CSCA67 TT3 Review Seminar

Held by AMACSS

Nov 15th (Sat) 13:00 - 15:30

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Simple Induction - AMACSS A67 TT3 Review Seminar - Fall 2025

General template for simple induction proofs:

Solution

Let P(n): "statement involving n". To prove $\forall n \geq b, P(n)$.
P(k): "same statement for k", P(k+1): "same statement for k+1"

Base Case ($n = b$): verify the P(b) statement. Thus, $P(b)$ holds.

Inductive Step: WTP $\forall k \geq b, P(k) \rightarrow P(k+1)$.

(1) Take arbitrary $k \geq b$

(2) Suppose P(k) [Induction Hypothesis]

(3) Copy the P(k) statement here [2, def. P]

(4) Start from one side of the P(k+1) statement and reach the other using IH

(5) P(k+1) [4, def. P]

(6) $P(k) \rightarrow P(k+1)$ [2, 5, implication]

(7) $\forall k \geq b, P(k) \rightarrow P(k+1)$ [1, 6, UG]

Therefore, $\forall n \geq b, P(n)$ by Simple Induction.

Problem 1. Prove that for all integers $n \geq 5$:

$$2^n > n^2 + n$$

Solution

Let P(n): "$2^n > n^2 + n$". To prove $\forall n \geq 5, P(n)$.
P(k): "$2^k > k^2 + k$"
P(k+1): "$2^{k+1} > (k+1)^2 + (k+1)$"

Base Case ($n = 5$): $2^5 = 32 > 30 = 5^2 + 5$. Thus, P(5) holds.

Inductive Step: WTP $\forall k \geq 5, P(k) \to P(k+1)$.

(1) Take arbitrary $k \geq 5$

(2) Suppose P(k) [Induction Hypothesis]

(3) $2^k > k^2 + k$ [2, def. P]

(4)
$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2(k^2 + k) && \text{[3, IH]} \\ &= 2k^2 + 2k \\ &= k^2 + 2k + k^2 - 1 + 1 \\ &= k^2 + 2k + (k+1)(k-1) + 1 \\ &= (k+1)^2 + (k+1)(k-1) \\ &> (k+1)^2 + (k+1) && \text{[Since } k - 1 \geq 5 - 1 = 4 > 0\text{]} \end{aligned}$$

(5) P(k+1) [4, def. P]

(6) $P(k) \to P(k+1)$ [2, 5, implication]

(7) $\forall k \geq 5, P(k) \to P(k+1)$ [1, 6, UG]

Therefore, $\forall n \geq 5, P(n)$ by Simple Induction.

Problem 2. We write $a \mid b$, read "a divides b", to stand for $\exists m \in \mathbb{Z}, b = am$. Prove that for all positive integers n:

$$5 \mid (n^5 - n)$$

Solution

Let P(n): "$5 \mid (n^5 - n)$". To prove $\forall n \geq 1, P(n)$.
P(k): "$5 \mid (k^5 - k)$"
P(k+1): "$5 \mid ((k+1)^5 - (k+1))$"

Base Case ($n = 1$): $1^5 - 1 = 0$ and $5 \mid 0$. Thus, P(1) holds.

Inductive Step: WTP $\forall k \geq 1, P(k) \to P(k+1)$.

(1) Take arbitrary $k \geq 1$

(2) Suppose P(k) [Induction Hypothesis]

(3) $5 \mid (k^5 - k)$ [2, def. P]

(4) $\exists m \in \mathbb{Z}, k^5 - k = 5m$ [3, def. divisibility]

(5) $k^5 - k = 5c$ [4, EI]

(6) $$\begin{aligned}(k+1)^5 - (k+1) &= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) - (k+1) && \text{[algebra]}\\ &= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)\\ &= 5c + (5k^4 + 10k^3 + 10k^2 + 5k) && \text{[5, IH]}\\ &= 5c + 5(k^4 + 2k^3 + 2k^2 + k)\\ &= 5(c + k^4 + 2k^3 + 2k^2 + k)\end{aligned}$$

(7) $(c + k^4 + 2k^3 + 2k^2 + k) \in \mathbb{Z}$ [closure of $\mathbb{Z}$ under mult. & add.]

