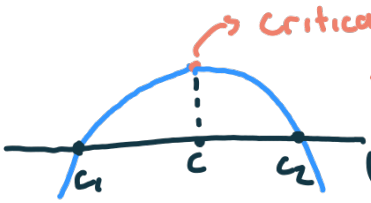
MATA31 AMACSS Review Seminar (Part 2)

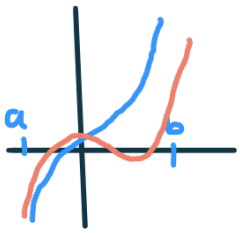
December 2025

1. Prove that if f is differentiable on $[a, b]$ and $f'(x) > 0$ for all $x \in (a, b)$, then f has at most one root in $[a, b]$.

f diff → f cont. on [a,b]



(Rolle's Theorem)



strictly increasing

Proof. Suppose f is differentiable on $[a,b]$ and $f'(x) > 0$ for all $x \in (a,b)$. Suppose on the contrary that there are at least two roots $c_1 < c_2$, where $c_1, c_2 \in [a,b]$

Since we know f is differentiable on $[a,b]$, then it must be that

1. f is continuous on $[a,b]$, which implies cont. on $[c_1,c_2]$ ✓
   (since $[c_1,c_2] \subseteq [a,b]$)
2. f is differentiable on (c_1,c_2) ✓

Also, since c_1, c_2 are roots of f, then $f(c_1) = f(c_2) = 0$. ✓

Hence, by Rolle's theorem, $\exists c \in (c_1, c_2)$ such that $f'(c) = 0$, but this is a contradiction, since $f'(x) > 0$ for all $x \in (a,b)$.

Therefore, f must have at most one root in $[a,b]$.

2. Sketch the graph of $f(x) = \dfrac{x-3}{\sqrt{x^2-9}}$ with derivatives

$$f'(x) = \frac{3(x-3)}{(x^2-9)^{3/2}}, \quad f''(x) = \frac{-3(x-3)(2x-3)}{(x^2-9)^{5/2}}$$

We find the following:

- **Domain**: denom: $\sqrt{x^2-9}$, $\neq 0 \Rightarrow x^2-9 \neq 0 \Rightarrow x \neq \pm 3$; $\sqrt{x^2-9} \geq 0$

  $x^2 - 9 \geq 0 \Rightarrow x^2 \geq 9$

  $\Rightarrow x \geq 3$ or $x \leq -3$

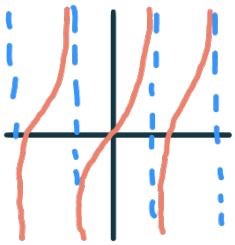
  D: $(-\infty, -3) \cup (3, \infty)$

- **x & y-intercepts**: x-intercept: $f(x)=0 \Rightarrow \dfrac{x-3}{\sqrt{x^2-9}} = 0$

  $\therefore$ no x-intercept $\Rightarrow x-3=0 \Rightarrow x=3$ not in domain

  y-intercept: $x=0$ not in domain

  $\therefore$ no y-intercept



- **Asymptotes**: horizontal:

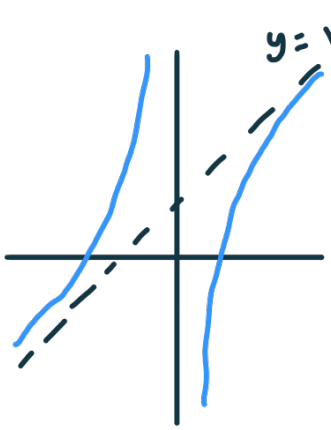
  $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{x-3}{\sqrt{x^2-9}} \approx \lim_{x\to-\infty} \dfrac{x}{|x|} = -1$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{x-3}{\sqrt{x^2-9}} = 1$

  $\boxed{y=-1}$ and $\boxed{y=1}$

  vertical:

  $\lim_{x\to-3} f(x) = \lim_{x\to-3} \dfrac{x-3}{\sqrt{x^2-9}} \to -\infty$, so $\boxed{x=-3}$

