CSCA67 Final Review Seminar

Held by AMACSS

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Induction

Question 1. Show that for all integers $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

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$i = 1 \quad \frac{1}{1(1+1)}$

Solution:

Let $P(n) : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$. We want to show $\forall n \geq 1, P(n)$.

Base Case: For $n = 1$: LHS $= \frac{1}{1(1+1)} = \frac{1}{2}$. RHS $= \frac{1}{1+1} = \frac{1}{2}$. Since LHS $=$ RHS, $P(1)$ is true.

Induction Step: We want to show $\forall k \geq 1, P(k) \rightarrow P(k+1)$.

Question 1. Show that for all integers $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(1) Take arbitrary k

(2) Suppose $k \geq 1$

(3) Suppose $P(k)$, i.e., $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ — Induction Hypothesis

(4) $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{1}{(k+1)((k+1)+1)}$ — Sigma Property

(5) $= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1}$ $P(k+1)$ — 3, IH

Question 1. Show that for all integers $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(5)	$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$	3, IH
(6)	$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$	5, algebra
(7)	$= \frac{k^2+2k+1}{(k+1)(k+2)}$	6, algebra
(8)	$= \frac{(k+1)^2}{(k+1)(k+2)}$	7, algebra
(9)	$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$	8, algebra
(10)	$P(k+1)$	9, def of P

Question 1. Show that for all integers $n \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

(11)	$P(k) \to P(k+1)$	3, 10, implication
(12)	$k \geq 1 \to (P(k) \to P(k+1))$	2, 11, implication
(13)	$\forall k \geq 1, P(k) \to P(k+1)$	1, 12, U.G.

$\therefore \forall n \geq 1, P(n)$ by Simple Induction.

Question 2. Prove that for every integer $n \geq 1$, $7^n - 2^n$ is a multiple of 5.

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Solution:

Let $P(n) : \exists m \in \mathbb{Z}, 7^n - 2^n = 5m$. We want to show $\forall n \geq 1, P(n)$.

Base Case: For $n = 1$: $7^1 - 2^1 = 5 = 5 * 1, 1 \in \mathbb{Z}, \exists m \in \mathbb{Z}, 7^1 - 2^1 = 5m$, by E.G. $P(1)$.

Induction Step: We want to show $\forall k \geq 1, P(k) \rightarrow P(k+1)$.

Question 2. Prove that for every integer $n \geq 1$, $7^n - 2^n$ is a multiple of 5.

(1)	Take arbitrary k	
(2)	Suppose $k \geq 1$	
(3)	Suppose $P(k)$	Induction Hypothesis
(4)	$\exists m \in \mathbb{Z}, 7^k - 2^k = 5m$	3, def of P, IH
(5)	$7^k - 2^k = 5a$ for some $a \in \mathbb{Z}$	4, E.I.
(6)	$7^k = 5a + 2^k$	5, algebra

Question 2. Prove that for every integer $n \geq 1$, $7^n - 2^n$ is a multiple of 5.

$7^k = 5a + 2^k$

(7)	$7^{k+1} - 2^{k+1} = 7 \cdot 7^k - 2 \cdot 2^k$	7, algebra
(8)	$= 7(5a + 2^k) - 2 \cdot 2^k$	6, def of 7^k
(9)	$= 35a + 7 \cdot 2^k - 2 \cdot 2^k$	8, algebra
(10)	$= 35a + 5 \cdot 2^k$	9, algebra
(11)	$= 5(7a + 2^k)$	10, algebra
(12)	Since $a \in \mathbb{Z}$, $k \in \mathbb{Z}$, and $k \geq 1$, let $b = 7a + 2^k$	
(13)	$\exists m \in \mathbb{Z}, 7^{k+1} - 2^{k+1} = 5m$	7, 12, E.G.

Question 2. Prove that for every integer $n \geq 1$, $7^n - 2^n$ is a multiple of 5.

(14)	$P(k+1)$	13, def of P
(15)	$P(k) \to P(k+1)$	3, 14, implication
(16)	$k \geq 1 \to (P(k) \to P(k+1))$	2, 15, implication
(17)	$\forall k \geq 1, P(k) \to P(k+1)$	1, 16, U.G.

$\therefore \forall n \geq 1, P(n)$ by Simple Induction.