(8) $\exists m \in \mathbb{Z}, (k+1)^5 - (k+1) = 5m$ [6, 7, EG]

(9) $5 \mid (k+1)^5 - (k+1)$ [8, def. divisibility]

(10) $P(k+1)$ [9, def. P]

(11) $P(k) \to P(k+1)$ [2, 10, implication]

(12) $\forall k \geq 1, P(k) \to P(k+1)$ [1, 11, UG]

Therefore, $\forall n \geq 1, P(n)$ by Simple Induction.

Problem 3. Define the sequence $\{a_n\}$ by:

$$a_1 = 1, \quad a_n = 2a_{n-1} + 3 \quad \text{for } n \geq 2$$

Prove that for all integers $n \geq 1$:

$$a_n = 2^{n+1} - 3$$

Solution

Let P(n): "$a_n = 2^{n+1} - 3$". To prove $\forall n \geq 1, P(n)$.
P(k): "$a_k = 2^{k+1} - 3$"
P(k+1): "$a_{k+1} = 2^{(k+1)+1} - 3$"

Base Case ($n = 1$): $2^{1+1} - 3 = 4 - 3 = 1 = a_1$. Thus, P(1) holds.

Inductive Step: WTP $\forall k \geq 1, P(k) \to P(k+1)$.

(1) Take arbitrary $k \geq 1$

(2) Suppose P(k) [Induction Hypothesis]

(3) $a_k = 2^{k+1} - 3$ [2, def. P]

(4) $a_{k+1} = 2a_k + 3$ [def. of sequence, since $k + 1 \geq 1 + 1 = 2$]
$= 2(2^{k+1} - 3) + 3$ [3, IH]
$= 2^{k+2} - 6 + 3$
$= 2^{k+2} - 3$
$= 2^{(k+1)+1} - 3$

(5) P(k+1) [4, def. P]

(6) $P(k) \to P(k+1)$ [2, 5, implication]

(7) $\forall k \geq 1, P(k) \to P(k+1)$ [1, 6, UG]

Therefore, $\forall n \geq 1, P(n)$ by Simple Induction.

A67 Strong Induction Solutions for TT3 Review Seminar

Question 1. *Consider the sequence of numbers $a_0, a_1, a_2, \ldots$ defined as follows:*

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_{n+2} &= 3a_{n+1} - a_n \quad \textit{for } n \in \mathbb{N} \end{aligned}$$

Show: $\forall n \in \mathbb{N}, n \geq 3 \implies a_n < 3^{n-1}$.

Let $P(n) : a_n < 3^{n-1}$.
We want to show $\forall n \geq 3, P(n)$. Thus, since we are using strong induction, we show

$$\forall k \geq 3, (\forall i, 3 \leq i < k, P(i)) \to P(k)$$

(1) Take arbitrary k.

(2) Suppose $k \geq 3$. (You can combine lines 1 and 2 if you've like)

(3) Suppose $\forall i, 3 \leq i < k, P(i)$. — Induction Hypothesis

(4) Suppose $k \geq 5$. — Case #1

(5) $k > k-1 > k-2 \geq 3$.

(6) $P(k-1)$. — 3, 5, U.M.P., IH

(7) $P(k-2)$. — 3, 5, U.M.P., IH

(8) $a_{k-1} < 3^{k-2}$. — 6, def of P

(9) $a_{k-2} < 3^{k-3}$. — 7, def of P

(10)
$$\begin{aligned} a_k &= 3a_{k-1} - a_{k-2} \\ &< 3 \cdot 3^{k-2} - a_{k-2} \\ &< 3^{k-1} \end{aligned}$$
by 8, as $a_{k-2} \geq 0$

(11) $P(k)$. — 10, def of P

(12) Suppose $k < 5$. — Case #2

(13) For $k = 3$: $a_3 = 3a_2 - a_1 = 3(3a_1 - a_0) - a_1 = 8 < 9 = 3^{3-1}$.

(14) For $k = 4$: $a_4 = 3a_3 - a_2 = 3(8) - 3 = 21 < 27 = 3^{4-1}$.

