

Agenda

- ▶ Classical logic
- ▶ Multi-valued logic
- ▶ Fuzzy logic
- ▶ Linguistic hedges
- ▶ Fuzzy inference



Classical Logic

- ▶ **“Liar Paradox”** in philosophy and logic
 - ▶ David: “I said in my alarm, Every man is a liar!”
 - ▶ If true, then David also is lying because he is a man!
 - ▶ If he too is lying, the statement consequently is not true!

Find the true statement and get a free beer!

The statement in
the right box is
true.

The statement in
the left box is
false.



Classical Logic

► Important concepts

Title	Description
Logic	Concluding based on some statements/propositions
Proposition	A statement than can be true or false
Propositional Logic	The logic based on a combination of several propositions
Logic Variable	The variable introducing a proposition
Logic Function	The function operating on one/several LV(s)

Classical Logic

Primitive

- ▶ Performs a logical operation
- ▶ Samples: OR, NOT, AND, Implication, etc.
- ▶ **Functional completeness:** a set of operators which can be used to express all possible truth tables by combining its members

Propositional Formula

- ▶ A type of syntactic formula which is well formed and has a truth value

$$(A \text{ and } B) \text{ or } (C \text{ and } D) \rightarrow E$$

Classical Logic

Tautology

- ▶ A formula or assertion that is true in every possible interpretation

Contradiction

- ▶ A formula or assertion that is false in every possible interpretation

a	b	$(a \rightarrow b)$	$(a \wedge (a \rightarrow b))$	$(a \wedge (a \rightarrow b)) \rightarrow b$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Tautology

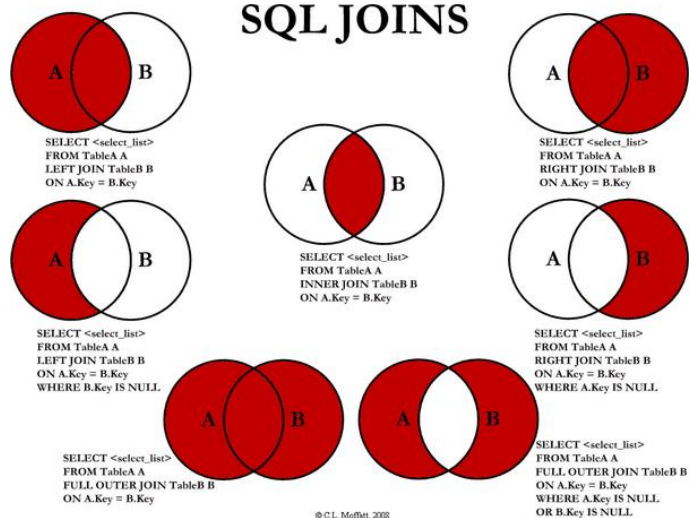
Classical Logic

Boolean Algebra

Law/Theorem	Law of Addition	Law of Multiplication
Identity Law	$x + 0 = x$	$x \cdot 1 = x$
Complement Law	$x + x' = 1$	$x \cdot x' = 0$
Idempotent Law	$x + x = x$	$x \cdot x = x$
Dominant Law	$x + 1 = 1$	$x \cdot 0 = 0$
Involution Law	$(x')' = x$	
Commutative Law	$x + y = y + x$	$x \cdot y = y \cdot x$
Associative Law	$x + (y + z) = (x + y) + z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
Distributive Law	$x \cdot (y + z) = x \cdot y + x \cdot z$	$x + y \cdot z = (x + y) \cdot (x + z)$
Demorgan's Law	$(x + y)' = x' \cdot y'$	$(x \cdot y)' = x' + y'$
Absorption Law	$x + (x \cdot y) = x$	$x \cdot (x + y) = x$

Classical Logic

Set Theory

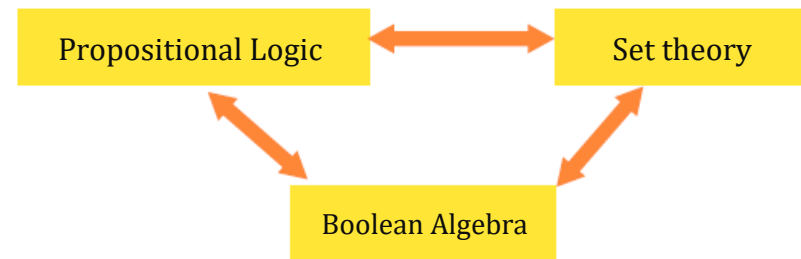


<i>Identity</i>	<i>Name</i>
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cap C) = (A \cup B) \cap C$ $A \cap (B \cup C) = (A \cap B) \cup C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Classical Logic

