MATA31 AMACSS Review Seminar (Part 1)

December 2025

1. True or False? If $\lim_{x\to 2} \frac{f(x)}{x^2-4}$ exists, then $\lim_{x\to 2} f(x)$ exists.

→ $\lim_{x\to 2} \frac{f(x)}{x^2-4}$ exists $= \lim_{x\to 2} f(x) \cdot \frac{1}{x^2-4}$

→ $\lim_{x\to 2} (x^2-4)$ exists x^2-4 is cont $\mathbb{R}$ b/c polyn.

∴ $\lim_{x\to 2} \frac{f(x)}{x^2-4}(x^2-4)$ exists $\lim_{x\to 2} f(x)$ by Cancellation Thm

2. True or False? If $S \subseteq \mathbb{R}$ is nonempty and bounded below, then there exists $\inf(S) \in S$.

$\Rightarrow \exists \inf(S) \in \mathbb{R}$ by GLB prop / Completeness

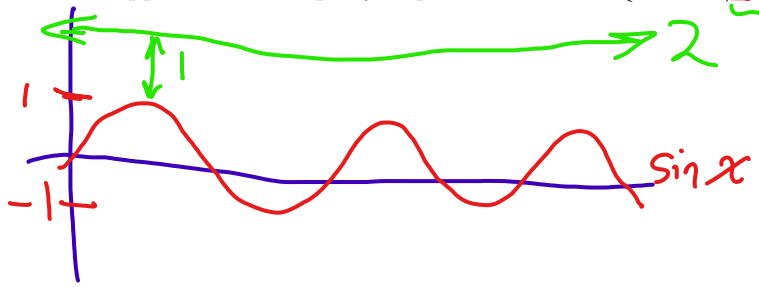
False – consider $S = (0,1) \subseteq \mathbb{R}$

nonempty

bounded below by 0

$\inf(S) = 0 \notin S$

3. True or False? The Least Upper Bound Property implies that the set $\{x \in \mathbb{R} : |2 - \sin(x)| \leq \frac{1}{2}\}$ has a supremum.



ineq has no soln's for $x \in \mathbb{R}$

∴ this set is empty

↑ LUB prop does not apply

4. Evaluate $\lim_{x\to-\infty} \frac{\sqrt{5x^2+5}}{3x-1}$.

$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $a, b>0$

$= \lim_{x\to-\infty} \frac{\frac{\sqrt{5x^2+5}}{x}}{\frac{3x+1}{x}}$ by Cancellation Thm

We can assume $x<0 \Rightarrow x = -\sqrt{x^2} = -|x|$

$= \lim_{x\to-\infty} \frac{\frac{\sqrt{5x^2+5}}{-\sqrt{x^2}}}{\frac{3x+1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{5x^2+5}{x^2}}}{\frac{3x+1}{x}}$

$5x^2+5 \geq 5 > 0$
$x^2 > 0$
square root prop

$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{5x^2}{x^2}+\frac{5}{x^2}}}{\frac{3x}{x}+\frac{1}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{5+\frac{5}{x^2}}}{3+\frac{1}{x}}$ ($\frac{5}{x^2} \to 0$, $\frac{1}{x} \to 0$)

$\frac{x^2}{x^2} = 1$ b/c $x<0$

$\frac{x}{x} = 1$ b/c $x<0$

$= \frac{-\sqrt{5+0}}{3+0}$ div, comp $\sqrt{\cdot}$, add lim rules

$\lim_{x\to-\infty} \frac{1}{x} = 0$

$= \frac{-\sqrt{5}}{3}$

5. Use the $\epsilon - \delta$ definition for limits to prove that $\lim_{x\to 3}(7 - 5x + x^2) = 1$.

WTS
$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $0 < |x-3| < \delta \Rightarrow |7-5x+x^2-1| < \varepsilon$

Let $\varepsilon > 0$ be arbitrary.

Choose $\delta = \min\{1/2, \frac{2}{3}\varepsilon\} > 0$ b/c $1/2 > 0$ and $\frac{2}{3}\varepsilon > 0$

$\Rightarrow \delta \leq \frac{1}{2}$ and $\delta \leq \frac{2}{3}\varepsilon$

Suppose $x \in \mathbb{R}$ s.t. $0 < |x-3| < \delta$.