(15) $a_k < 3^{k-1}$. 13-14, cases

(16) $P(k)$. 15, def of P

(17) $P(k)$. 4, 11, 12, 16, cases

(18) $(\forall i, 3 \leq i < k, P(i)) \rightarrow P(k)$. 3, 17, implication

(19) $k \geq 3 \rightarrow (\forall i, 3 \leq i < k, P(i)) \rightarrow P(k)$. 2, 18, implication

(20) $\forall k \geq 3, (\forall i, 3 \leq i < k, P(i)) \rightarrow P(k)$. 1, 19, U.G.

$\therefore \forall n \geq 3, P(n)$ by Strong Induction.

Question 2. *Consider the sequence of real numbers $a_1, a_2, \ldots$ defined by:*

$$a_n = \begin{cases} 1 & \text{if } n = 1 \\ (a_{\lfloor\sqrt{n}\rfloor})^2 + 2a_{\lfloor\sqrt{n}\rfloor} & \text{if } n \geq 2 \end{cases}$$

Note: $\lfloor x \rfloor$ equals x rounded down to the nearest whole number.
Show that $\forall n \geq 2$, a_n is a multiple of 3.
You may use the fact that $\forall x > 1, x > \sqrt{x}$ and $\forall b \in \mathbb{N}, (x \geq b \Rightarrow \lfloor x \rfloor \geq b)$ in your proof.

Let $P(n) : \exists m \in \mathbb{Z}, a_n = 3m$.
We want to show $\forall n \geq 2, P(n)$. Thus, since we are using strong induction, we show

$$\forall k \geq 2, (\forall i, 2 \leq i < k, P(i)) \rightarrow P(k)$$

(1) Take arbitrary k.

(2) Suppose $k \geq 2$.

(3) Suppose $\forall i, 2 \leq i < k, P(i)$. — Induction Hypothesis

(4) Suppose $k \geq 4$. — Case #1

(5) $\sqrt{k} \geq 2$. — by 4 (sqrt both sides)

(6) $k > \sqrt{k} \geq 2$. — by 5, as $k > \sqrt{k}, \forall k \geq 2$

(7) $k > \lfloor\sqrt{k}\rfloor \geq 2$. — by 6, as $x \geq 2 \implies \lfloor x \rfloor \geq 2$

(8) $P(\lfloor\sqrt{k}\rfloor)$. — 3, 7, U.M.P., IH

(9) $\exists m \in \mathbb{Z}, a_{\lfloor\sqrt{k}\rfloor} = 3m$. — 8, def. of P

(10) $a_{\lfloor\sqrt{k}\rfloor} = 3b, b \in \mathbb{Z}$. — 9, E.I.

(11) $a_k = (a_{\lfloor\sqrt{k}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{k}\rfloor}$. — by def of a_k for $k \geq 2$

(12) $a_k = (3b)^2 + 2(3b)$.

(13) $a_k = 9b^2 + 6b$.

(14) $a_k = 3(3b^2 + 2b)$.

(15) $3b^2 + 2b \in \mathbb{Z}$. — by 10, closure

(16) $\exists m \in \mathbb{Z}, a_k = 3m$. — 14, 15, E.G.

(17) $P(k)$. — 16, def. of P

(18) Suppose $k < 4$. — Case #2

(19) $a_2 = (a_{\lfloor\sqrt{2}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{2}\rfloor} = (a_1)^2 + 2a_1 = 1^2 + 2(1) = 3$.

(20) $a_3 = (a_{\lfloor\sqrt{3}\rfloor})^2 + 2 \cdot a_{\lfloor\sqrt{3}\rfloor} = (a_1)^2 + 2a_1 = 1^2 + 2(1) = 3.$

(21) $\exists m \in \mathbb{Z}, a_2 = 3m.$ by 19

(22) $\exists m \in \mathbb{Z}, a_3 = 3m.$ by 20

(23) $P(2) \wedge P(3).$ 21, 22, Conj., def of P

(24) $P(k).$ 23, 18

(25) $P(k).$ 4, 17, 18, 24, cases

(26) $(\forall i, 2 \leq i < k, P(i)) \rightarrow P(k).$ 3, 25, implication

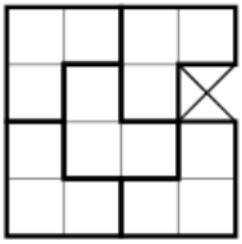
(27) $k \geq 2 \rightarrow (\forall i, 2 \leq i < k, P(i)) \rightarrow P(k).$ 2, 26, implication

(28) $\forall k \geq 2, (\forall i, 2 \leq i < k, P(i)) \rightarrow P(k).$ 1, 27, U.G.

