University of Guilan

Computational Intelligence

Subject6: An Introduction to Fuzzy Systems

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Computational Intelligence - Ali Tourani - Fall 2020-2021

Agenda

- Intro
- Fuzzy sets
- Fuzzy membership functions
- Fuzzy sets in practice
- Fuzzy Alpha-cut

Intro

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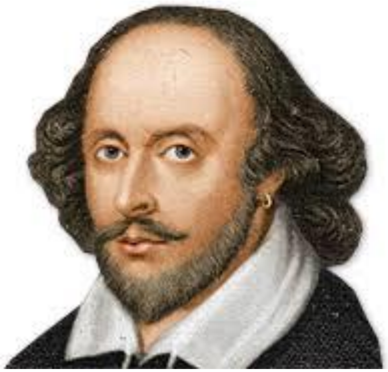
Some students are good at Python!

- Can computers understand this?!
- The world is not a **binary** or a **Boolean** system!

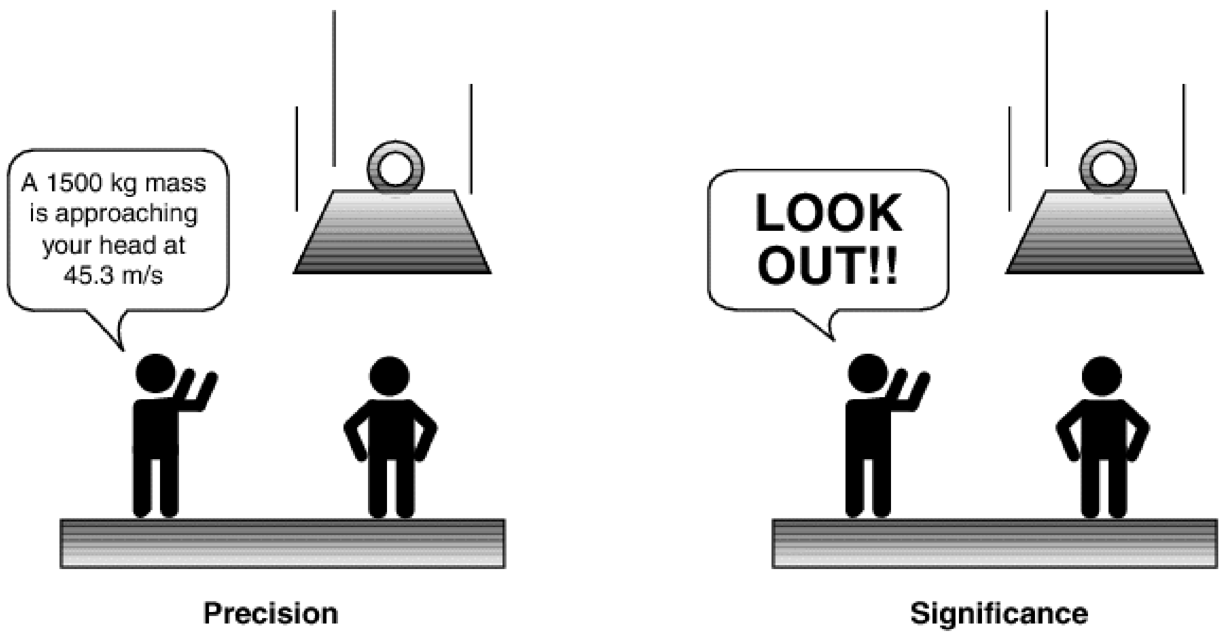


“To be, or not to be, that is the question.”

- **Fuzzy logic**
  - An approach to computing based on "degrees of truth“

Intro

How much is it important?!



Intro

- Can a computer understand this?

Most of the students could not produce a good and clean code in Python

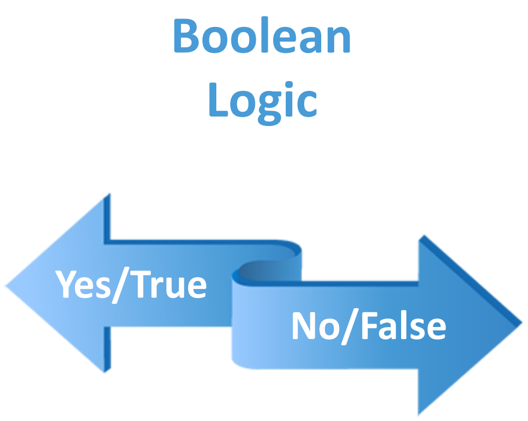
- Fuzzy Logic (FL)
  - Our life is full of probabilities, imprecise structures, and vague notions
  - A method of reasoning that resembles human reasoning
  - Imitates the way of decision making in humans
    - With all intermediate possibilities between YES and NO

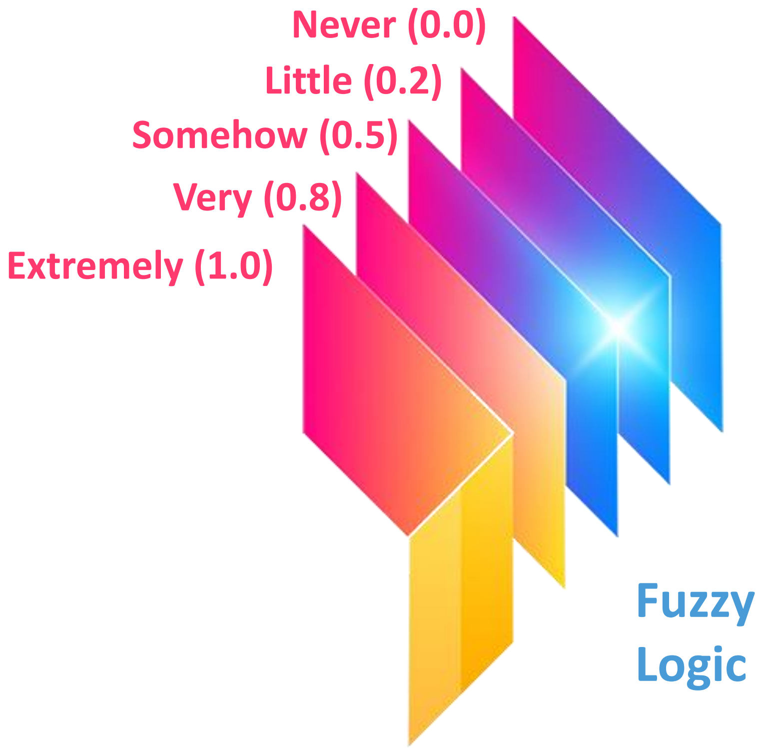
Intro

- Fuzzy Logic (FL)
  - Works on the levels of possibilities of input to achieve the definite output

Intro

Is John a polite person?