Question 3. Consider the sequence of numbers $a_1, a_2, \ldots$ defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{aligned}$$

Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

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Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

Solution:

Let $P(n) : a_n < 2^n$. We want to show $\forall n \geq 1, P(n)$. Thus, since we are using strong induction, we show

$$\forall k \geq 1, (\forall i, 1 \leq i < k, P(i)) \to P(k)$$

Question 3. Consider the sequence of numbers $a_1, a_2, \ldots$ defined as follows:

$$a_1 = 1$$
$$a_2 = 3$$
$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3$$

Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

(1)	Take arbitrary k	
(2)	Suppose $k \geq 1$	
(3)	Suppose $\forall i, 1 \leq i < k, P(i)$	Induction Hypothesis
(4)	Suppose $k \geq 3$	Case #1
(5)	$k > k-1 > k-2 \geq 1$	
(6)	$P(k-1)$	3, 5, U.M.P., IH
(7)	$P(k-2)$	3, 5, U.M.P., IH

Question 3. Consider the sequence of numbers $a_1, a_2, \ldots$ defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{aligned}$$

Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

(8)	$a_{k-1} < 2^{k-1}$	6, def of P
(9)	$a_{k-2} < 2^{k-2}$	7, def of P
(10)	$a_k = a_{k-1} + a_{k-2}$	4, def of a_k
	$< 2^{k-1} + 2^{k-2}$	by 8, 9
	$= 2 \cdot 2^{k-2} + 1 \cdot 2^{k-2}$	algebra
	$= 3 \cdot 2^{k-2} < 4 \cdot 2^{k-2} = 2^2 \cdot 2^{k-2} = 2^k$	algebra
(11)	$P(k)$	10, def of P

$a_k < 2^k$

Question 3. Consider the sequence of numbers $a_1, a_2, \ldots$ defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{aligned}$$

Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

(12) Suppose $k = 1$: $a_1 = 1 < 2 = 2^1$, so $P(1)$ Case #2

(13) Suppose $k = 2$: $a_2 = 3 < 4 = 2^2$, so $P(2)$ Case #3

Question 3. Consider the sequence of numbers $a_1, a_2, \ldots$ defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_n &= a_{n-1} + a_{n-2} \quad \text{for } n \geq 3 \end{aligned}$$

Show: $\forall n \in \mathbb{N}, n \geq 1 \implies a_n < 2^n$.

(14)	$P(k)$	4, 11, 12, 13, cases
(15)	$(\forall i, 1 \leq i < k, P(i)) \to P(k)$	3, 20, implication
(16)	$k \geq 1 \to (\forall i, 1 \leq i < k, P(i)) \to P(k)$	2, 21, implication
(17)	$\forall k \geq 1, (\forall i, 1 \leq i < k, P(i)) \to P(k)$	1, 22, U.G.

$\therefore \forall n \geq 1, P(n)$ by Strong Induction.

Quantifiers

What's wrong with this inference?

x's birthday

(1) $\exists x, P(x)$ → it's someone's birthday

(2) Let c be arbitrary. ⇒ pick a random person

(3) $P(c)$ (E.I.), (1) ⇒ it's that person's birthday

(1) $\exists x, (P(x) \to Q(x))$

(2) $P(c)$

(3) $\therefore Q(c)$

What's wrong with this inference?

I have brown eyes

(1) Suppose $P(x)$

(2) Suppose $Q(y)$ my mother blue

(3) ...

(4) $Q(z)$ $\rightarrow Q(y) \rightarrow Q(z)$ father brown

(5) ~~$P(x) \rightarrow Q(z)$~~ (1, 4, implication) ✗

$P(x) \rightarrow (Q(y) \rightarrow Q(z))$ ✓

If I have brown eyes, if my mother has blue eyes, then my father has brown eyes

Practice!



1. Consider the statements $\exists x, (P(x) \to \exists y, Q(y))$ and $(\exists x, P(x)) \to \exists y, Q(y))$. Are the two logically equivalent? If yes, prove it; if no, disprove with a counterexample.

NOT equiv.

Solution

$(\exists x, P(x)) \Rightarrow \exists y, Q(y)$

$\exists x, (P(x) \Rightarrow \exists y, Q(y))$

→ want to be false

so that one statement can be false

The statements are not equivalent. We disprove with a counterexample:

1. Consider a universe with two elements $\{a, b\}$.
   (Thought process: we want to be as simple as possible!)
2. Let $Q(a) = F, Q(b) = F$.

   Let $P(a) = F, P(b) = T$.