$\therefore \forall k \geq 2, P(k)$ by Strong Induction.

Question 3. *(Challenge Question, it's also from the textbook) Show: $\forall n \geq 1$, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled using triominoes.*

A triomino is an L-shaped tile, like this: 
Example of possible tiling:



Let $P(n)$: The $2^n \times 2^n$ checkerboard can be tiled using triominoes with any one square removed.
We want to show $\forall n \geq 1, P(n)$. Thus, since we are using strong induction, we show

$$\forall k \geq 1, (\forall i, 1 \leq i < k, P(i)) \rightarrow P(k)$$

(1) Take arbitrary k.

(2) Suppose $k \geq 1$.

(3) Suppose $\forall i, 1 \leq i < k, P(i)$. Induction Hypothesis

(4) Suppose $k \geq 2$. Case #1

(5) $k > k - 1 \geq 1$.

(6) $P(k-1)$. 3, 5, U.M.P, I.H.

(7) Then we know a $2^{k-1} \times 2^{k-1}$ checkerboard can be tiled using triominoes with any one square removed by the definition of $P(k-1)$.

(8) Recall the $2^k \times 2^k$ checkboard is made of four $2^{k-1} \times 2^{k-1}$ checkboards. The one square removed lies in one of these four sub-boards. Place a triomino at the centre of the board to cover one corner square from each of the other three sub-boards. By line 7, we can fill these three boards (as they now each have one square missing). The first sub-board can also be filled by line 7. This fulfills $P(k)$.

(9) Suppose $k = 1$. Case #2

(10) Then a $2^1 \times 2^1$ board with one square removed can be filled as it is the shape of a single triomino. Thus, $P(k)$ by def of P.

(11)	$P(k)$.	4, 8, 9, 10, cases
(12)	$(\forall i, 1 \leq i < k, P(i)) \rightarrow P(k)$.	3, 11, implication
(13)	$k \geq 1 \rightarrow (\forall i, 1 \leq i < k, P(i)) \rightarrow P(k)$.	2, 12, implication
(14)	$\forall k \geq 1, (\forall i, 1 \leq i < k, P(i)) \rightarrow P(k)$.	1, 13, U.G.

$\therefore \forall n \geq 1, P(n)$ by Strong Induction.

Proof by Contrapositive

Introduction to the contrapositive

- Sometimes, it is hard to prove a fact directly
  - e.g. x is prime, x != 0, no real number has some property, etc.
- Sometimes, one equivalent form of an implication is easier to prove than the other
- Luckily for us, we have the contrapositive!

$$A \rightarrow B$$
$$\Leftrightarrow \neg B \rightarrow \neg A$$

- If an implication is hard to prove one direction, we can equivalently try to prove its contrapositive instead

Contrapositive structure

WTS $A \to B$

1. Suppose $\neg B$

    2. Prove $\neg A$

3. $\neg B \to \neg A$

4. $\therefore A \to B$

Practice!

Show that if $x^3y = x^2 + 5x + y$, then $(y \neq 0 \rightarrow x \neq 0)$.

Solution

Show that if $x^3y = x^2 + 5x + y$, then $(y \neq 0 \to x \neq 0)$.

Solution: This is very hard to prove directly! Instead, we'll use the contrapositive.

(1) Suppose $\neg(y \neq 0 \to x \neq 0)$.

(2) $\neg(\neg y \neq 0 \vee x \neq 0)$ (1, cond.)

(3) $y \neq 0 \wedge x = 0$ (DM)

(4) $x = 0$ (3, simpl.)