Isomorphism

- ▶ If there are equivalent concepts in different systems



Operator	Propositional Logic	Boolean Algebra	Set Theory
Union	\cup	$+$	\vee
Intersection	\cap	\cdot	\wedge
Empty set	\emptyset	\circ	\circ
Subsequence	\subseteq	\leq	\rightarrow

Classical Logic

► Propositions, Subjects, and Predicates

Iran is in the south-west of Asia

Subject

Predicate

The predicate

- Must contain a verb
- Applies on the subject
- Is a characteristic/set indicator function

Subject is the person or thing performing the action

Predicate indicates the action performed by the subject

Usually a noun, pronoun or noun phrase

Contains the verb, objects, and other elements

Usually precedes the predicate

Usually comes after the subject

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Classical Logic

- ▶ Predicate - other properties

- ▶ Extendable for more than one variable

X is a student of the class.

$X = \{ \text{Mohammadi, Rezaee, Alizadeh} \}$

- ▶ Extendable using existential and universal quantifiers

$\exists x P(x) = \bigvee P(x)$ *"there exists", "there is at least one", or "for some"*

$\forall x P(x) = \bigwedge P(x)$ *"given any" or "for all"*

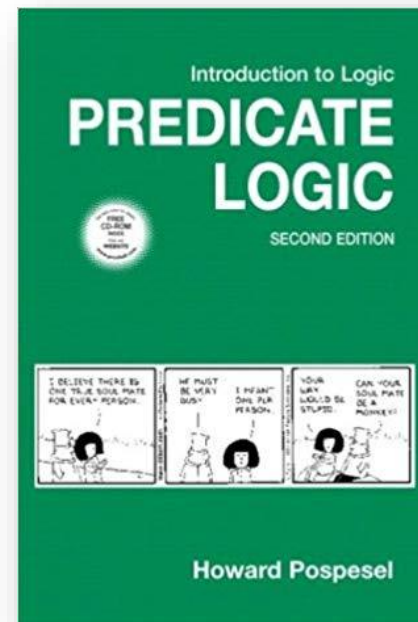


Classical Logic

Predicate Logic

Covers more concepts, including:

- ▶ Constants, like “Ali” or 100
- ▶ Variables, like x and y
- ▶ Predicates
- ▶ Universal and existential quantifiers
- ▶ Functions to describe objects



Classical Logic

Inference

$$(a \rightarrow b) \equiv (\neg a \vee b)$$

Modus Ponens

$$\begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array}$$

If it is Tuesday, I will go to the gym
Today is Tuesday
So, I will go to the gym

Modus Tollens

$$\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \neg P \end{array}$$

If it is Tuesday, I will go to the gym
Today I didn't go to the gym
So, it is not Tuesday

Chain Rule

$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array}$$

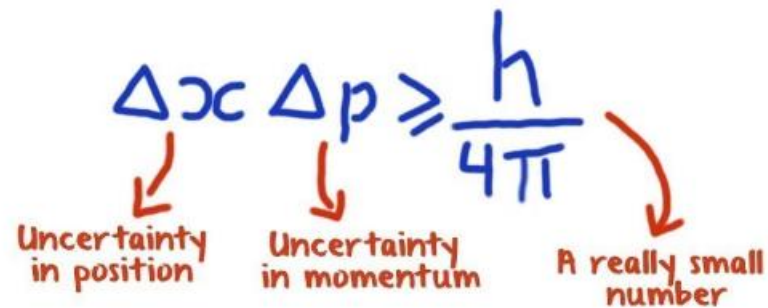
If it is Tuesday, I will go to the gym
If I go to the gym, I will see Joe
So, If it is Tuesday, I will see Joe

Multi-valued Logic

- ▶ Classical Logic has two outcomes: true or false
- ▶ Based on the Heisenberg's uncertainty principle
- ▶ Three-valued logic:
 - ▶ Including: True, False, and Unsure

$$\Delta x \Delta p \geq \frac{h}{4\pi}$$

Uncertainty in position Uncertainty in momentum A really small number



Multi-valued Logic

Three-valued logic

(F, false; U, unknown; T, true)