Fuzzy Sets

Classical Sets

- A collection of distinct objects, digits, people, etc.
  - For instance: Computer Engineering students
- Can be represented as a list or a rule
- Each individual entity in a set is called a member or an element of the set

$A = \{3,7,9,16,21\}$

$B = \{9,18,31\}$

$C = \{x \mid x \in \mathbb{R}, x < 10\}$

- Crisp set
  - An element is either a member of the set or not
    - John is a member of football team, but not tennis team
  - Membership operators
    - **John** $\in$ **FootballTeam**, but **John** $\notin$ **TennisTeam**

Fuzzy Sets

Classical Sets

Representation	Operation
$A \subseteq B$	Subset (B contains the members of A and other elements)
$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$	Equity (A and B have the same elements)
$A \neq B$	Inequity (A and B do not have the same elements)
$A \subset B \Leftrightarrow A \subseteq B \text{ and } A \neq B$	Proper Subset (B contains the members of A, but A and B are not equal)
$\mathcal{P}(A)$	Power set (the set of all subsets of A, including the empty set and A itself)
$\lvert \mathcal{P}(A) \rvert = 2^{\lvert A \rvert}$	Cardinality (the number of elements contained)

Fuzzy Sets

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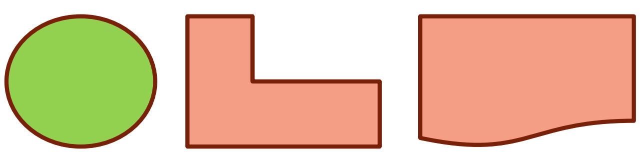
Classical Sets

Representation	Operation
$B - A = \{x \mid x \in B, x \notin A\}$	Complement (the complement of A with respect to B)
U	Universal Set
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$	Union (the set of all elements that are in either A or B)
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$	Intersection (the set of all elements that are in both A and B)
$A \cap B = \phi$	Isolated sets (A and B do not have a shared elements)
$\pi(A)$	Partition (a grouping of its elements into non-empty subsets)

Fuzzy Sets

Classical Sets

Representation	Operation
$Cover\ (A)$	Cover (a collection of sets whose union is U)
$A \times B = \{(a,b) \mid a \in A\ , b \in B\}$ $A \times B \neq B \times A$	Cartesian product
	Convex and Non-convex sets

Fuzzy Sets

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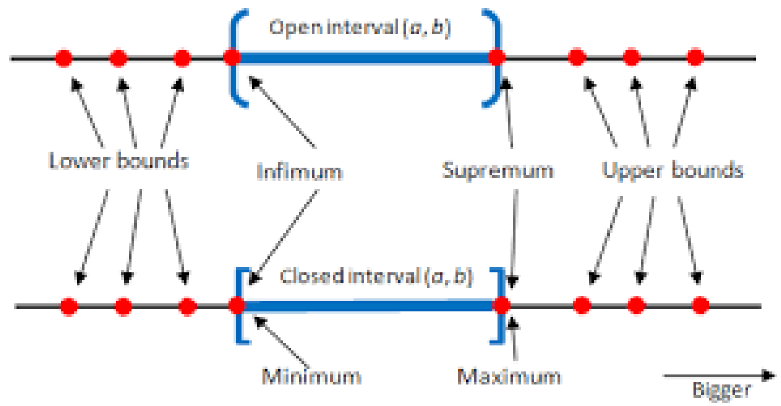
Classical Sets

Consider a set *M* of real numbers;

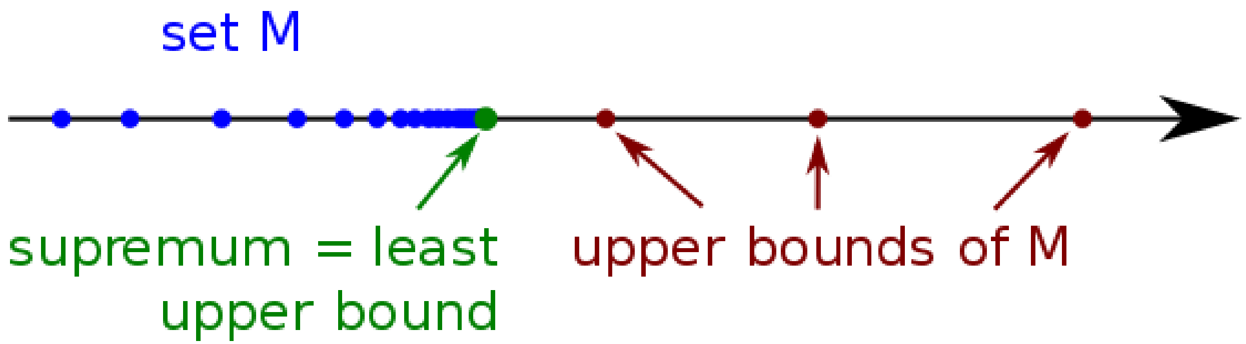


- **Bounded from above**
  - There exists some real number *k* such that k ≥ s for all *m* in *M*
  - **Upper Bound**: the number *k* with respect to *M*
- Likewise: bounded from below and lower bound
- A set of real numbers is bounded if it is contained in a finite interval

Fuzzy Sets

Classical Sets

- **Supremum**: the least upper bound (LUB)
- **Infimum**: the greatest lower bound (GLB)

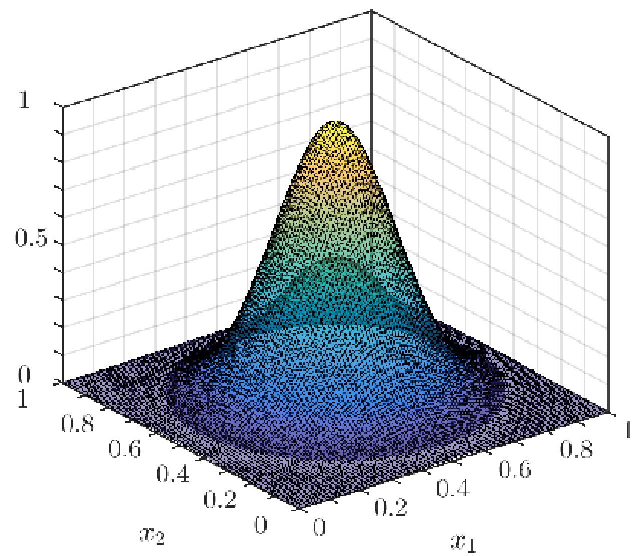


Fuzzy Sets

In contrast, in a Fuzzy Set:

