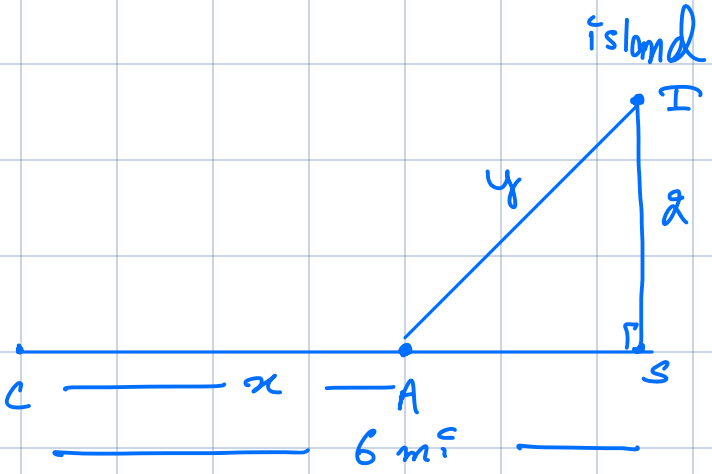
Optimization

- Make a function(s) for the problem → Convert it into one-variable function
  (optimization / constraint; usually the hardest/main part)
- Know the interval / bounds on our variable
- Determine the absolute extrema (critical points / endpoints)

Q1 An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?



Let I be the island

Let S be the closest point to the island on the shoreline

Let C represent the cabin

Let A be the unknown point of where the visitor starts swimming

distance = speed × time

time = $\frac{\text{distance}}{\text{speed}}$

time taken to reach A from C = $\frac{x}{8}$

time taken to reach I from A = $\frac{y}{3}$

total time taken = $T = \frac{x}{8} + \frac{y}{3}$ → Minimize

From pythagoras theorem on ΔISA,

$$(2)^2 + (6-x)^2 = y^2$$

$$\Rightarrow 4 + (6-x)^2 = y^2$$

$$\Rightarrow \sqrt{4+(6-x)^2} = y \quad [\text{since } y > 0]$$

$$\therefore T(x) = \frac{x}{8} + \frac{\sqrt{4+(6-x)^2}}{3}$$

Constraints on x, $x \geq 0$, $x \leq 6$

∴ Finding absolute min. of $T(x)$ on $[0, 6]$

$T(x)$ is a cont. function over $[0, 6]$

$$T'(x) = \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2\sqrt{4+(6-x)^2}} \cdot 2(6-x)(-1)$$