A	$\neg A$
F	T
U	U
T	F

$A \wedge B$		B		
		F	U	T
A	F	F	F	F
	U	F	U	U
	T	F	U	T

$A \vee B$		B		
		F	U	T
A	F	F	U	T
	U	U	U	T
	T	T	T	T

Multi-valued Logic

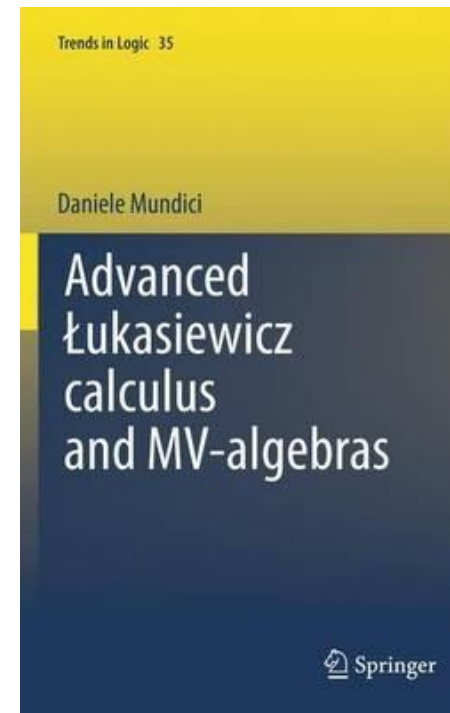
Multi-valued logic by Lukasiewicz

- ▶ Members have a degree of truth

$$T_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\}$$

Lukasiewicz Isomorphism

- ▶ Multi-valued logic \rightarrow Fuzzy set
- ▶ Binary-valued element \rightarrow Crisp set

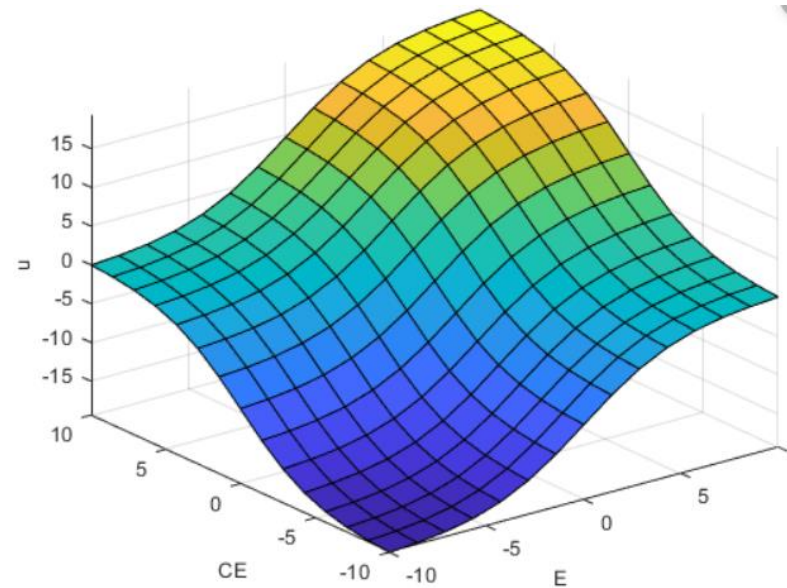


Fuzzy Logic

- ▶ Here, instead of true/false, the membership degrees are used

Different types of propositions

- ▶ Unconditional and unqualified
- ▶ Unconditional and qualified
- ▶ Conditional and unqualified
- ▶ Conditional and qualified