- There is at least ONE fuzzy element
- An object may belong to this set with varying membership degrees
  - Range [0,1]
    - Zero means lack of membership
    - One means full membership
  - Might be defined as a function
    - Called Membership Function

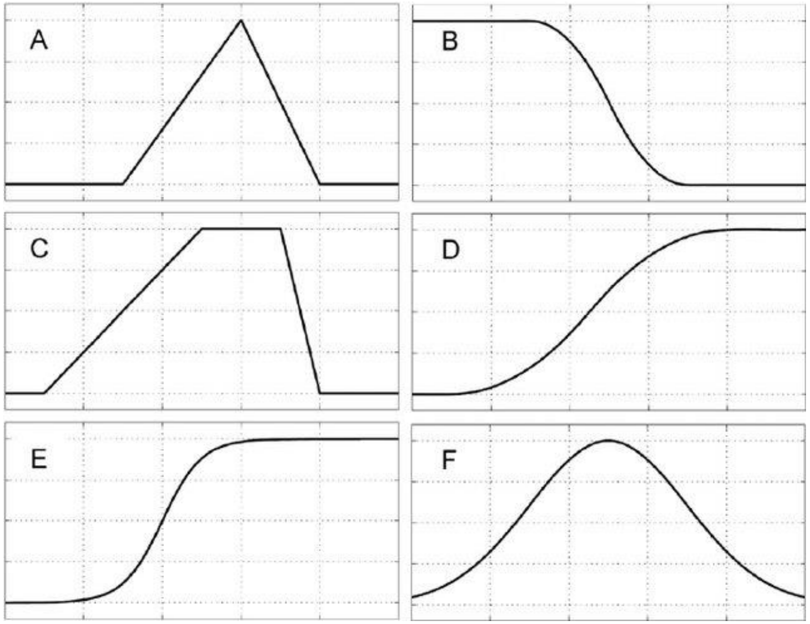
$$\mu_A : X \to [0,1]$$



Fuzzy Membership Functions

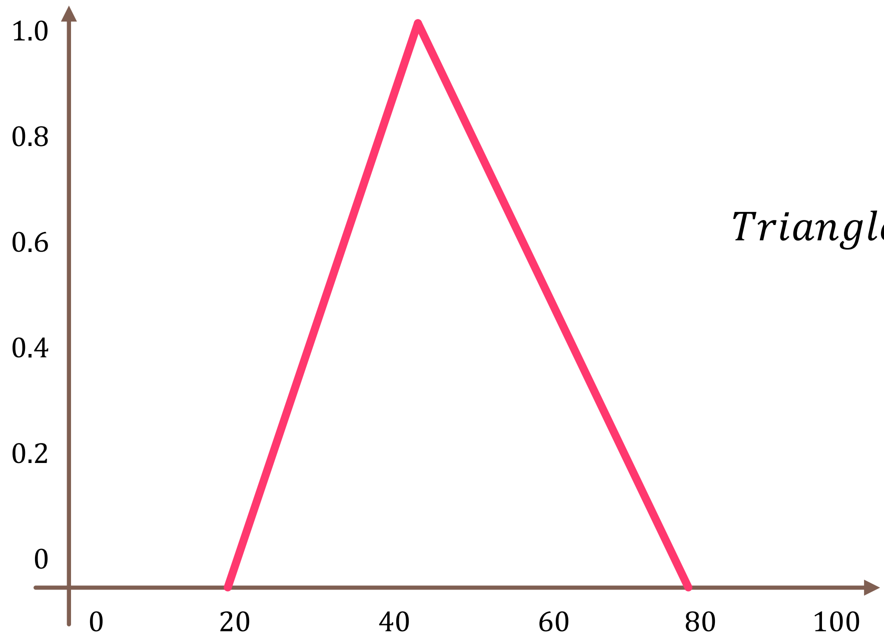
Some of the most common MFs are:

A. Triangle
B. Z-Shaped
C. Trapezoid
D. S-Shaped
E. Sigmoid
F. Gaussian



Fuzzy Membership Functions

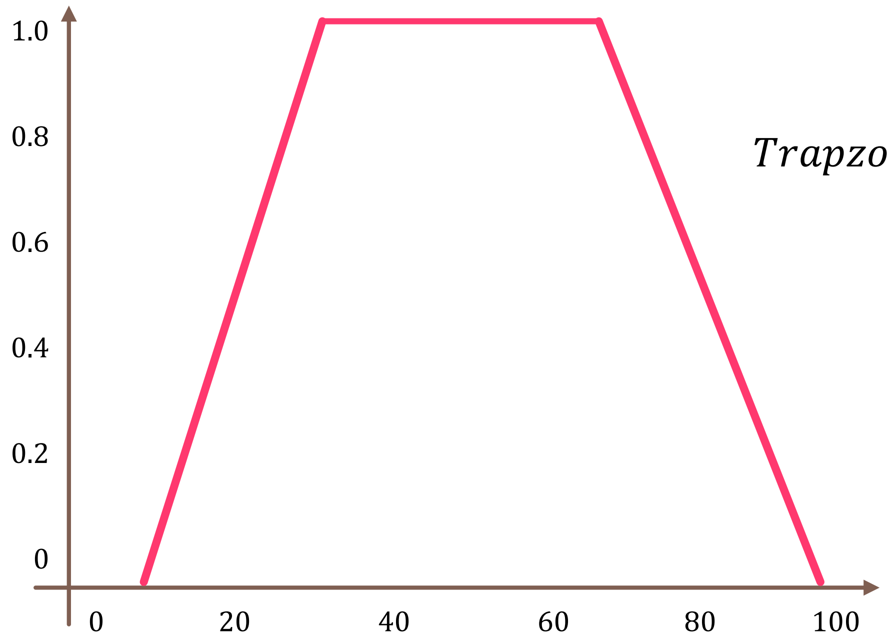
Some of the most common MFs are:



$$Triangle(a,b,c) = \begin{cases} 0, & x \le a. \\ \dfrac{x-a}{b-a}, & a \le x \le b. \\ \dfrac{c-x}{c-b}, & b \le x \le c. \\ 0, & c \le x. \end{cases}$$