Then $|x-2| = |x-3+1| \leq |x-3| + |1|$ by Δ-ineq
$< \delta + 1$
$\leq \frac{1}{2} + 1$ choice δ from above
$= 3/2$ (*)

Consider $|7-5x+x^2-1| = |x^2-5x+6|$
$= |(x-3)(x-2)| = |x-3| \cdot |x-2|$ by 1.1 prop
$< \delta \cdot |x-2|$ by our assumption
$< \delta \cdot \frac{3}{2}$ by (*)
$\leq \frac{2}{3}\varepsilon \cdot \frac{3}{2}$ by choice of δ
$= \varepsilon$

6. Use the definition of derivative to prove $\frac{d}{dx}\sin x = \cos x$. You may use the following known limits:

$$\begin{cases} \lim_{h\to 0} \dfrac{\sin h}{h} = 1 \\ \lim_{h\to 0} \dfrac{\cos h - 1}{h} = 0 \end{cases}$$

Let $x \in \mathbb{R}$.

$$\frac{d}{dx}\sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \quad \text{defn of deriv.}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} \quad \text{sin add rule}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)\cdot 1 + \sin(h)\cos(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\left(\cos(h) - 1\right) + \sin(h)\cos(x)}{h}$$

$$= \lim_{h\to 0} \sin(x)\frac{\cos(h)-1}{h} + \lim_{h\to 0} \frac{\sin(h)}{h}\cos(x) \quad \text{lim add, note both lims exist}$$

$$= \left(\lim_{h\to 0}\sin(x)\right)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\cos(x) \quad \text{Lim mult, note both lims exist}$$

$$= \sin(x)\cdot 0 + 1\cdot\cos(x) \quad \text{known Limits}$$

$$= \cos(x)$$

$$\ln(a^b) = b\ln(a)$$

7. Find the derivative of $y = (3x^2+1)^{3x^2+1}$.

ln both sides

$$\Rightarrow \quad \ln(y) = \ln\left((3x^2+1)^{3x^2+1}\right)$$

note ln is defined here, $3x^2+1 \geq 1 > 0$

$$\Rightarrow \quad \ln(y) = (3x^2+1)\ln(3x^2+1) \qquad \text{ln prop}$$

$\frac{d}{dx}$ both sides

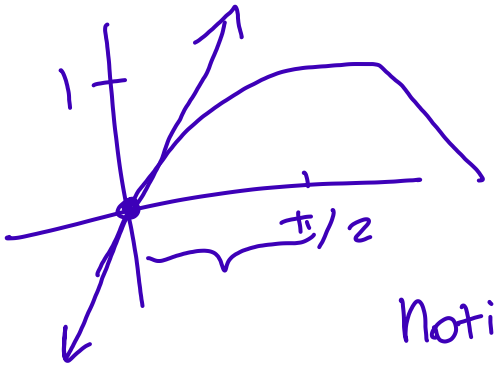
$$\Rightarrow \quad \frac{1}{y}\cdot y' = (6x)\ln(3x^2+1) + \cancel{(3x^2+1)\cdot\frac{1}{3x^2+1}\cdot 6x}$$

chain rule, prod rule

$$\Rightarrow \quad y' = y\left(6x\ln(3x^2+1) + 6x\right) \qquad \text{note } 3x^2+1 \neq 0$$

$$= (3x^2+1)^{3x^2+1}\cdot 6x\left(\ln(3x^2+1)+1\right)$$

8. Prove that the equation $\sin x = x$ has ~~exactly one~~ real solution for $x \in [0, \pi/2]$.



Consider a function

$g(x) = \sin x - x = 0$ over $[0, \pi/2]$

Note its roots correspond to sol'ns.

Notice $g(0) = \sin 0 - 0 = 0$ is a root.

For $x \in (0, \pi/2]$,

$g'(x) = \cos x - 1$ deriv rules

$< 1 - 1$ $\cos x < 1$ for $x \in (0, \frac{\pi}{2}]$

$= 0$

Therefore g is strictly decreasing over $(0, \frac{\pi}{2}]$.

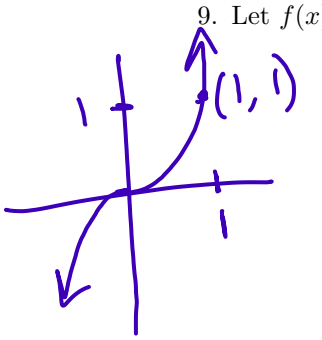
$\Rightarrow$ g is strictly decreasing over $[0, \frac{\pi}{2}]$.

For contradiction, sps $\exists c \in (0, \frac{\pi}{2}]$ s.t. $g(c) = 0$

Notice: $c > 0$, but $g(c) = g(0) = 0$ which contradicts the def of strictly increasing.

$\therefore$ $x = 0$ is the only root/sol'n in $[0, \frac{\pi}{2}]$

9. Let $f(x) = x^3$ on $[0, 1]$. Prove there exists $c \in (0, 1)$ such that $f'(c) = 3/2$.



$\Rightarrow f'(x) = 3x^2$ by deriv rules

Note $f'(0) = 3 \cdot 0^2 = 0$

$f'(1) = 3 \cdot 1^2 = 3$

$f'(x)$ is cont over $[0, 1]$ b/c it's a polyn.

Note $f'(0) = 0 < \frac{3}{2} < 3 = f'(1)$

By IVT, $\exists c \in (0, 1)$ s.t.

$f'(c) = \frac{3}{2}$