Fuzzy Logic

Unconditional and unqualified

The **temperature** is very **high** today

$P: \mu$ is F

$$T(P) = F(v)$$

P = a fuzzy preposition

μ = a variable of a universal set V

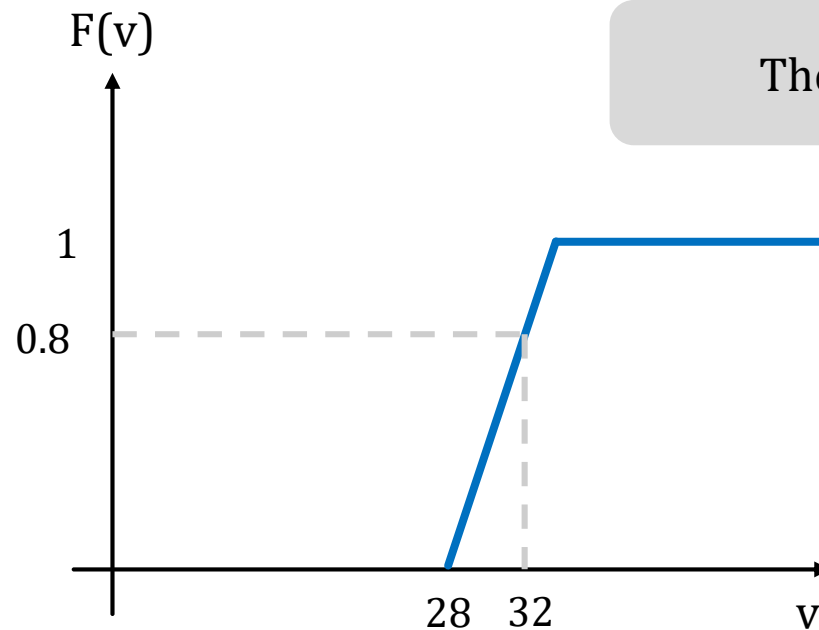
F = a fuzzy set on V

$T(P)$ = the degree of truth of P

$F(v)$ = the membership grade of F

Fuzzy Logic

Unconditional and unqualified



The temperature is very high today

$$v = 32$$

$$F(32) = 0.8$$

$$T(P) = 0.8$$

Fuzzy Logic

Unconditional and qualified

The **possibility** of rain is **20%**, which is **ignorable!**

$P: \mu$ is F is S

$$T(P) = S(F(v))$$

P = a fuzzy preposition

μ = a variable of a universal set V

F = a fuzzy set on V

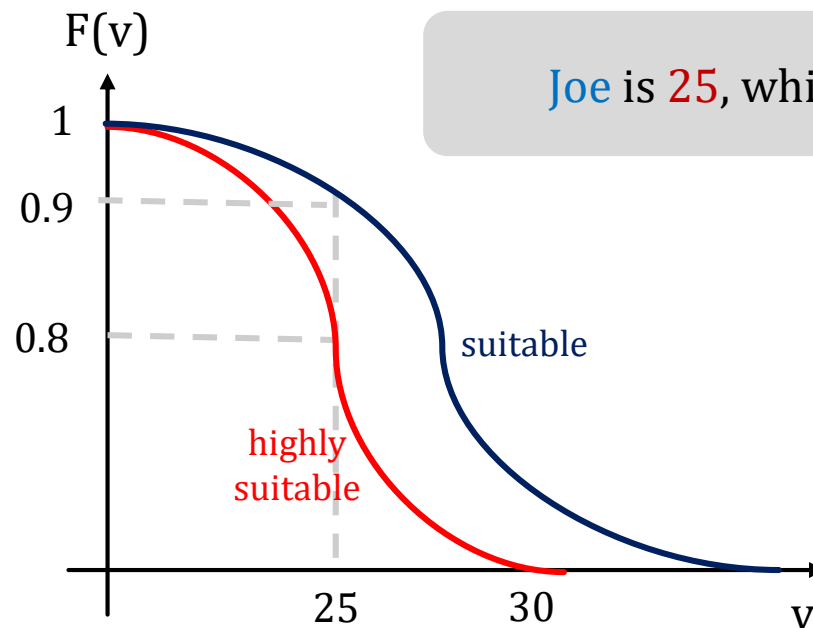
S = a fuzzy truth qualifier

$T(P)$ = the degree of truth of P

$F(v)$ = the membership grade of F

Fuzzy Logic

Unconditional and qualified



Joe is 25, which makes him suitable for the job!

$$v = 25$$

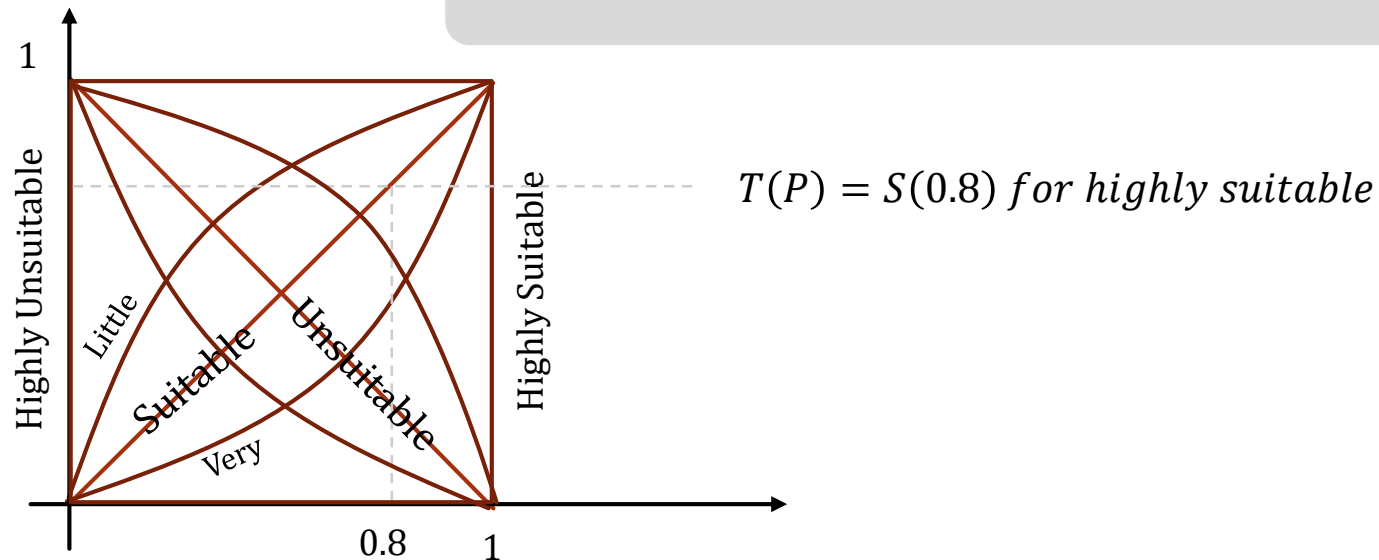
$$F(25) = 0.8 \text{ in highly suitable}$$