  $\lim_{x\to3} \dfrac{x-3}{\sqrt{x^2-9}} \overset{L'H}{=} \lim_{x\to3} \dfrac{1}{\frac{1}{2\sqrt{x^2-9}} \cdot 2x} = \lim_{x\to3} \dfrac{\sqrt{x^2-9}}{x} = 0$



slant: since $\lim_{x\to\pm\infty} f(x) \neq \pm\infty$, then no slant asymptote

1. $\lim_{x\to\pm\infty} f(x) = \pm\infty$ ✓
2. $\lim_{x\to\pm\infty} \dfrac{f(x)}{x} = m \in \mathbb{R}$ ✓
3. $\lim_{x\to\pm\infty} (f(x) - mx) = b$

$\therefore$ slant asymptote: $y = mx + b$

- **Symmetry**: 1. even functions $[f(x) = f(-x)]$

  2. odd functions $[-f(x) = f(-x)]$

  3. periodic functions $[f(x) = f(x+a)]$

- Critical points: $f'(x)=0 \Rightarrow \frac{3(x-3)}{(x^2-9)^{3/2}}=0 \Rightarrow x=3$ (not in domain)

$\therefore$ no critical points

- Intervals of inc/dec: increase: $f'(x)>0$, decrease: $f'(x)<0$

(First Derivative Test) Note denominator $(x^2-9)^{3/2}>0$ for all x in domain

$\therefore f'(x)>0 \iff 3(x-3)>0 \Rightarrow x>3$

$f'(x)<0 \iff 3(x-3)<0 \Rightarrow x<3$

Hence, inc at $(3,\infty)$ and dec at $(-\infty,-3)$

- Concavity: check $f''(x)=0 \Rightarrow \frac{-3(x-3)(2x-3)}{(x^2-9)^{5/2}}=0 \Rightarrow x=3$ or $x=\frac{3}{2}$

(Second Derivative Test)