Fuzzy Membership Functions

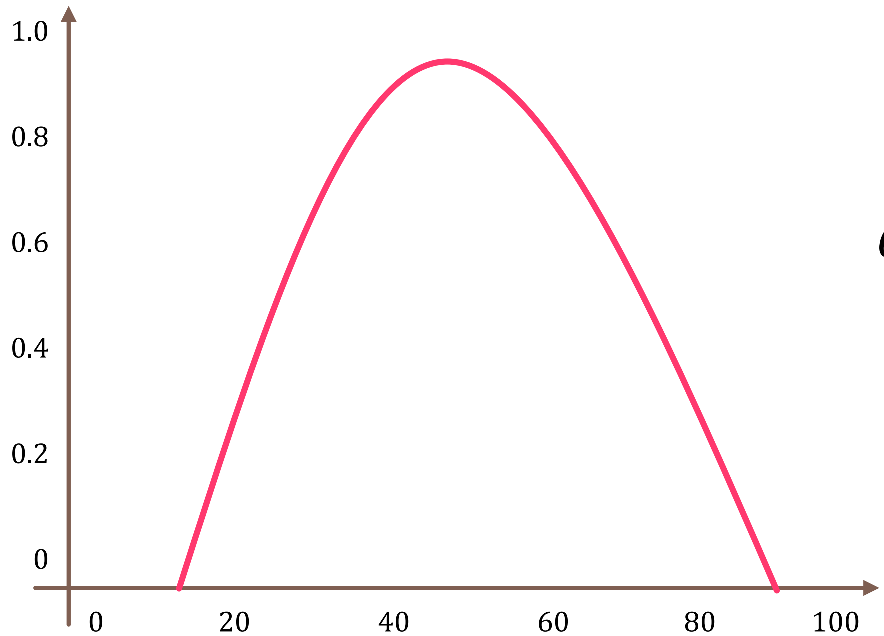
Some of the most common MFs are:



$$Trapzoid(a,b,c) = \begin{cases} 0, & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & b \le x \le c. \\ \frac{d-x}{d-c}, & c \le x \le d. \\ 0, & d \le x. \end{cases}$$

Fuzzy Membership Functions

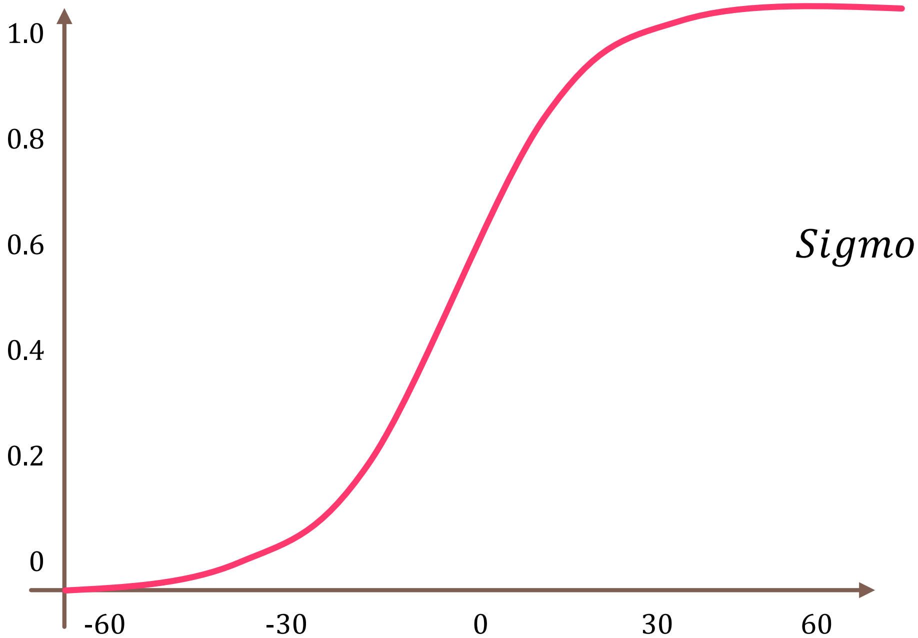
Some of the most common MFs are:



$$Gaussian(x,c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Fuzzy Membership Functions

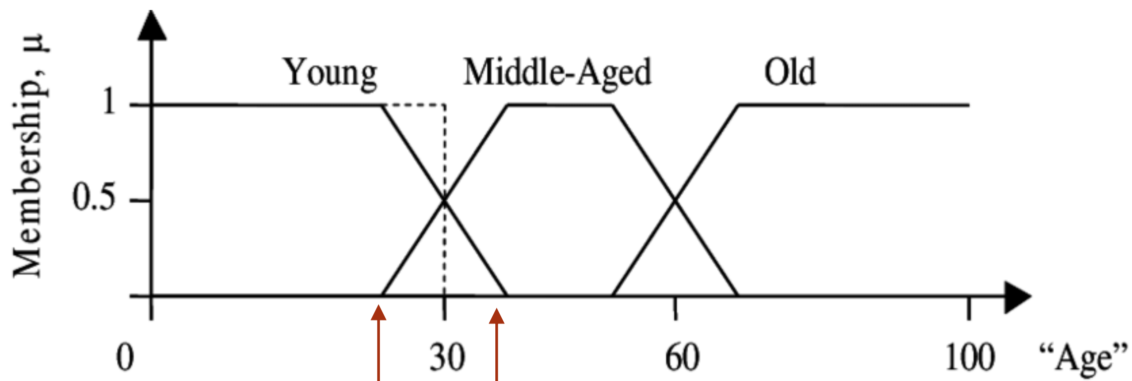
Some of the most common MFs are:



$$Sigmoid(x, a, c) = \frac{1}{1 + \exp[-a(x - c)]}$$

Fuzzy Sets in Practice

Sample: Age classification

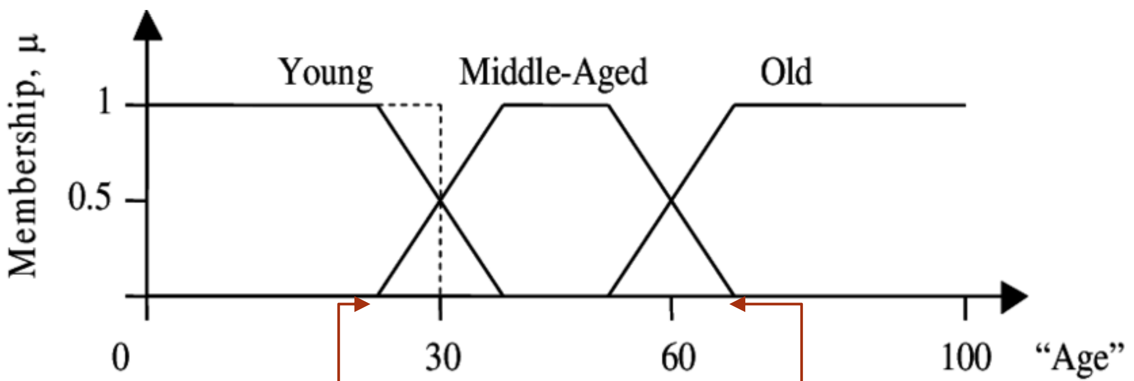


$$Young = \begin{cases} 1 & x \leq 20 \\ \dfrac{40 - x}{20} & 20 < x < 40 \\ 0 & x \geq 40 \end{cases}$$

$$Old = \begin{cases} 0 & x \leq 50 \\ \dfrac{x - 50}{20} & 50 < x < 70 \\ 1 & x \geq 70 \end{cases}$$