$$F(25) = 0.9 \text{ in suitable}$$

Fuzzy Logic

Unconditional and qualified

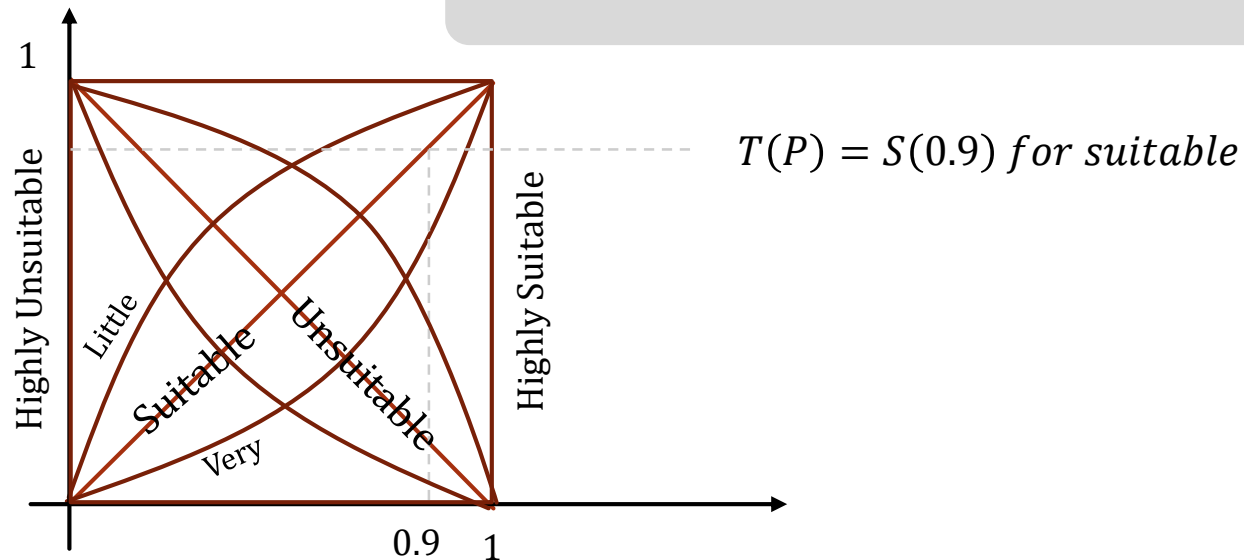
Joe is 20, which makes him **suitable** for the job!



Fuzzy Logic

Unconditional and qualified

Joe is 20, which makes him **suitable** for the job!



Fuzzy Logic

Conditional and unqualified

If **the temperature** is **low**, then **the speed** of the fan is **high**!

P: if X is A, then Y is B

P = a fuzzy preposition

X&Y = variables of universal sets

A&B = fuzzy sets on X and Y

$$R(x, y) = \gamma[A(x), B(y)]$$

*R(x, y) = a fuzzy set on X * Y*

Fuzzy Logic

Conditional and unqualified

- ▶ Consider these two Fuzzy numbers with a relation as γ :

$$A = \frac{0.1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3} \quad B = \frac{0.5}{y_1} + \frac{1}{y_2} \quad \gamma[A(x), B(y)] = \min(1, 1 - a + b)$$

- ▶ Thus:

$$R(x, y) = \gamma[A(x), B(y)] = \frac{1}{x_1 y_1} + \frac{1}{x_1 y_2} + \frac{0.7}{x_2 y_1} + \frac{1}{x_2 y_2} + \frac{0.5}{x_3 y_1} + \frac{1}{x_3 y_2}$$

$$\min(1, 1 - (0.1) + 0.5) \text{ for } x_1 \text{ and } y_1 = 1$$

$$\min(1, 1 - (0.8) + 0.5) \text{ for } x_1 \text{ and } y_1 = 1$$

Fuzzy Logic

Conditional and qualified

- ▶ A combination of the previous methods ...