$F \Rightarrow F \iff T$

$T \Rightarrow F \iff F$

$\exists x, P(x) \iff P(a) \vee P(b)$

Solution

$$P(a) = F, P(b) = T.$$
$$Q(a) = F, Q(b) = F.$$

$$\begin{aligned}
&(\exists x, P(x)) \to (\exists y, Q(y)) \\
\equiv\ &(P(a) \vee P(b)) \to (\exists y, Q(y)) \\
\equiv\ &(P(a) \vee P(b)) \to (Q(a) \vee Q(b)) \\
\equiv\ &(T \vee T) \to (F \vee F) \\
\equiv\ &T \to F \\
\equiv\ &F
\end{aligned}$$

$$\begin{aligned}
&\exists x, (P(x) \to \exists y, Q(y)) \\
\equiv\ &(P(a) \to (\exists y, Q(y)) \vee (P(b) \to (\exists y, Q(y)) \\
\equiv\ &(P(a) \to (Q(a) \vee Q(b))) \vee (P(b) \to (Q(a) \vee Q(b))) \\
\equiv\ &(F \to (F \vee F)) \vee (T \to (F \vee F)) \\
\equiv\ &(F \to F) \vee (T \to F) \\
\equiv\ &T \vee F \\
\equiv\ &T
\end{aligned}$$

As this truth assignment makes one statement true and the other false, they are not logically equivalent

Practice!

2. Prove that $\forall x, (P(x) \to \exists y, Q(y))$ implies $(\exists x, P(x)) \to \exists y, Q(y))$.

Solution

Prove that $\forall x, (P(x) \to \exists y, Q(y))$ implies $(\exists x, P(x)) \to \exists y, Q(y))$.

(1) Suppose $\forall x, (P(x) \to \exists y, Q(y))$

(2) Suppose $\exists x, P(x)$.

(3) $P(d)$ (2, E.I.)

(4) $P(d) \to \exists y, Q(y)$ (1, U.I.)

(5) $\exists y, Q(y)$ (3, 4, M.P.)

M.P.

(6) $\exists x, P(x) \to \exists y, Q(y)$ (2, 5, implication)

(7) $\forall x, (P(x) \to \exists y, Q(y)) \to (\exists x, P(x)) \to \exists y, Q(y)$ (1, 6, implication)

as wanted.

Counting

AMACSS A67 Final Exam Review Seminar - Fall 2025

Counting & Probability

1 Counting

Problem 1. In how many 6 digit numbers with distinct digits, the first, second and third digits (from the left) are odd?

Solution

The first 3 digits are a 3 digit permutation of the digits 1, 3, 5, 7, and 9. So, there are $P(5,3)$ possibilities. Then we need a 3 digit permutation of unused digits from 0-9, which gives us $P(7,3)$.
$\therefore P(5,3) \cdot P(7,3)$ by product rule

Problem 2. In how many permutations of the letters of the word "universal", there is exactly one letter between each of the pairs of letters u & i and a & e?

Solution

Two cases:

1. The letters between u & i and e & a both consonants (two separate "uxi/ixu" and "exa/axe" blocks)
   - Choose one consonant from n, v, r, s, l for between u & i: 5
   - Choose one from the remaining consonants for e & a: 4
   - Order of letters within each pair: $2! \cdot 2! = 2 \cdot 2$
   - Permute remaining 3 consonants + 2 blocks: 5!

   $\therefore 5 \cdot 4 \cdot 2 \cdot 2 \cdot 5! = 80 \cdot 5!$ by product rule
2. One of e & a between u & i and one of u & i between e & a (the blocks joined together)
   - Choose which pair comes first: 2
   - Order of letters within each pair: $2! \cdot 2! = 2 \cdot 2$
   - Permute remaining 5 consonants + 1 vowel block: 6!

   $\therefore 2 \cdot 2 \cdot 2 \cdot 6! = 8 \cdot 6!$ by product rule

Therefore, by the sum rule, the answer is $80 \cdot 5! + 8 \cdot 6!$.

Problem 3. In how many ways can we put 10 people in 8 distinct rooms such that no room is empty?

Solution

We have two cases:

1. **One room has 3 people and the rest have 1 each**
   We choose those 3 people using combinations. Then we line up everyone (including the 3 people set) in some fixed order (e.g. sorted by a specific attribute), and only permute the rooms to match the people.

   $\therefore C(10,3) \cdot 8!$ by product rule

2. **Two rooms have 2 people and the rest have 1 each**
   We choose 2 out of 10 people for the first 2-people room, and then 2 out of the remaining 8 for the second one. Notice that we're counting each possibility twice: e.g. $(\{a,b\},\{c,d\})$ and $(\{c,d\},\{a,b\})$, show the same arrangement. So we need to divide by 2 to account for our overcounting.
   Then, we permute the 8 rooms, just like we did in case 1.