(5) $x^3y = 0^3y$ (4)

$= 0$

(6) $x^2 + 5x + y = 0^2 + 5(0) + y = 0 + y$ (4)

$= y$

(7) $y \neq 0$ (3, simpl.)

(8) $x^3y \neq x^2 + 5x + y$ (5, 6, 7)

(9) $\neg(y \neq 0 \to x \neq 0) \to x^3y \neq x^2 + 5x + y$ (1, 8, implication)

(10) $x^3y = x^2 + 5x + y \to (y \neq 0 \to x \neq 0)$ (9, contrapositive)

as wanted.

Practice!

What are the two errors in the following proof by contrapositive? Is the theorem correct?

WTS: $(A \subseteq C \wedge B \subseteq C \wedge x \in A) \rightarrow x \in B$

(1) Suppose $x \notin B$

(2) $A \subseteq C \wedge B \subseteq C \wedge x \in A$ (given)

(3) $B \subseteq C$ (2, simplification)

(4) $x \notin C$ (1, 3, subset defn.)

(5) $x \in A$ (2, simplification)

(6) $x \in A \wedge x \notin C$ (4, 5, conjunction)

(7) $\neg(\neg(x \in A) \vee x \in C)$ (6, DM)

(8) $\neg(x \in A \rightarrow x \in C)$ (7, cond.)

(9) $\neg(A \subseteq C)$ (8, subset defn.)

(10) $\neg(A \subseteq C) \vee \neg(B \subseteq C) \vee \neg(x \in A)$ (9, addition)

(11) $\neg(A \subseteq C \wedge B \subseteq C \wedge x \in A)$ (10, DM)

(12) $x \notin B \rightarrow \neg(A \subseteq C \wedge B \subseteq C \wedge x \in A)$ (1, 11, implication)

(13) $(A \subseteq C \wedge B \subseteq C \wedge x \in A) \rightarrow x \in B$ (12, contrapositive)

Solution

WTS: $(A \subseteq C \land B \subseteq C \land x \in A) \rightarrow x \in B$

(1) Suppose $x \notin B$

~~(2) $A \subseteq C \land B \subseteq C \land x \in A$ (given)~~

(3) $B \subseteq C$ (2, simplification)

~~(4) $x \notin C$ (1, 3, subset defn.)~~

(5) $x \in A$ (2, simplification)

(6) $x \in A \land x \notin C$ (4, 5, conjunction)

(7) $\neg(\neg(x \in A) \lor x \in C)$ (6, DM)

(8) $\neg(x \in A \rightarrow x \in C)$ (7, cond.)

(9) $\neg(A \subseteq C)$ (8, subset defn.)

(10) $\neg(A \subseteq C) \lor \neg(B \subseteq C) \lor \neg(x \in A)$ (9, addition)

