AMACSS MATA31 Term Test 2 Review

November 6, 2025

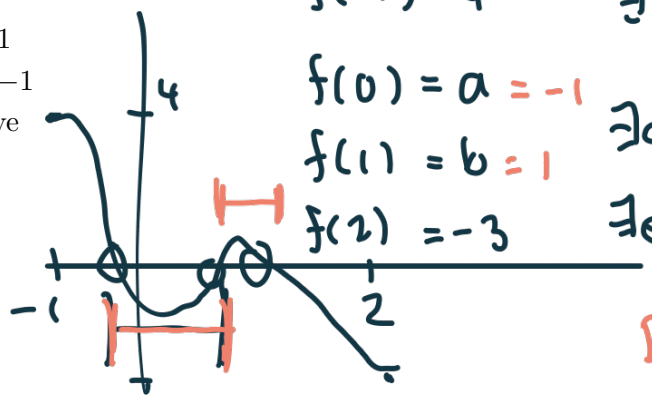
1. Choose the correct answer.

   (a) Assume f is continuous on $[-1, 2]$ and has values listed in the table below for some $a, b \in \mathbb{R}$.

x	-1	0	1	2
$f(x)$	4	a	b	-3

   Which of the following values of a and b guarantees that f has at least three real roots?

   i. $a = -1$ and $b = 1$
   ii. $a = 2$ and $b = 1$
   iii. $a = 1$ and $b = -1$
   iv. $a = -2$ and $b = -1$
   v. None of the above



$f(-1) = 4$

$f(0) = a = -1$

$f(1) = b = 1$

$f(2) = -3$

$\exists c \in (-1, 0),\ f(c) = 0$

$\exists d \in (0, 1),\ f(d) = 0$

$\exists e \in (1, 2),\ f(e) = 0$

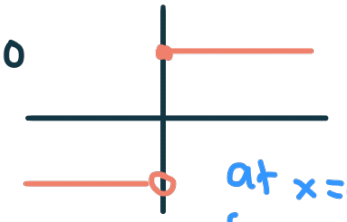
By IVT

IVT: f is continuous $f(a) > 0 > f(b)$, for some a, b, then $\exists c \in [a, b]$ such that $f(c) = 0$

(b) True or false? $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]$ can exist even if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

True. $f(x) = g(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$

$\frac{f(x)}{g(x)} = \begin{cases} \frac{1}{1} = 1, & x \geq 0 \\ \frac{-1}{-1} = 1, & x < 0 \end{cases}$



at $x=0$ $f(x)$ is discontinuous

$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = 1$ ($\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]$ does not exist, $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist?

(c) True or false? The function $f(x) = |x|^3$ is differentiable at x = 0.

Clue: formal definition of derivatives

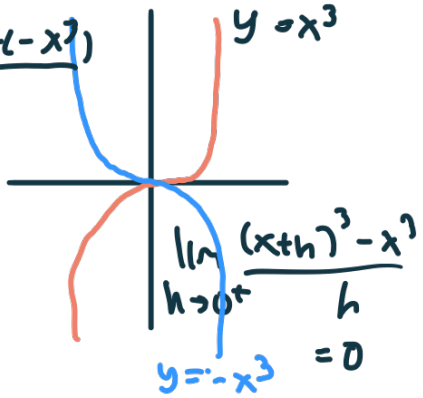
$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$\lim_{h\to 0^-} \frac{-(x+h)^3 - (-x^3)}{h} = 0$

$|x|^3 = \begin{cases} x^3, & \text{if } x \geq 0 \\ -x^3, & \text{if } x < 0 \end{cases}$ $\longrightarrow$ for $x \geq 0$,

$\lim_{h\to 0^+} \frac{(x+h)^3 - x^3}{h} = 0$

$\frac{d}{dx}|x|^3$



True

(d) True or false? If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f'(4) = 5$, then $\lim_{x\to 4} xf(x)$ exists

True. f differentiable $\Rightarrow$ f continuous!