If **the temperature** is **low**, then “**the fan speed** becomes **25rpm**” is a **good choice!**

If **Joe** is **young**, then “**his age** is **20**” is a **true sentence!**

Linguistic Hedges

How to extend the propositions?

- ▶ Using some simple tools, called LH
 - ▶ Allows speakers and writers to signal caution, or probability
 - ▶ Samples: Very, more, less, often, rather, etc.

Joe likes to do the job himself

Joe **often** likes to do the job himself

Joe **often** likes to do **most of** the jobs himself

Linguistic Hedges

Important notes:

- ▶ LHs can be applied on Fuzzy sets, not Crisp sets
 - ▶ Sample: The line is too straight !!?
- ▶ Use “power” for some words like “often” or “very”
- ▶ Use “Square root” for some words like “sometimes” or “little”

The **temperature** is **high** today

$$v = 32 \rightarrow F(32) = 0.8 \rightarrow T(P) = 0.8$$

→	$T(P) = 0.8^2 = 0.64$	Very high
→	$T(P) = \sqrt{0.8} = 0.89$	A little high

Fuzzy Inference

- Based on the Classical sets inference rules

$$(a \rightarrow b) \equiv (\neg a \vee b)$$

Modus Ponens

$$\begin{array}{c} P \rightarrow Q \\ P \\ \hline Q \end{array}$$

If it is Tuesday, I will go to the gym
Today is Tuesday
So, I will go to the gym

Modus Tollens

$$\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \neg P \end{array}$$

If it is Tuesday, I will go to the gym
Today I didn't go to the gym
So, it is not Tuesday

Chain Rule

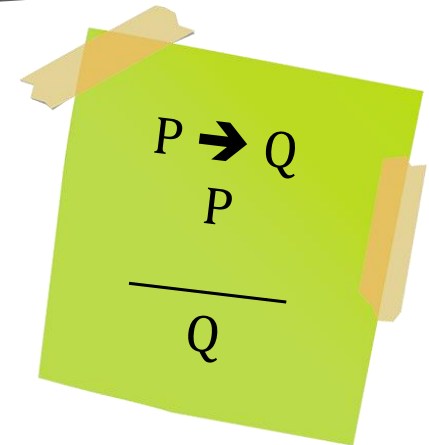
$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline P \rightarrow R \end{array}$$

If it is Tuesday, I will go to the gym
If I go to the gym, I will see Joe
So, If it is Tuesday, I will see Joe

Fuzzy Inference

Modus Ponens (Classical)

- ▶ **Rule:** if P is true, then Q is concluded
- ▶ **Phases:** observation and conclusion
- ▶ No new knowledge is produced (exact inference)

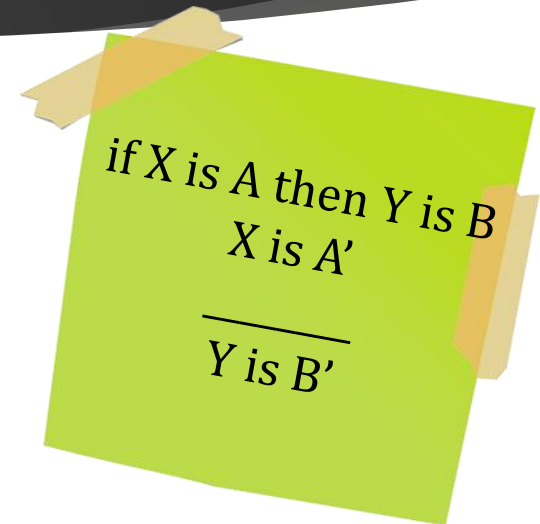


If Joe passes the exam, he will get the job
Observation: Joe passed the exam
Conclusion: He will get the job

Fuzzy Inference

Modus Ponens (Fuzzy)

- ▶ **Rule:** if P is true, then Q is concluded
- ▶ **Phases:** observation and conclusion
- ▶ New knowledge is produced
 - ▶ The outcome is not definite (approximate inference)