   $\therefore \frac{1}{2} \cdot C(10,2) \cdot C(8,2) \cdot 8!$ by product rule and division rule

Thus, by sum rule, the answer is $C(10,3) \cdot 8! + \frac{1}{2} \cdot C(10,2) \cdot C(8,2) \cdot 8!$

Problem 4. In how many ways can we put 8 people in 4 distinct rooms A, B, C, and D such that no room is empty and the number of people in rooms A and B are equal?

Solution

Let the number of people in rooms A, B, C, D be a, b, c, d respectively. We have $a = b$, $a, b, c, d \geq 1$, and $a + b + c + d = 8$, so $2a + c + d = 8$. Since $c, d \geq 1$, we have $c + d \geq 2$, so $2a \leq 6$, hence $a \leq 3$. Also $a \geq 1$, so $a \in \{1, 2, 3\}$.
For each possible a, we count the number of ways to assign the 8 distinct people to the rooms such that room A gets a people, room B gets a people, and rooms C and D each get at least one person. We can count by first choosing and assigning people to rooms A and B, then assigning the remaining people to rooms C and D with no empty rooms.

1. **Case $a = 1$:** We choose 2 people out of 8 to go into rooms A and B: $\binom{8}{2}$ ways. Then assign one of these two to room A and the other to room B: $\binom{2}{1}$ ways.

   After assigning 2 people to A and B, there are 6 people left. They must be assigned to rooms C and D such that neither room is empty. The total number of assignments of 6 distinct people to two distinct rooms is $2^6 = 64$. Subtract the 2 assignments where all 6 go to C or all go to D. So $64 - 2 = 62$ valid assignments to C and D.

$\therefore$ for $a = 1$, total ways: $\binom{8}{2} \cdot 2 \cdot 62$ by product rule.

2. **Case** $a = 2$**:** Choose 4 people out of 8 for rooms A and B together: $\binom{8}{4}$ ways. From these 4, choose 2 for room A: $\binom{4}{2}$ ways, and the remaining 2 go to room B.

   After assigning 4 people to A and B, there are 4 people left. Assign them to C and D with no empty rooms: total assignments $2^4 = 16$, minus 2 where all to C or all to D gives 14.

   $\therefore$ for $a = 2$, total: $\binom{8}{4} \cdot \binom{4}{2} \cdot 14$ by product rule.

3. **Case** $a = 3$**:** Choose 6 people out of 8 for rooms A and B: $\binom{8}{6}$ ways. From these 6, choose 3 for room A: $\binom{6}{3}$ ways, the rest to B.

   After assigning 6 people to A and B, there are 2 people left. Assign them to C and D with no empty rooms: each room must get exactly one person: $2! = 2$. Alternatively, $2^2 = 4$ total, minus 2 (both to C or both to D) gives 2.

   $\therefore$ for $a = 3$, total: $\binom{8}{6} \cdot \binom{6}{3} \cdot 2$ by product rule.

By the sum rule, the total number of ways is $(\binom{8}{2} \cdot 2 \cdot 62) + (\binom{8}{4} \cdot \binom{4}{2} \cdot 14) + (\binom{8}{6} \cdot \binom{6}{3} \cdot 2)$

Problem 5. In how many permutations of the letters of the word "message", the letter m is next to at least one e?

Solution

We count the total permutations of the word "message" and subtract those where m is not adjacent to any e.
The word "message" has 7 letters: m (1), e (2), s (2), a (1), g (1). Total permutations: $\frac{7!}{2!2!}$.
To count permutations where m is not adjacent to any e: first arrange the other letters (s, s, a, g) in $\frac{4!}{2!}$ ways. These create 5 gaps. Place m in one of these gaps: 5 ways. Then place the two identical e's into the remaining 4 gaps (not the gap containing m); the number of ways is the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 2$, which is $C(4 + 2 - 1, 2) = C(5, 2) = \frac{5!}{2!3!} = 10$.

Alternatively, we can consider 2 cases for placing e's in the gaps between the other letters (other than the gap containing m):

1. Both in one gap: $C(4, 1) = 4$
2. In separate gaps: $C(4, 2) = \frac{4!}{2!2!} = 6$

$\therefore$ By the sum rule, we have $4 + 6 = 10$ ways to place e's.
Note that the first approach might be more straightforward in more complicated problems (possibly with more restrictions, letters, or repetitions).