Fuzzy Sets in Practice

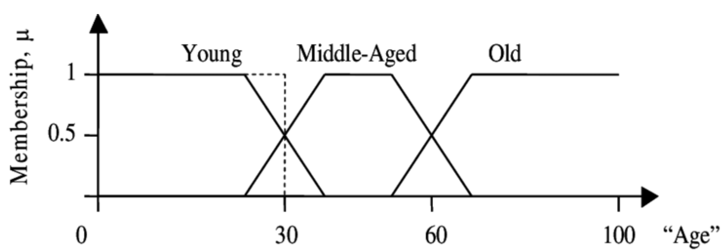
Sample: Age classification



$$M.A. = \begin{cases} 0 & x \leq 20 \quad OR \quad x \geq 70 \\ \dfrac{x-20}{20} & 20 < x < 40 \\ \dfrac{70-x}{20} & 50 < x < 70 \\ 1 & 40 \leq x \leq 50 \end{cases}$$

Fuzzy Sets in Practice

Sample: Age classification



$$Young = \begin{cases} 1 & x \le 20 \\ \dfrac{35 - x}{15} & 20 < x < 35 \\ 0 & x \ge 35 \end{cases}$$

$$M.A. = \begin{cases} 0 & x \le 20 \quad OR \quad x \ge 70 \\ \dfrac{x - 20}{15} & 20 < x < 35 \\ \dfrac{60 - x}{15} & 45 < x < 70 \\ 1 & 35 \le x \le 45 \end{cases}$$

$$Old = \begin{cases} 0 & x \le 45 \\ \dfrac{x - 45}{15} & 45 < x < 70 \\ 1 & x \ge 70 \end{cases}$$

Samples:

- x=23 ➔ 0.8 young and 0.2 M.A
- x=34 ➔ 0.07 young and 0.93 M.A
- x=40 ➔ 0 young and 1 M.A
- x=71 ➔ 1 Old

Sum = 1

Fuzzy Sets in Practice

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- Fuzzy set $\tilde{A}$ in the Universal set U is in fact a set of ordered pairs, each element with a membership degree $\mu_{\tilde{A}}(y)$

$$\tilde{A} = \{(y, \mu_{\tilde{A}}(y)) \,|y \in U\}$$

- Where $\mu_{\tilde{A}}(y)$ is a real number between 0 and 1

$$\mu_{\tilde{A}}(y) \in [0, 1] .$$

- **Sample** - successful students set:

{(Alex, 0.9), (John, 0.2), (Jack, 0.5)}

Fuzzy Sets in Practice

- In a discrete universal space :

$$\widetilde{A} = \left\{ \frac{\mu_{\widetilde{A}}(y_1)}{y_1} + \frac{\mu_{\widetilde{A}}(y_2)}{y_2} + \frac{\mu_{\widetilde{A}}(y_3)}{y_3} + \dots \right\}$$

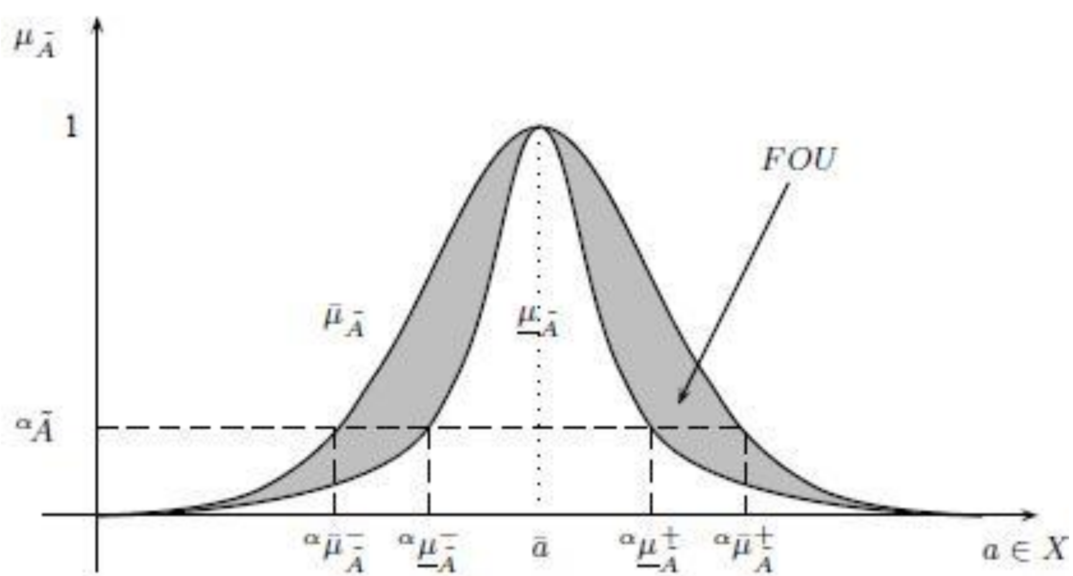
$$= \left\{ \sum_{i=1}^{n} \frac{\mu_{\widetilde{A}}(y_i)}{y_i} \right\}$$