If Joe study hard, he will get a good score

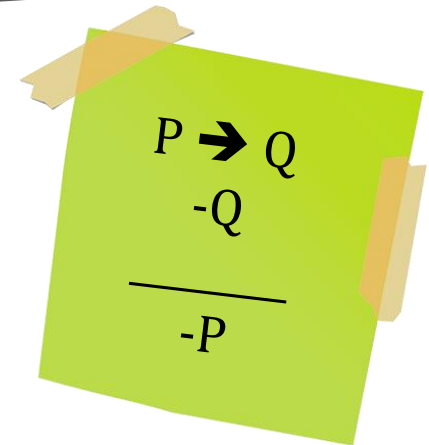
Observation: Joe studies **very hard**

Conclusion: He will get a **great score**

Fuzzy Inference

Modus Tollens (Classical)

- ▶ **Rule:** if P is true, then Q is concluded
- ▶ **Phases:** observation and conclusion
- ▶ No new knowledge is produced (exact inference)

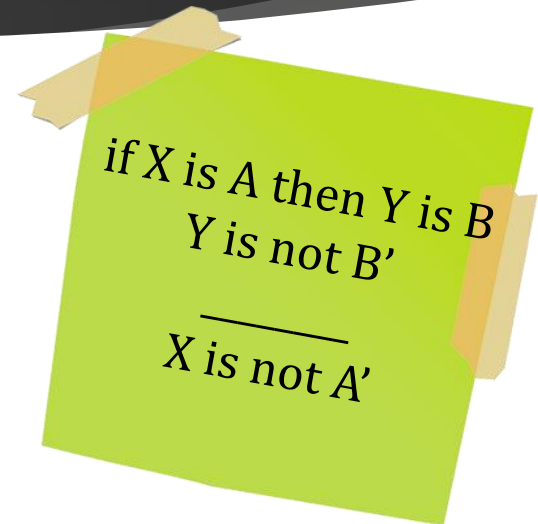


If Joe passes the exam, he will get the job
Observation: Joe didn't get a good job
Conclusion: He didn't pass the exam

Fuzzy Inference

Modus Tollens (Fuzzy)

- ▶ **Rule:** if P is true, then Q is concluded
- ▶ **Phases:** observation and conclusion
- ▶ New knowledge is produced
 - ▶ The outcome is not definite (approximate inference)
- ▶ To calculate A' we need:
 - ▶ The relation between X and Y
 - ▶ The value of B'



$$A'(y) = \sup \min[B'(y), R(x, y)]$$

$$R(x, y): \gamma(A(x), B(y))$$

Fuzzy Inference

Modus Tollens (Fuzzy)

▶ For instance, if: $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$ $B = \frac{1}{y_1} + \frac{0.4}{y_2}$

▶ And R is defined as: $R = \frac{1}{x_1y_1} + \frac{0.9}{x_1y_2} + \frac{1}{x_2y_1} + \frac{0.4}{x_2y_2} + \frac{1}{x_3y_2} + \frac{0.8}{x_3y_2}$

▶ Let's assume: $B' = \frac{0.9}{y_1} + \frac{0.7}{y_2}$

$A'(y) = \sup \min[B'(y), R(x, y)]:$

$A'(x_1) = \sup \min[B'(y), R(x_1, y)] = \max [\min(0.9, 1), \min(0.7, 0.9)] = 0.9$

$A'(x_2) = \sup \min[B'(y), R(x_2, y)] = \max [\min(0.9, 1), \min(0.7, 0.4)] = 0.9$

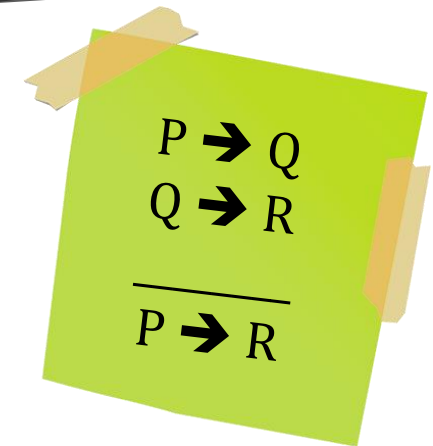
$A'(x_3) = \sup \min[B'(y), R(x_3, y)] = \max [\min(0.9, 1), \min(0.7, 0.8)] = 0.9$

$A'(y) = \frac{0.9}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3}$

Fuzzy Inference

Chain Rule (Classical)

- ▶ Several rules connected together
- ▶ Conclusion can be met using other statements



