar{t}}$ is Milnor-closed, using Lemma 4.1. Thus, if v is nontrivial on $\mathbb{Q}$, and p is the prime for which $v|_\mathbb{Q}$ is the p-adic valuation, we would have $1+p \in H_t \cap \mathrm{U}_v^1$, while $\mathcal{D}^\perp = \mathcal{U}_v^1$, hence the image of $1+p$ would be a nontrivial element of $\mathcal{H}_{\bar{t}} \cap \mathcal{D}^\perp$.

If $t^n = -2$ for some nonzero integer n, then we have $2 \in H_t$, so we may assume this is not the case. By replacing t with t^n for some nonzero integer n (recall that this doesn't change H_t), we may assume that $1+t$ and $(2+t)/t$ are also not roots of unity (note that our assumptions ensure that $1+t^n$ and $(2+t^n)/t^n$ are both nonzero for any nonzero integer n).

To see this, consider the subfield $\mathbb{Q}(t)$ of K generated by t. Assume first that there exists a complex embedding $\sigma : \mathbb{Q}(t) \to \mathbb{C}$ such that $|\sigma(t)| = 1$, and write $\sigma(t) = a + ib$, with a, b real numbers. In this case, we show that $1+t$ and $(2+t)/t$ cannot be roots of unity, so that $n = 1$ works. Indeed, if $1+t$ is a root of unity then

$$|\sigma(1+t)|^2 = (1+a)^2 + b^2 = 1 = a^2 + b^2,$$

which implies that $a = -1/2$ and $b = \pm\sqrt{3}/2$, so that t must be a root of unity. Similarly, if $(2+t)/t$ is a root of unity, then

$$|\sigma((2+t)/t)|^2 = \frac{(2+a)^2 + b^2}{a^2+b^2} = (2+a)^2 + b^2 = 1 = a^2 + b^2,$$

which implies that $a = -1$ and $b = 0$, so that again t would have to be a root of unity.

We can thus assume that for every complex embedding $\sigma : \mathbb{Q}(t) \to \mathbb{C}$, one has $|\sigma(t)| \neq 1$. Let σ be such an embedding. Replacing t with t^{-1} if needed, assume that $|\sigma(t)| > 1$ hence $|\sigma(t^n)| \to \infty$ as $n \to \infty$ and thus $1 + t^n$ is not a root of unity for all sufficiently large n. Replace t with such a t^n with n positive, and note that $t^{n \cdot k}$ would have also worked for any positive k.

Now if $(2+t)/t$ is a root of unity, then $|2+\sigma(t)| = |\sigma(t)|$ which forces $\sigma(t) = -1 + b \cdot i$ for some nonzero real b. If $(2+t^2)/t^2$ is also a root of unity then $\sigma(t^2)$ must have real part -1 as well, hence $-1 = 1 - b^2$ so that $b = \pm\sqrt{2}$. But in this case the real part of $\sigma(t^3)$ is 5, so that $(2+t^3)/t^3$ cannot be a root of unity.

Having made this replacement, we now use Lemma 4.1 repeatedly to find that $1+t \in H_t$, $2+t \in H_t$, $(2+t)/t \in H_t$ and $(2+t)/t - 1 \in H_t$ as well. Thus, $2 = t \cdot ((2+t)/t - 1) \in H_t$. In any case, we have obtained that $2 \in H_t$, which implies that v is trivial on k as noted above.

We deduce that $\mathcal{K}_k$ is indeed contained in the intersection in question. On the other hand, if t is transcendental over k, then we may extend t to a transcendence base

$$\mathcal{B} = \{t\} \cup \mathcal{B}_0$$

for $K|k$. Put $L := k(\mathcal{B}_0)$ and $M = k(\mathcal{B}) = L(t)$. Let v be an extension of the t-adic valuation on M to K and note that v is visible by Proposition 3.10 and our assumption on $\operatorname{trdeg}(K|k)$. However, the image of t is not contained in $\mathcal{U}_v = \mathcal{Z}_K(\mathcal{D}_v)^\perp$. On the other hand, letting $s \in \mathcal{B}_0$ be any element (which is nonempty by our assumption on $\operatorname{trdeg}(K|k)$), we have $\mathcal{D}_v \in \mathscr{V}_{\overline{s}}$ where $\overline{s}$ is the image of s, by arguing as in the proof of Theorem 4.6. Thus, the image of t is not contained in the intersection in question, and so the assertion of the proposition follows. □

Proof of Theorem 5.4. Let k denote the relative algebraic closure of the prime subfield of K, and l the relative algebraic closure of the prime subfield of L. The algebras $\mathcal{K}_*(K)$ and $\mathcal{K}_*(L)$ are isomorphic, while K has absolute transcendence degree ≥ 5. Such an isomorphism induces an isomorphism of structures $\mathscr{A}(K) \cong \mathscr{A}(L)$ by Fact 2.1, hence also an isomorphism $\mathcal{G}_K \cong \mathcal{G}_L$ which is compatible with alternating pairs.

