MATA31 TT2 Review

November 5, 2025

1 Assignment 5 (Supremum, Def'n of Limits)

1. (×) Show by example that $\lim_{x\to c}[f(x)+g(x)]$ can exist even if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

2. Prove that if
$$\lim_{x\to a} f(x) = L$$
then
$$\lim_{x\to a} |f(x)| = |L|.$$

3. Prove that for the function
$$h(x) = \begin{cases} x, & \text{if } x \text{ rational} \\ 0, & \text{if } x \text{ irrational} \end{cases}$$
$\lim_{x\to 0} h(x) = 0$.

4. Find the supremum and infimum of the following sets if they exist.

   (a)
   $$A = \left\{ \frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0 \right\}$$

   (b)
   $$B = \left\{ x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2} \right\}$$

   (c)
   $$C = \left\{ x \in \mathbb{R} : x^2 + x + 1 \geq 0 \right\}$$

5. Let S be a non-empty subset of $\mathbb{R}$, and let $\alpha \in \mathbb{R}$ be an upper bound for S. Prove that α is the supremum of S if and only if for every $\varepsilon > 0$, there exists $x \in S$ such that $x > \alpha - \varepsilon$. (Can do the same for infimum)

6. (×) Let f be a function defined on an open interval containing a, and suppose that f is continuous at a with $f(a) > 0$. Using the precise definition of the limit, show that there exists an open interval centered at a such that $f(x) > 0$ for all x in that interval.

2 Assignment 6 (IVT, EVT)

1. For which values of the constant c is the function defined by

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2 \\ x^3 - cx, & \text{if } x \geq 2 \end{cases}$$

continuous on $(-\infty, +\infty)$?

2. Use the Intermediate Value Theorem to show that there is at least one root of the equation $\cos\sqrt{x} = e^x - 2$ in the interval $(0, 1)$.

3. ($\times$) Show that for a continuous function f which is defined on the interval $[0, 1]$ with the property that $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$, there is a point $c \in [0, 1]$ such that $f(c) = c$. [Basically proving IVT]

4. Use the Intermediate Value Theorem to show that every real number has a cube root.

5. The function

$$f(x) = \begin{cases} \frac{1}{x^2}, & \text{if } 0 < x \leq 2 \\ 5, & \text{if } x = 0 \end{cases}$$

does not take on a maximum value. Explain why this does not violate the Extreme Value Theorem.

6. ($\times$) Let f be continuous on $[a, b]$ with $f(a) < 0 < f(b)$. Show there exists $c \in (a, b)$ such that $f(c) = 0$ (without immediately using the Intermediate Value Theorem).

7. ($\times$) Use the Intermediate Value Theorem to prove that, at any given time, there exist two antipodal points on the Earth's surface (i.e., exactly opposite each other) with exactly the same temperature. (Hint: consider defining a function that takes the temperature at any point and subtracts the temperature at its antipodal point.)

3 Assignment 7 (Def'n of Derivatives)

1. Let $m \in \mathbb{R}$. Use the definition of the derivative to determine $f'(x)$ if $f(x) = \sqrt{3x + m}$.

2. Let $a, b \in \mathbb{R}$ and suppose that f and g are functions defined on $\mathbb{R}$ such that $g(x) = ax + bx^2 f(x)$ for all $x \in \mathbb{R}$. Prove using the definition that if f is differentiable on $\mathbb{R}$, then g is differentiable on $\mathbb{R}$, and in addition, $g'(x) = a + bx^2 f'(x) + 2bxf(x)$.

3. Show that

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not differentiable at $x = 0$.

4. ($\times$) Use the definition of the derivative to prove that the derivative of an odd function is an even function.

5. (×) The greatest integer function is define by

$$f(x) = \lfloor x \rfloor = \text{the largest integer that is less than or equal to } x$$

For example $\lfloor 4 \rfloor = 4, \lfloor 2.8 \rfloor = 2, \lfloor -1 \rfloor = -1, \lfloor -3.5 \rfloor = -4$. Sketch the curve of f. At which points is the function not differentiable? Why? Find a formula for f' and sketch the curve of f' for the values for which the function is differentiable.

6. Suppose that f is a function defined on $\mathbb{R}$ such that $|f(x)| \leq |x|^m$ for all $x \in \mathbb{R}$, where $m > 1$. Prove that f is differentiable at $x = 0$.

4 Fall 2024 TT2

1. (a) Assume f is continuous on $[-1, 2]$ and has values listed in the table below for some $a, b \in \mathbb{R}$.

x	-1	0	1	2
$f(x)$	4	a	b	-3

Which of the following values of a and b guarantees that f has at least three real roots?

i. $a = -1$ and $b = 1$
ii. $a = 2$ and $b = 1$
iii. $a = 1$ and $b = -1$
iv. $a = -2$ and $b = -1$
v. None of the above

(b) True or false? If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f'(4) = 5$, then $\lim_{x \to 4} xf(x)$ exists.

(c) Give an example of a set $S \subseteq \mathbb{Q}$ such that S has a supremum but S does not have a maximum.

(d) True or false? There is a real number that is exactly one less than its cube.

(e) True or false? The function $f(x) = |x|^3$ is differentiable at $x = 0$.

2. Compute the following, if they exist. Briefly justify each answer.

(a) $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{x^2 + 5x^4} - 1}{1 - 3x^2 - x}$

(b) $\displaystyle\lim_{x \to \infty} 4^{(\sin(x) - x)}$

3. Use the ε-δ definition for continuity to prove $f(x) = |4x + 5|$ is continuous on $\mathbb{R}$.

4. Apply the Squeeze Theorem to prove $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. Let $S = \left\{ 4 - \frac{5}{n} : n \in \mathbb{N} \right\}$. (Here, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers.)

(a) Show $\sup(S)$ exists.

(b) Use contradiction and the Archimedean property to prove the following statement: If b is an upper bound of S, then $b \geq 4$.

(c) Prove $\sup(S) = 4$ using the definition of supremum.

6. Let $c \in \mathbb{R}$ and f be a positive function with domain $\mathbb{R}$ that is differentiable at c.
Prove using the definition of derivative that $g(x) = \dfrac{1}{f(x)}$ is differentiable at c, and furthermore, $g'(c) = -\dfrac{f'(c)}{f(c)^2}$.