If Joe passes the exam, he will get the job

If Joe has a good job, he can buy a car

Conclusion: If Joe passes the exam, he can buy a car

Fuzzy Inference

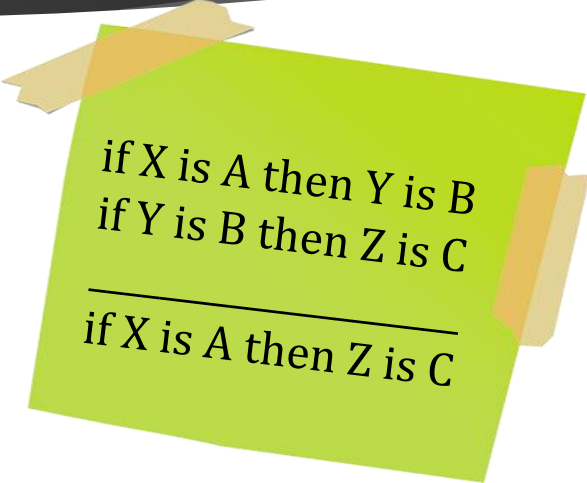
Chain Rule (Fuzzy)

- ▶ Several rules connected together
- ▶ There are three relations for all Fuzzy sets

$$R1(x, y): \gamma(A(x), B(y))$$

$$R2(y, z): \gamma(B(y), C(z))$$

$$R3(x, z): \gamma(A(x), C(z)) = \sup \min[R1(x, y), R2(x, y)]$$



if X is A then Y is B
if Y is B then Z is C

if X is A then Z is C

Fuzzy Inference

Chain Rule (Fuzzy)

► For instance, if: $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$ $B = \frac{1}{y_1} + \frac{0.4}{y_2}$ $C = \frac{0.2}{z_1} + \frac{1}{z_2}$

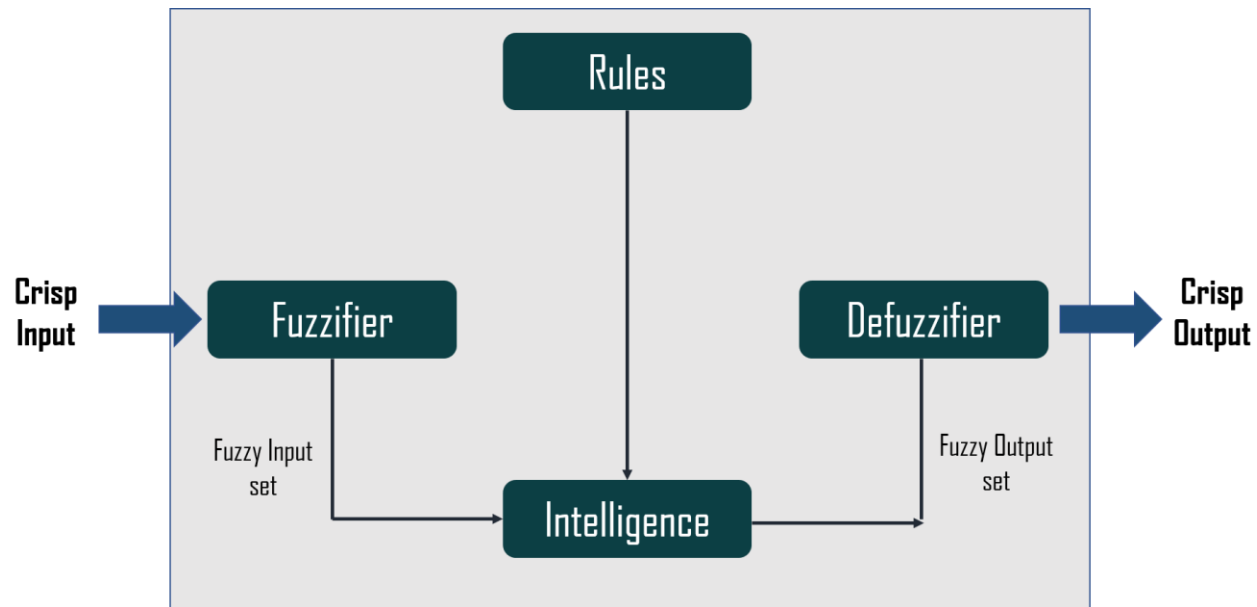
► And J defines the relations: $J(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$

► We will have:

$$\begin{array}{c}
 \begin{array}{ccc}
 & y_1 & y_2 \\
 R1(x, y) = & 1 & 0.4 \\
 & 1 & 0.4 \\
 & 1 & 0.4
 \end{array}
 \quad
 \begin{array}{ccc}
 & y_1 & y_2 \\
 R2(y, z) = & 0.2 & 1 \\
 & 0.2 & 1
 \end{array}
 \quad
 \begin{array}{ccc}
 & z_1 & z_2 \\
 R3(x, z) = & 0.2 & 1 \\
 & 0.2 & 1 \\
 & 0.2 & 1
 \end{array}
 \end{array}$$

What's Next?

► Fuzzy Controllers



Questions?