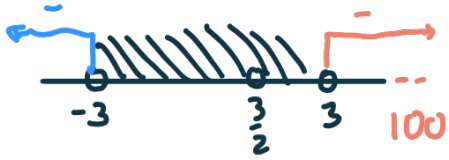
$f''(x)>0$

concave up

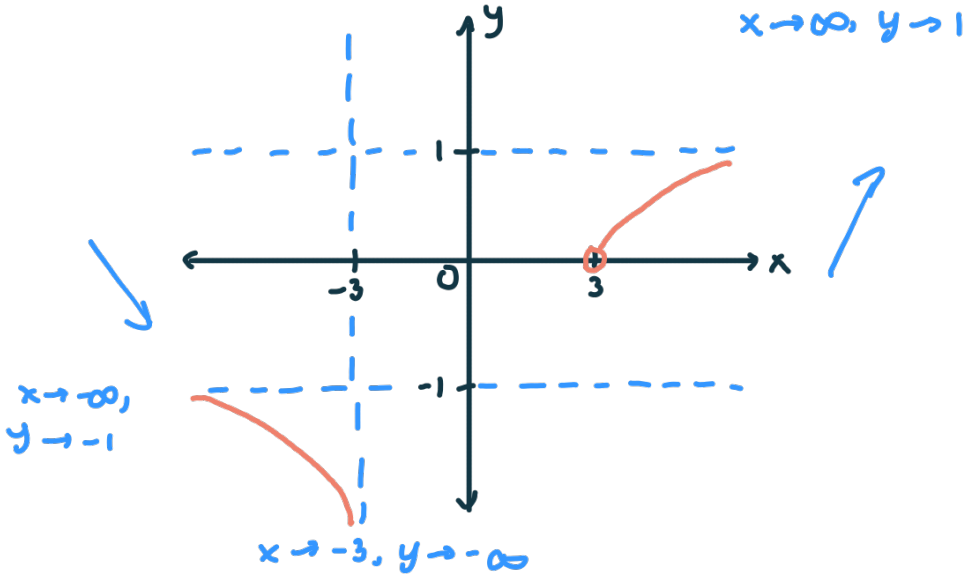


$f''(x)<0$

concave down



concave down at $(3,\infty)$ and $(-\infty,-3)$



$x \to \infty$, $y \to 1$

$x \to -\infty$, $y \to -1$

$x \to -3$, $y \to -\infty$

3. Sketch the graph of $f(x) = (x-1)^4(3x-8)$ with derivatives

$$f'(x) = 5(x-1)^3(3x-7), \quad f''(x) = 60(x-1)^2(x-2)$$

- **Domain**: $\mathbb{R}$
- **x-intercept**: $f(x)=0 \Rightarrow x=1$ or $x=\frac{8}{3}$
- **y-intercept**: $y=f(0)=(-1)^4\cdot(-8)=-8$
- **Symmetry**: no symmetry
- **Asymptotes**: - Horizontal: $\lim_{x\to\infty} \underbrace{(x-1)^4}_{+\infty}\underbrace{(3x-8)}_{+\infty} = \infty$ | $\lim_{x\to-\infty} \underbrace{(x-1)^4}_{+\infty}\underbrace{(3x-8)}_{-\infty} = -\infty$
  - Vertical: domain is $\mathbb{R}$, so no VA
  - Slant: for $x\to\infty$, $\lim_{x\to\infty} \frac{(x-1)^4(3x-8)}{x} = \infty$ | for $x\to-\infty$, $\lim_{x\to-\infty} \frac{(x-1)^4(3x-8)}{x} = \infty$,

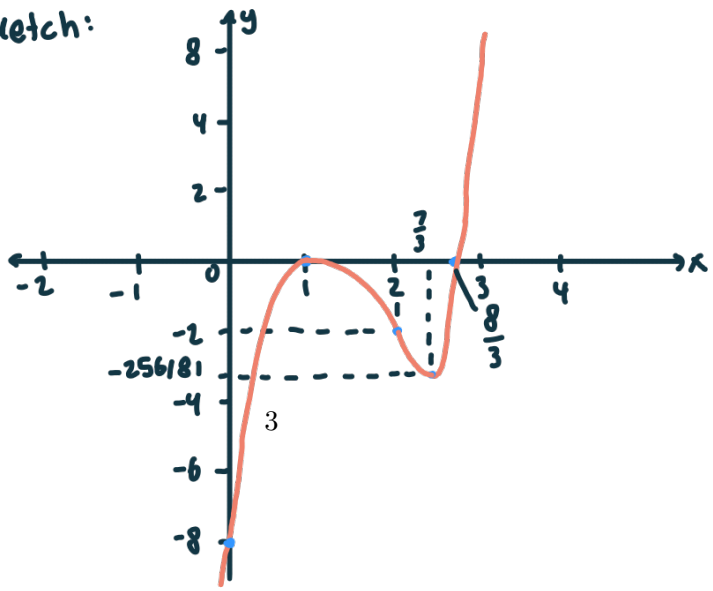
    so, no slant asymptotes
- **Critical points**: $f'(x)=0 \Rightarrow 5(x-1)^3(3x-7)=0 \Rightarrow x=1$ or $x=\frac{7}{3}$
- **Interval of increase/decrease**: $f'(x)$: + (1) − ($\frac{7}{3}$) + ∴ increasing on $(-\infty,1)$ and $(\frac{7}{3},\infty)$; decreasing on $(1,\frac{7}{3})$

  max at $x=1$, min at $x=\frac{7}{3}$
- **Concavity**: consider $f''(x)=0 \Rightarrow 60(x-1)^2(x-2)=0 \Rightarrow x=1$ or $x=2$

  $f''(x)$: − (1) − (2) + ∴ concave up on $(2,\infty)$

  inflection point at $x=2$; concave down on $(-\infty,1)$ and $(1,2)$
- [Roughly] compute the important points:
  - critical point: $f(\frac{7}{3}) = (\frac{7}{3}-1)^4(3\cdot\frac{7}{3}-8) = -\frac{256}{81} \approx -3.1$
  - inflection point: $f(2) = (2-1)^4(3\cdot2-8) = -2$

Sketch:



good practice problems:

1. $f(x) = x + \sqrt{x^2-1}$
2. $f(x) = e^{2x} - 3e^x + 2$
   (Note: $\ln(2) \approx 0.693$)
3. $f(x) = \frac{\sqrt{x^2-25}}{x-10}$

Draw first, then open Desmos.