- In a indiscrete universal space:

$$\widetilde{A} = \left\{ \int \frac{\mu_{\widetilde{A}}(y)}{y} \right\}$$

Fuzzy Sets in Practice

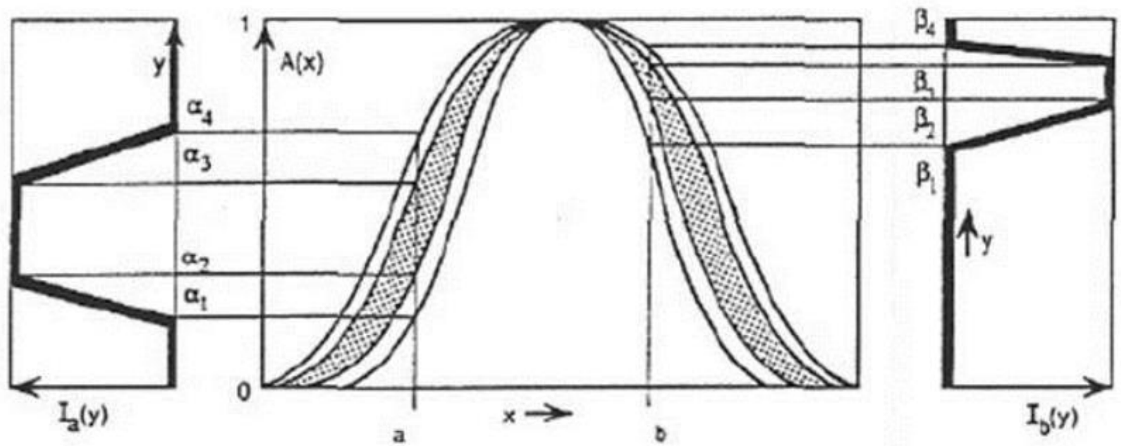
- In some cases, fuzzy sets are **interval-valued**:
  - Membership function provides a **range of values** instead of an individual value (Footprint of Uncertainty)



Fuzzy Sets in Practice

- In some cases, fuzzy sets are **Type-2**:
  - Membership function provides a **function** instead of an individual value

$$A: X \rightarrow F([0,1])$$



Fuzzy Alpha-Cut

- Alpha-cut
  - A list of all elements of Fuzzy set A with membership grades greater than or equal to a given **Alpha (α)**
- Strong Alpha-cut
  - A list of all elements of Fuzzy set A with membership grades greater than a given **Alpha (α)**

- **Note that** $\alpha \in [0,1]$

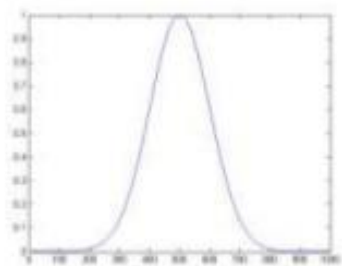
Fuzzy Alpha-Cut

Alpha Cut

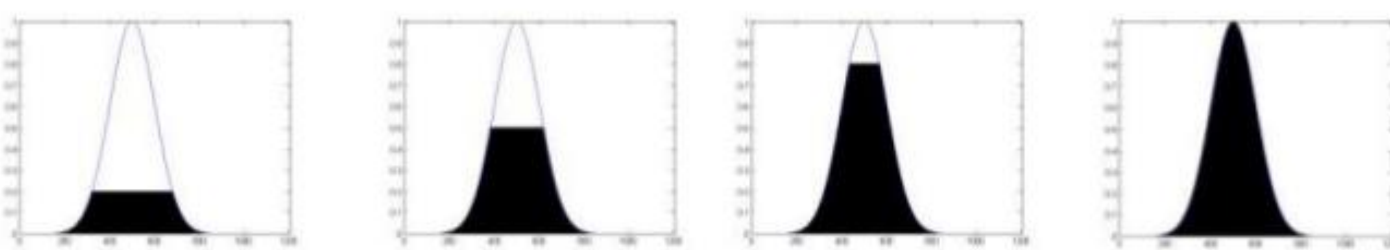
$$A_{\alpha} = \; x \in X \,\middle|\, \mu_A \;\; x \;\; \geq \alpha$$

Strong Alpha Cut

$$A_{\alpha'} = \; x \in X \,\middle|\, \mu_A \;\; x \;\; > \alpha$$

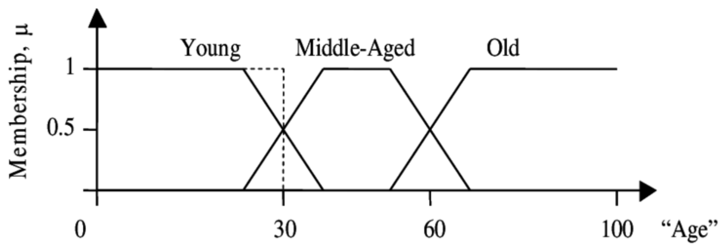


$\alpha = 0$



$\alpha = 0.2$ $\alpha = 0.5$ $\alpha = 0.8$ $\alpha = 1$

Fuzzy Alpha-Cut



$$Young = \begin{cases} 1 & x \le 20 \\ \dfrac{35 - x}{15} & 20 < x < 35 \\ 0 & x \ge 35 \end{cases}$$

$$M.A. = \begin{cases} 0 & x \le 20 \quad OR \quad x \ge 70 \\ \dfrac{x - 20}{15} & 20 < x < 35 \\ \dfrac{60 - x}{15} & 45 < x < 70 \\ 1 & 35 \le x \le 45 \end{cases}$$

$$Old = \begin{cases} 0 & x \le 45 \\ \dfrac{x - 45}{15} & 45 < x < 70 \\ 1 & x \ge 70 \end{cases}$$

$$Young_{\alpha=0} = X$$

$$M.A._{\alpha=0} = X$$

$$Old_{\alpha=0} = X$$

Fuzzy Alpha-Cut

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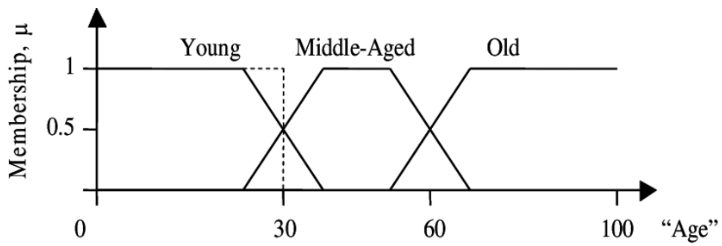


$$Young_{\alpha^+=1} = \emptyset$$

$$M.A._{\alpha^+=1} = \emptyset$$

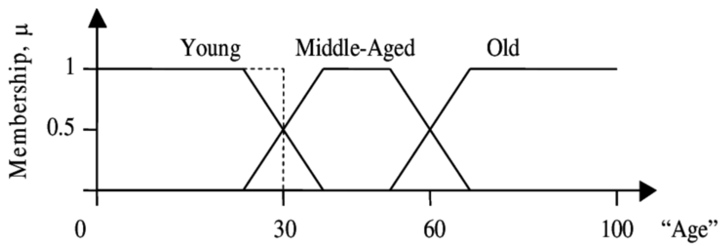
$$Old_{\alpha^+=1} = \emptyset$$

$$Young = \begin{cases} 1 & x \leq 20 \\ \dfrac{35-x}{15} & 20 < x < 35 \\ 0 & x \geq 35 \end{cases}$$

$$M.A. = \begin{cases} 0 & x \leq 20 \quad OR \quad x \geq 70 \\ \dfrac{x-20}{15} & 20 < x < 35 \\ \dfrac{60-x}{15} & 45 < x < 70 \\ 1 & 35 \leq x \leq 45 \end{cases}$$

$$Old = \begin{cases} 0 & x \leq 45 \\ \dfrac{x-45}{15} & 45 < x < 70 \\ 1 & x \geq 70 \end{cases}$$