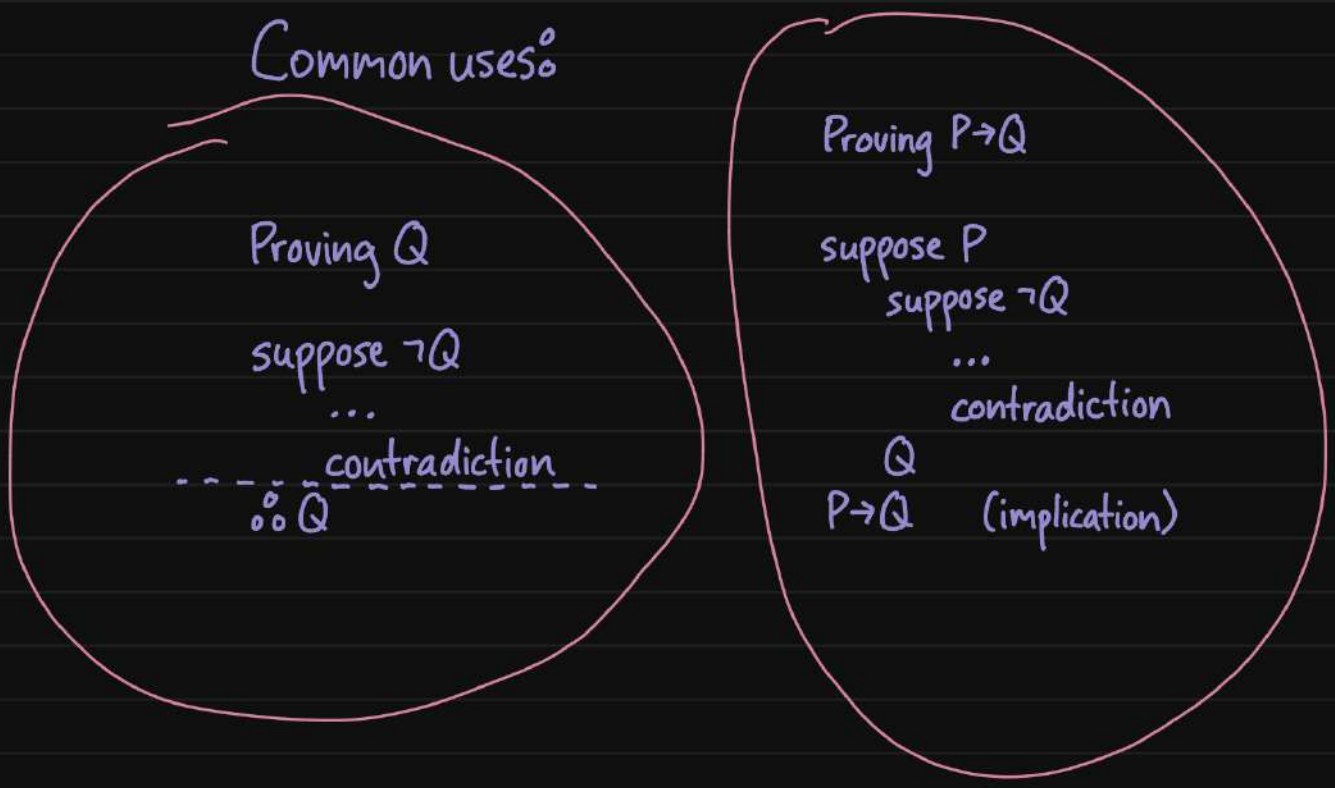
(11) $\neg(A \subseteq C \land B \subseteq C \land x \in A)$ (10, DM)

(12) $x \notin B \rightarrow \neg(A \subseteq C \land B \subseteq C \land x \in A)$ (1, 11, implication)

(13) $(A \subseteq C \land B \subseteq C \land x \in A) \rightarrow x \in B$ (12, contrapositive)

Theorem is not correct

Contradiction



Contradiction

Prove the following:
For all real numbers x and y, if $x \neq y$, $x>0$, and $y>0$, then $\frac{x}{y}+\frac{y}{x}>2$.

Logical form: $\forall x,y \in \mathbb{R}, (x \neq y \wedge x>0 \wedge y>0) \rightarrow (\frac{x}{y}+\frac{y}{x}>2)$

Proof:

(1)	take arbitrary $x,y \in \mathbb{R}$	
(2)	suppose $x \neq y$, $x>0$, and $y>0$	
(3)	suppose $\frac{x}{y}+\frac{y}{x} \leq 2$ for contradiction	
(4)	$(xy)(\frac{x}{y}+\frac{y}{x}) \leq 2xy$	(3) multiply by xy, and $xy>0$ by (2)
(5)	$x^2+y^2 \leq 2xy$	(4) algebra
(6)	$x^2-2xy+y^2 \leq 0$	(5) algebra
(7)	$(x-y)^2 \leq 0$	(6) factoring
(8)	$(x-y)^2 \geq 0$	(7) anything squared ≥ 0
(9)	$(x-y)^2 = 0$	(7)(8) algebra
(10)	$x-y=0$	(9) sqrt both sides
(11)	$x=y$	(10) algebra
(12)	contradiction	(2) (11) cannot both be true
(13)	$\frac{x}{y}+\frac{y}{x}>2$	(3)(12) negation of (3)
(14)	$(x \neq y \wedge x>0 \wedge y>0) \rightarrow (\frac{x}{y}+\frac{y}{x}>2)$	(2)(13) implication
(15)	$\forall x,y \in \mathbb{R}, (x \neq y \wedge x>0 \wedge y>0) \rightarrow (\frac{x}{y}+\frac{y}{x}>2)$	(1)(14) UG