$$= \frac{1}{8} - \frac{1}{3} \cdot \frac{(6-x)}{\sqrt{4+(6-x)^2}}$$

To find critical points, $T'(x) = 0$

$$\Rightarrow \frac{1}{8} - \frac{1}{3}\cdot\frac{(6-x)}{\sqrt{4+(6-x)^2}} = 0$$

$$\Rightarrow \frac{1}{8} = \frac{1}{3}\cdot\frac{(6-x)}{\sqrt{4+(6-x)^2}}$$

$$\Rightarrow \left(\frac{3}{8}\right)^2 = \frac{(6-x)^2}{4+(6-x)^2}$$

$$\Rightarrow \frac{64}{9} = \frac{4+(6-x)^2}{(6-x)^2}$$

$$\Rightarrow \frac{64}{9} = \frac{4}{(6-x)^2} + 1$$

$$\Rightarrow \frac{64}{9} - 1 = \frac{4}{(6-x)^2}$$

$$\Rightarrow \frac{55}{9} = \frac{4}{(6-x)^2}$$

$$\Rightarrow (6-x)^2 = \frac{36}{55}$$

$$\Rightarrow (x-6)^2 = \frac{36}{55}$$

$$\Rightarrow x = 6 \pm \sqrt{\frac{36}{55}}$$

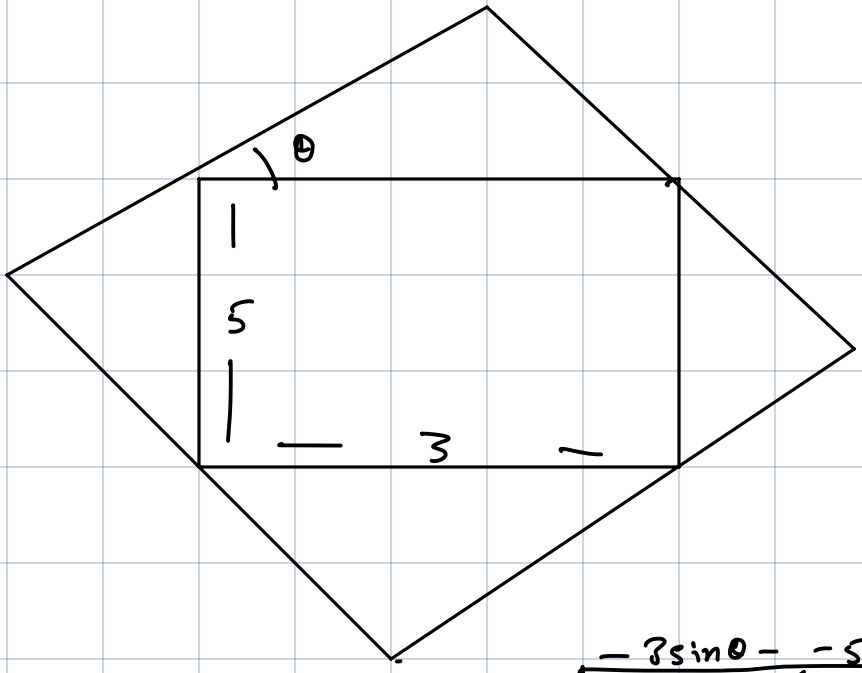
Note, $x = 6 + \sqrt{\frac{36}{55}} \notin [0, 6]$, hence

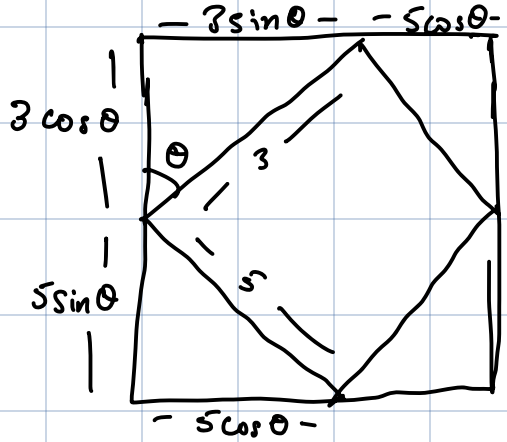
$$x = 6 - \sqrt{\frac{31}{55}}$$

$T(0)$, $T\left(6 - \sqrt{\frac{31}{55}}\right)$, $T(6)$

$= \sqrt{\frac{10}{3}}$ ≈ 1.368 $\frac{6}{8} + \sqrt{\frac{4}{3}}$

Ques 2 Find the max. area of a rectangle circumscribed around a rectangle of sides $L = 5$ and $H = 3$





$$A(\theta) = (5\sin\theta + 3\cos\theta)\cdot(3\sin\theta + 5\cos\theta)$$

$$\theta \in \left[0, \frac{\pi}{2}\right]$$

$$= 15\sin^2\theta + 25\sin\theta\cos\theta + 9\cos\theta\sin\theta + 15\cos^2\theta$$

$$= 15 + 17\sin(2\theta)$$

$$A'(\theta) = 17\cos(2\theta)\cdot(2)$$

$$= 34\cos(2\theta)$$

$$A'(\theta) = 0 \quad , \quad \cos(2\theta) = 0 \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$A(0) = 15$$

$$A\left(\frac{\pi}{4}\right) = \left(\frac{8}{\sqrt{2}}\right)\left(\frac{8}{\sqrt{2}}\right) = \frac{64}{2} = 32$$

$$A\left(\frac{\pi}{2}\right) = 15$$

A rectangular box with a square base, an open top, and a volume of 216 inch cubic is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Let x be the side of the square base

y be the height of the square base

$$S = x^2 + 4xy$$

$$V = x^2 y = 216 \quad , x>0, y>0$$

$$\Rightarrow y = \frac{216}{x^2}$$

$$\therefore S = x^2 + 4x\left(\frac{216}{x^2}\right)$$

$$= x^2 + \frac{864}{x}$$

$$x \in (0, \infty)$$

$$S'(x) = 2x - \frac{864}{x^2}$$

$$S'(x) = 0$$

$$\Rightarrow \quad 2x = \frac{864}{x^2}$$

$$\Rightarrow \quad x^3 = 432$$

$$\Rightarrow \quad x = \sqrt[3]{432} = 6\sqrt[3]{2}$$

$$S''(x) = 2 + \frac{864(2)}{x^3}$$

$$S''(6\sqrt[3]{2}) > 0$$

$\therefore$ $S(x)$ has a local min. at $6\sqrt[3]{2}$

By only critical point in town test, we can conclude $S(x)$ has an absolute min. at $6\sqrt[3]{2}$

$$\therefore \quad x = 6\sqrt[3]{2}, \quad y = \frac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}$$

$$\therefore \quad S(6\sqrt[3]{2}) = 108\sqrt{}$$