Fuzzy Alpha-Cut



$$Young_\alpha = [0, 35 - 15\alpha]$$

$$M.A._\alpha = [20 + 15\alpha, 60 - 15\alpha]$$

$$Old_\alpha = [45 + 15\alpha, 100]$$

$$Young = \begin{cases} 1 & x \le 20 \\ \dfrac{35 - x}{15} & 20 < x < 35 \\ 0 & x \ge 35 \end{cases}$$

$$M.A. = \begin{cases} 0 & x \le 20 \quad OR \quad x \ge 70 \\ \dfrac{x - 20}{15} & 20 < x < 35 \\ \dfrac{60 - x}{15} & 45 < x < 70 \\ 1 & 35 \le x \le 45 \end{cases}$$

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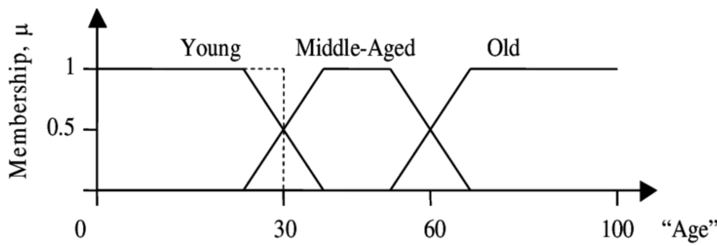
Fuzzy Alpha-Cut



$$Young = \begin{cases} 1 & x \le 20 \\ \dfrac{35-x}{15} & 20 < x < 35 \\ 0 & x \ge 35 \end{cases}$$

$$M.A. = \begin{cases} 0 & x \le 20 \;\; OR \;\; x \ge 70 \\ \dfrac{x-20}{15} & 20 < x < 35 \\ \dfrac{60-x}{15} & 45 < x < 70 \\ 1 & 35 \le x \le 45 \end{cases}$$

$$Old = \begin{cases} 0 & x \le 45 \\ \dfrac{x-45}{15} & 45 < x < 70 \\ 1 & x \ge 70 \end{cases}$$

$$Young_{\alpha^+} = (0, 35 - 15\alpha)$$

$$M.A._{\alpha^+} = (20 + 15\alpha, 60 - 15\alpha)$$

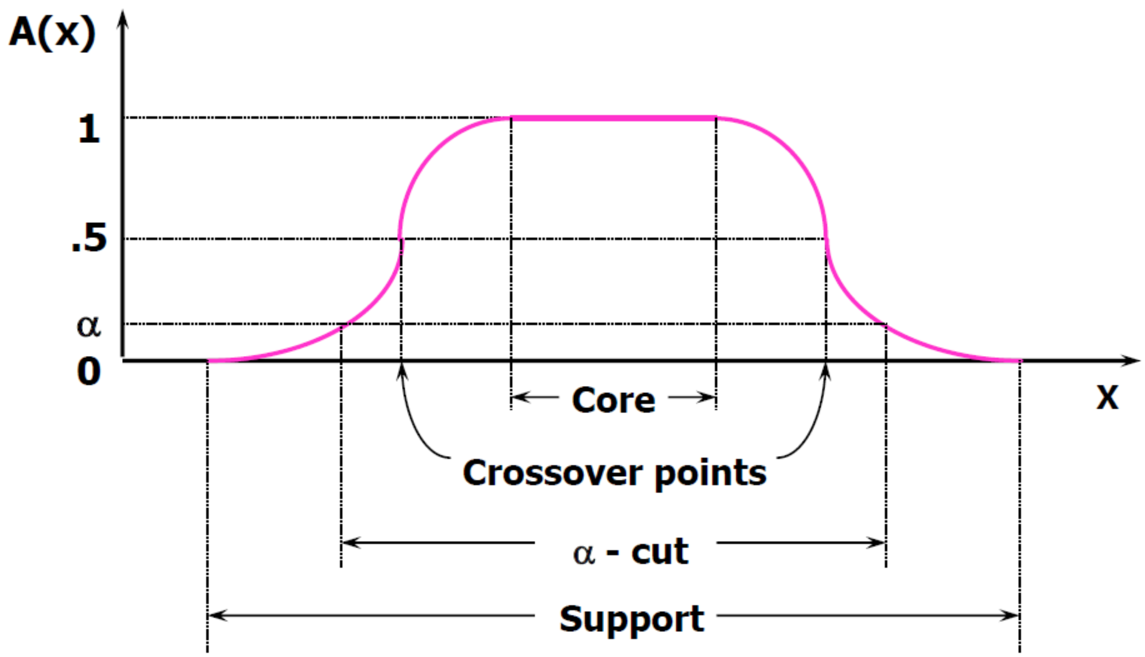
$$Old_{\alpha^+} = (45 + 15\alpha, 100)$$

Fuzzy Alpha-Cut

$$Support(A) = A_{\alpha^+=0}$$

$$Core(A) = A_{\alpha=1}$$

Support and core sets



Fuzzy Alpha-Cut

Height of a Fuzzy set

- The maximum membership degree in a fuzzy set
- Actually, Supremum for Alpha where $A_\alpha = \emptyset$
- If Height(A)=1, the set is **Normal**
- If Height(A)<1, the set is **Sub-normal**