(can be its own step)

(3.5) $xy>0$ by (2) ☆ $x>0, y>0$

The stmt has been proven, as req. ▨

Contradiction

Prove the following:
For all $n \in \mathbb{Z}$, (n^3+5 is odd) → (n is even)

Logical form: $\forall n \in \mathbb{Z}$, (n^3+5 is odd → n is even)

Proof:

(1)	take arbitrary $n \in \mathbb{Z}$	
(2)	suppose n^3+5 is odd	
(3)	suppose n is odd for contradiction	
(4)	$\exists k \in \mathbb{Z}, n^3+5=2k+1$	(2) def^n of odd
(5)	$\exists \ell \in \mathbb{Z}, n=2\ell+1$	(3) def^n of odd
(6)	$n^3+5=2s+1$, for some $s \in \mathbb{Z}$	(4) EI
(7)	$n=2r+1$, for some $r \in \mathbb{Z}$	(5) EI
(8)	$(2r+1)^3+5=2s+1$	(6)(7) substitution
(9)	$(2r)^3+3(2r)^2+3(2r)+1+5=2s+1$	(8) expanding
(10)	$8r^3+12r^2+6r+5=2s$	(9) alg
(11)	$4r^3+6r^2+3r+\frac{5}{2}=s$	(10) divide by 2
	idea: this $\in \mathbb{Z}$	
(12)	$m+\frac{5}{2}=s$, for some $m \in \mathbb{Z}$	$2m+5=2s$, $2(m+2)+1=2s$ (odd, even)
(13)	$s \notin \mathbb{Z}$	(11) $4r^3+6r^2+3r \in \mathbb{Z}$ since $r \in \mathbb{Z}$ (7); (12) integer + $\frac{5}{2}$ = non-integer
(14)	contradiction	(6)(13) can't both be true
(15)	n is even	(3)(14) negation of line 3
(16)	(n^3+5 is odd) → (n is even)	(2)(15) implication
(17)	$\forall n \in \mathbb{Z}$, (n^3+5 is odd) → (n is even)	(1)(16) UG

The stmt has been proven, as req. ▨

Contradiction



Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers.

Logical form: $\exists a_k \in S,\ a_k \geq v \iff \exists k,\ a_k \in S \wedge a_k \geq v$

$\forall a_k \in S,\ a_k < v$

$\neg(\exists a_k \in S,\ a_k \geq v) \leftrightarrow \forall a_k \in S,\ \neg(a_k \geq v)$

Proof: ☆ take arbitrary $S = \{a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n \in \mathbb{R}$

(1)	~~let $S = \{a_1, a_2, \ldots, a_n\}$ where $a_1, a_2, \ldots, a_n \in \mathbb{R}$~~	
(2)	let v be the average of the numbers in S.	WTP: $\exists a_k \in S,\ a_k \geq v$
(3)	to derive a contradiction, suppose $\forall a_k \in S,\ a_k < v$	
(4)	$\forall a_k \in S,\ a_k < \frac{a_1+a_2+\ldots+a_n}{n}$	(1)(2)(3) def'n of average
(5)	$a_1 < \frac{a_1+a_2+\ldots+a_n}{n}$ $a_2 < \frac{a_1+a_2+\ldots+a_n}{n}$ ... $a_n < \frac{a_1+a_2+\ldots+a_n}{n}$	(1)(4) universal instantiation n times
(6)	$a_1+a_2+\ldots+a_n < \frac{a_1+a_2+\ldots+a_n}{n} \cdot n$	(5) adding up inequalities
(7)	$a_1+a_2+\ldots+a_n < a_1+a_2+\ldots+a_n$	(6) alg
(8)	contradiction our assumption that $\forall a_k \in S,\ a_k < v$ was false	(7) cannot be true
(9)	$\neg\forall a_k \in S,\ a_k < v$	(8) rewritten
(10)	$\exists a_k \in S,\ \neg(a_k < v)$	(9) quantifier negation
(11)	$\exists a_k \in S,\ a_k \geq v$	(10) def'n of $\geq$

∴ at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers, as required. ▨