Thus the number of permutations where m is not adjacent to any e is $\frac{4!}{2!} \cdot 5 \cdot 10$. Therefore, the desired number is:

$$\frac{7!}{2!2!} - \frac{4!}{2!} \cdot 50.$$

Problem 6. How many 5 letter permutations of the letters of the word "management" are there?

Solution

The word "management" has letters: m (2), a (2), n (2), e (2), g (1), t (1). We need to form 5-letter permutations. We consider cases based on the multiplicities of the chosen letters.

1. **All 5 letters distinct:** Choose 5 types from the 6 available: $C(6,5)$. Then arrange them: 5!. So $C(6,5) \cdot 5!$.
2. **One letter appears twice, the other three distinct:** Choose the double letter from $\{m, a, n, e\}$: 4 ways. Choose 3 distinct types from the remaining 5: $C(5,3)$. Then arrange the 5 letters (one letter twice, three once): $\frac{5!}{2!}$. So $4 \cdot C(5,3) \cdot \frac{5!}{2!}$.
3. **Two letters appear twice, one letter appears once:** Choose the two double letters from $\{m, a, n, e\}$: $C(4,2)$. Choose the single letter from the remaining 4 types: $C(4,1)$. Then arrange the 5 letters (two letters twice, one once): $\frac{5!}{2!2!}$. So $C(4,2) \cdot C(4,1) \cdot \frac{5!}{2!2!}$.

By sum rule, the total number is:

$$C(6,5) \cdot 5! \; + \; 4 \cdot C(5,3) \cdot \frac{5!}{2!} \; + \; C(4,2) \cdot C(4,1) \cdot \frac{5!}{2!2!}.$$

Problem 7. Use combinatorics to answer the following:

(a) In how many ways can we give 40 identical candies to 6 kids?

(b) In how many ways each kid gets at least 3 candies?

(c) In how many ways exactly 4 kids get candies?

(d) In how many ways at most 4 kids get candies?

Solution

(a) The number of nonnegative integer solutions to $x_1 + \cdots + x_6 = 40$ is $C(40+6-1, 40) = C(45, 40)$.

(b) After giving each kid 3 candies, we distribute the remaining $40-18 = 22$ candies without restriction: $C(22+6-1, 22) = C(27, 22)$.

(c) Choose the 4 kids who get candies: $C(6, 4)$. Then distribute 40 candies to these 4 so that each gets at least 1: let $y_i \geq 1$ and $\sum y_i = 40$. The number of positive integer solutions is $C(40-1, 4-1) = C(39, 3)$. So total: $C(6, 4) \cdot C(39, 3)$.

(d) "At most 4 kids get candies" means exactly k kids get at least one candy, for $k = 1, 2, 3, 4$. For a fixed k, choose the k kids: $C(6, k)$, then distribute 40 candies to these k so that each gets at least 1: $C(40-1, k-1) = C(39, k-1)$. So total:

$$\sum_{k=1}^{4} C(6, k) \cdot C(39, k-1).$$

Problem 8. In how many 5 digit numbers, the product of all the digits is divisible by 6?

Solution

A 5-digit number has product divisible by 6 if it contains at least one even digit and at least one multiple of 3, or if it contains the digit 0 (making the product 0). We count the complement: numbers whose product is not divisible by 6. These are numbers that do not contain 0 and also have no even digit or no multiple of 3.
Total 5-digit numbers: $9 \cdot 10^4$.
Let A be the set of 5-digit numbers with no even digit (digits from $\{1, 3, 5, 7, 9\}$): $|A| = 5^5$. Let B be the set of 5-digit numbers with no multiple of 3 and no 0 (digits from $\{1, 2, 4, 5, 7, 8\}$): $|B| = 6^5$. The intersection $A \cap B$ consists of numbers with digits from $\{1, 5, 7\}$: $|A \cap B| = 3^5$.
Then the number of 5-digit numbers whose product is not divisible by 6 is $|A \cup B| = 5^5 + 6^5 - 3^5$.
Thus the desired number is:

$$9 \cdot 10^4 - (5^5 + 6^5 - 3^5).$$

Problem 9. In how many ways can 3 teachers and 8 students sit around a circular table so that no two teachers sit next to each other?