By Lemma 4.8, there exists a closed subspace $\mathcal{D} \subset \mathcal{G}_{K|k} \subset \mathcal{G}_K$ such that $\mathcal{Z}_K(\mathcal{D}) \neq \mathcal{D}$ and $4 \leq \dim(\mathcal{Z}_K(\mathcal{D}))$. Transferring $\mathcal{D}$ across $\mathcal{G}_K \cong \mathcal{G}_L$, we obtain a closed subspace $\mathcal{D}' \subset \mathcal{G}_L$ such that $\mathcal{Z}_L(\mathcal{D}') \neq \mathcal{D}'$ and $4 \leq \dim(\mathcal{Z}_L(\mathcal{D}'))$. By Proposition 3.7, $\mathcal{I} := \mathcal{Z}_L(\mathcal{D}')$ is valuative, and if v denotes the valuation associated to $\mathcal{I}$, it follows that $\mathbb{Q} \otimes_{\mathbb{Z}} vL$ has dimension ≥ 4. Since l the algebraic over a prime field, it follows that $\mathbb{Q} \otimes_{\mathbb{Z}} vL/vl$ has dimension ≥ 3, and thus $3 \leq \operatorname{trdeg}(L|l)$ by Abhyankar's inequality.

In any case, we may thus apply Proposition 5.5 to both K and L. This shows that the isomorphism $\varphi_1 : \mathcal{K}_K \cong \mathcal{K}_L$ sends $\mathcal{K}_k$ to $\mathcal{K}_l$, and thus φ_* descends to an isomorphism

$$\overline{\varphi}_* : \mathcal{K}_*(K|k) = \frac{\mathcal{K}_*(K)}{\langle \mathcal{K}_k \rangle} \cong \frac{\mathcal{K}_*(L)}{\langle \mathcal{K}_l \rangle} = \mathcal{K}_*(L|l).$$

By Theorem 5.3, we obtain an isomorphism of fields $K^i \cong L^i$ and a rational number $r \in \mathbb{Q}^\times$ satisfying the conditions of our theorem. □

5.3. **The finitely-generated case.** In the case where $K|k$ is finitely generated, and k is perfect, we use the work of [4] to obtain a better result.

Theorem 5.6. *Let k be a perfect field and K a regular function field over k of transcendence degree ≥ 2. Then $K|k$ are determined, up-to isomorphism, from $\mathrm{K}^{\mathrm{M}}_*(K|k)$. More precisely, if $L|l$ is another regular function field over a perfect field of transcendence degree ≥ 2 and $\varphi_* : \mathrm{K}^{\mathrm{M}}_*(K|k) \cong \mathrm{K}^{\mathrm{M}}_*(L|l)$ is an isomorphism, then there exists an isomorphism $\Phi : K \cong L$ of fields satisfying $\Phi(k) = l$ and some $\varepsilon \in \{\pm 1\}$ such that $\varepsilon \cdot \varphi_1$ is the isomorphism $K^\times/k^\times \cong L^\times/l^\times$ induced by Φ. If $\operatorname{trdeg}(K|k) \geq 3$, then the assumption on $\operatorname{trdeg}(L|l)$ can be dropped.*

Proof. The isomorphism $\varphi_1 : K^\times/k^\times \cong L^\times/l^\times$ is compatible with algebraic dependence by Theorem 4.6. Thus the claim follows from [4, Theorem 4]. In the case where $\operatorname{trdeg}(K|k) \geq 3$, the fact that $\operatorname{trdeg}(L|l) \geq 2$ follows from Lemma 4.10. □

Theorem 5.7. *Let K be a finitely-generated field whose absolute transcendence degree is at least two. Then the isomorphism type of K is determined by $\mathrm{K}_*^{\mathrm{M}}(K)$. More precisely, suppose that L is any other finitely-generated field of absolute transcendence degree ≥ 2, $\varphi_* : \mathrm{K}_*^{\mathrm{M}}(K) \cong \mathrm{K}_*^{\mathrm{M}}(L)$ is an isomorphism, k denotes the relative algebraic closure of the prime subfield of K and l the relative algebraic closure of the prime subfield of L. Then φ_* induces an isomorphism*

$$\overline{\varphi}_* : \mathrm{K}_*^{\mathrm{M}}(K|k) \cong \mathrm{K}_*^{\mathrm{M}}(L|l),$$

and there exists an isomorphism of fields $\Phi : K \cong L$ and some $\varepsilon \in \{\pm 1\}$, such that $\varepsilon \cdot \overline{\varphi}_1 : K^\times/k^\times \cong L^\times/l^\times$ is the isomorphism induced by Φ. If K has absolute transcendence degree ≥ 4, then the assumption on the transcendence degree of L can be dropped.

Proof. Use the characterization of $\mathcal{K}_k$ in $\mathcal{K}_K$ from Proposition 5.5 and the fact that the preimage of $\mathcal{K}_k$ in $K^\times$ is $k^\times$ to observe that $\varphi_1 : K^\times \cong L^\times$ sends $k^\times$ to $l^\times$. Conclude by using Theorem 5.6. If $\operatorname{trdeg}(K|k) \geq 4$, we may argue as in the proof of Theorem 5.4 to see that L has absolute transcendence degree ≥ 2. □

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