Fuzzy Alpha-Cut

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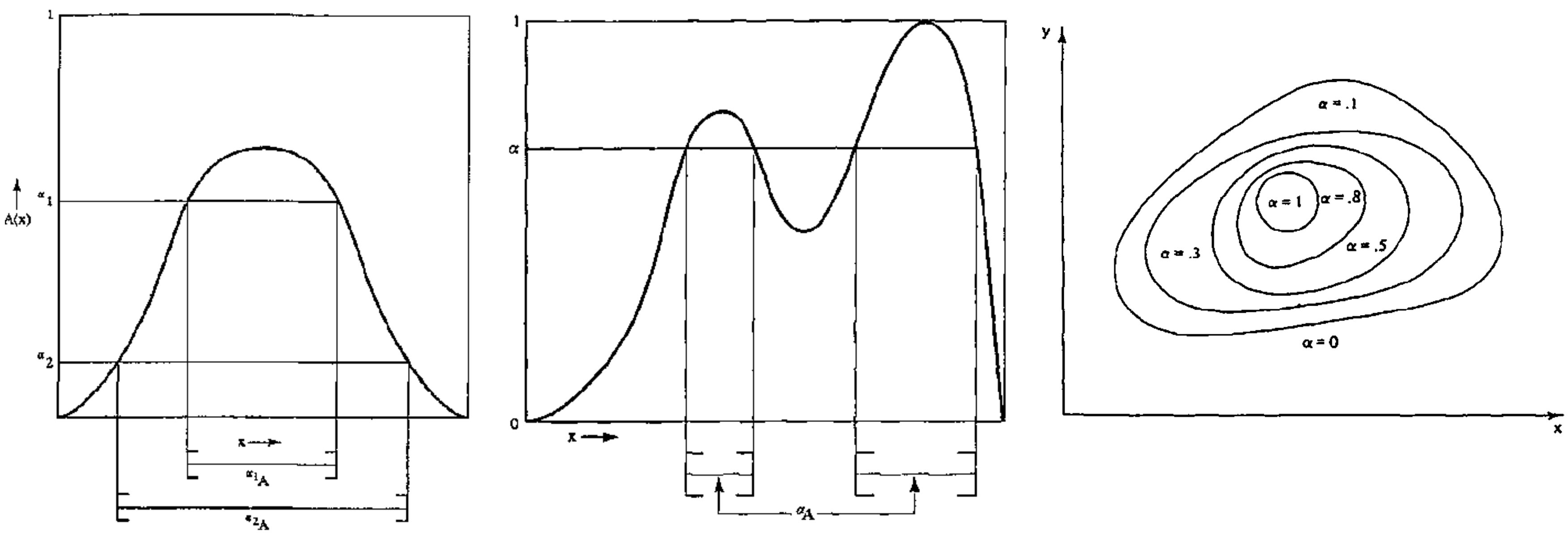
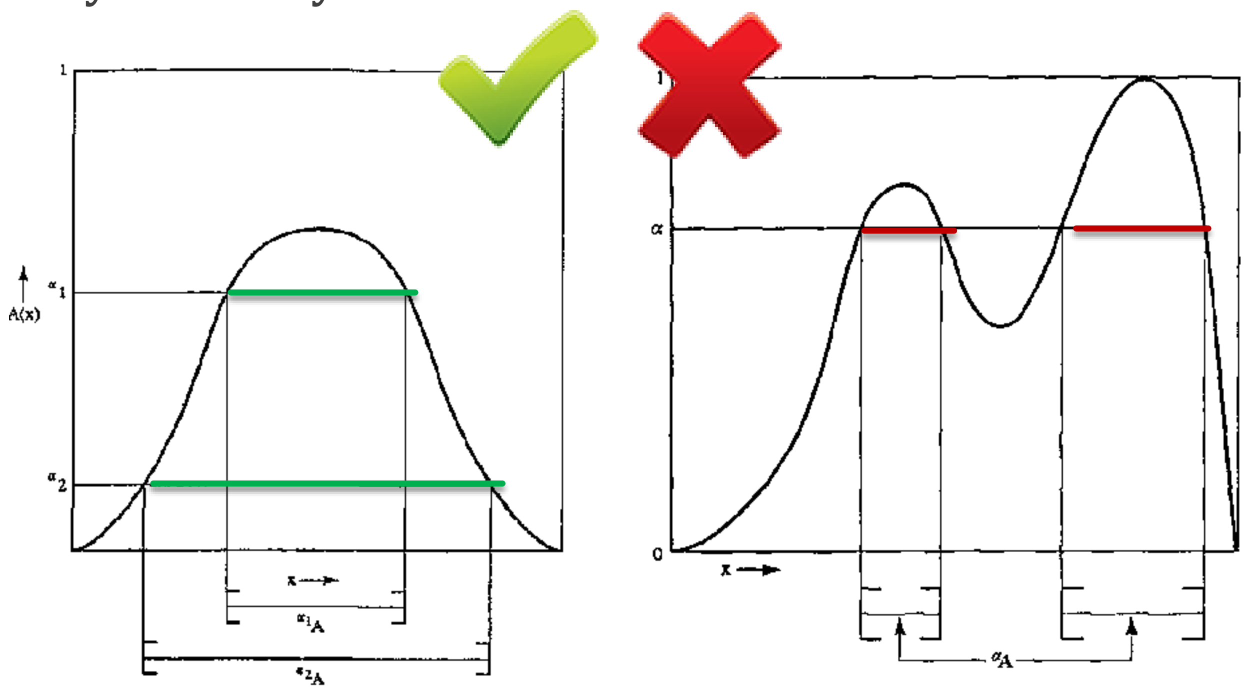
Height of a Fuzzy set



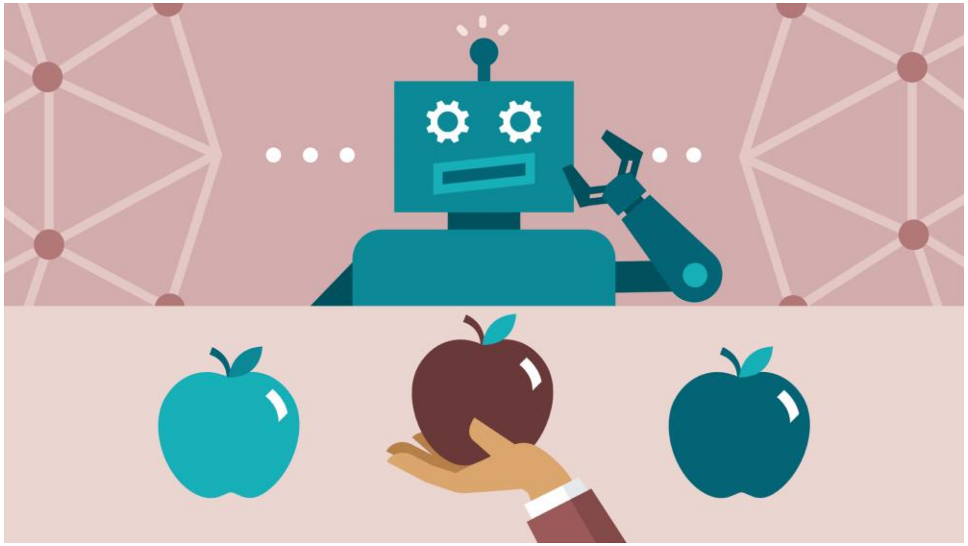
Fuzzy Alpha-Cut

Convexity of a Fuzzy set



What's Next?

- Fuzzy Operators and Relations



Questions?