Solution

First, arrange the 8 students around the table. Since rotations are identical, the number of circular arrangements of 8 distinct students is $(8-1)! = 7!$.
After the students are seated, there are 8 gaps between them. Choose 3 of these gaps for the teachers: $C(8,3)$. Then assign the 3 distinct teachers to these gaps: $3!$.
By the product rule, the total number of ways is:

$$7! \cdot C(8,3) \cdot 3!.$$

2 Probability

Problem 10. A die is weighted so that the probability of rolling a number is proportional to that number. That is, $P(1) = k$, $P(2) = 2k$, ..., $P(6) = 6k$. Find the probability of rolling an even number.

Solution

First, find k using the fact that the probabilities sum to 1:

$$k + 2k + 3k + 4k + 5k + 6k = 21k = 1 \quad \Rightarrow \quad k = \frac{1}{21}.$$

The even outcomes are 2, 4, 6. Their probabilities:

$$P(2) = \frac{2}{21}, \quad P(4) = \frac{4}{21}, \quad P(6) = \frac{6}{21}.$$

Thus,

$$P(\text{even}) = \frac{2}{21} + \frac{4}{21} + \frac{6}{21} = \frac{12}{21} = \frac{4}{7}.$$

Problem 11. Two cards are drawn from a standard 52-card deck without replacement. Given that the first card is a heart, what is the probability that the second card is a king?

Solution

Let A: first card is a heart, B: second card is a king. We want $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.
First, $P(A) = \frac{13}{52} = \frac{1}{4}$.
Now, $P(A \cap B)$: first card is a heart AND second card is a king.
Case 1: First card is the king of hearts. Then the second card must be a king from the remaining 51 cards (3 kings left). Probability:

$$\frac{1}{52} \times \frac{3}{51}.$$

Case 2: First card is a heart that is not the king (12 such hearts). Then the second

card must be one of the 4 kings. Probability:

$$\frac{12}{52} \times \frac{4}{51}.$$

Adding:

$$P(A \cap B) = \frac{1}{52} \cdot \frac{3}{51} + \frac{12}{52} \cdot \frac{4}{51} = \frac{3+48}{52 \cdot 51} = \frac{51}{52 \cdot 51} = \frac{1}{52}.$$

Thus,

$$P(B \mid A) = \frac{1/52}{1/4} = \frac{1}{52} \cdot 4 = \frac{1}{13}.$$

Problem 12. In a school, 60% of students play soccer, 50% play basketball, and 30% play both. What is the probability that a randomly selected student plays at least one of these sports?

Solution

Let S: student plays soccer, B: student plays basketball. Given: $P(S) = 0.6,\ P(B) = 0.5,\ P(S \cap B) = 0.3$. Using the sum rule:

$$P(S \cup B) = P(S) + P(B) - P(S \cap B) = 0.6 + 0.5 - 0.3 = 0.8.$$

Thus, the probability is 80%.

Problem 13. In a group of 5 people, what is the probability that at least two were born on the same day of the week? Assume each day (Monday through Sunday) is equally likely for a birthday.

Solution

Let the sample space be all sequences of 5 weekdays: $|S| = 7^5$. Let E: at least two share a weekday. It's easier to compute the complement $\overline{E}$: all 5 birthdays are on different weekdays.

$$|\overline{E}| = P(7,5) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520.$$

Thus,

$$P(\overline{E}) = \frac{2520}{7^5} = \frac{2520}{16807} \approx 0.1499.$$

Therefore,

$$P(E) = 1 - P(\overline{E}) \approx 1 - 0.1499 = 0.8501.$$

(Exact fraction: $1 - \frac{2520}{16807} = \frac{14287}{16807}$.)

Problem 14. Two fair distinct dice are rolled. What is the probability that the sum is at least 10, given that at least one of the dice shows a 5?

Solution

Let A: at least one die shows a 5, B: sum is at least 10.
We want $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$.
First, count outcomes for A.
There are $6 + 6 - 1 = 11$ outcomes with at least one 5: (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5).

Alternatively, we can count using combinatorics:
We can consider the event where no die shows a 5, and use the subtraction rule to get our desired event.
Total possibilities for two dice: $6^2 = 36$
No die shows a 5: $5^2 = 25$
$\therefore$ At least one die shows a 5: $36 - 25 = 11$

So $|A| = 11,\ |S| = 36,\ P(A) = \frac{11}{36}$.

Now, $A \cap B$: at least one 5 **and** sum ≥ 10. From the list above, those with sum ≥ 10 are: (5,5) sum=10, (5,6) sum=11, (6,5) sum=11. So $|A \cap B| = 3$. Thus,

$$P(B \mid A) = \frac{3/36}{11/36} = \frac{3}{11}.$$