(f not continuous $\Rightarrow$ f not differentiable)!

- f is diff. at $x=4$ $\Rightarrow$ f is continuous at $x=4$ $\Rightarrow \lim_{x\to 4} f(x)$ exists
- $\lim_{x\to 4} x = 4$, exists

$\therefore \lim_{x\to 4} xf(x) = \lim_{x\to 4} x \cdot \lim_{x\to 4} f(x)$

Exists

$|(3f(x)-4)+1| = |3f(x)-3|$

$= 3 \cdot |f(x)-1|$

$< 3 \cdot \frac{\varepsilon}{3}$

$= \varepsilon$

2. (a) Using the definition of limits, show that if

$$\lim_{x\to 5} f(x) = 1$$

then

$$\lim_{X\to 5} [3f(x) - 4] = -1.$$

Proof. WTP

$\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < |x-5| < \delta \Rightarrow |(3f(x)-4)-(-1)| < \varepsilon$

Take arbitrary $\varepsilon > 0$. Then

$\exists \delta_1 > 0$ such that $0 < |x-5| < \delta_1 \Rightarrow |f(x)-1| < \frac{\varepsilon}{3}$ ←

Choose $\delta = \delta_1$.

Suppose $0 < |x-5| < \delta$. Then $0 < |x-5| < \delta_1$.

Hence, $|(3f(x)-4)-(-1)| = |3f(x)-3|$

$= 3 \cdot |f(x)-1|$ by abs. value prop.

$< 3 \cdot \frac{\varepsilon}{3}$

$= \varepsilon$ as required to show. ∎

(b) Evaluate the limit

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+5x^4}-1}{1-3x^2-x}$$

$$\frac{\sqrt{x^2+5x^4}-1}{1-3x^2-x} = \frac{\sqrt{5x^4\left(\frac{1}{5x^2}+1-\frac{1}{5x^4}\right)}}{-3x^2\left(-\frac{1}{3x^2}+1+\frac{1}{3x}\right)} = -\frac{\sqrt{5}x^2 \cdot \sqrt{\frac{1}{5x^2}+1-\frac{1}{5x^4}}}{3x^2 \cdot \left(-\frac{1}{3x^2}+1+\frac{1}{3x}\right)}$$

$$= \frac{\sqrt{5}}{3} \cdot \frac{\sqrt{\frac{1}{5x^2}+1-\frac{1}{5x^4}}}{-\frac{1}{3x^2}+1+\frac{1}{3x}}$$

$\frac{1}{5x^2} \to 0$, $\frac{1}{5x^4} \to 0$, $\frac{1}{3x^2} \to 0$, $\frac{1}{3x} \to 0$

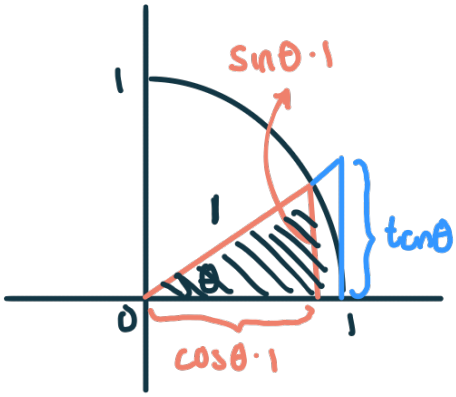
$$\lim_{x\to\pm\infty} \frac{1}{x} = 0$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+5x^4}-1}{1-3x^2-x} = -\frac{\sqrt{5}}{3} \cdot \frac{\sqrt{0+1-0}}{0+1+0}$$

$$= \boxed{-\frac{\sqrt{5}}{3}}$$

6. Evaluate and prove the value of

(Hint: Use squeeze theorem) $\lim_{x\to 0} \frac{\sin(3x)}{3x}$



Area $\triangle$ $= \frac{1}{2}\sin\theta\cos\theta$

Area $\triangle$ $= \frac{1}{2}\cdot 1\cdot\tan\theta$

Area $\triangle$ $= \frac{\theta}{2}$

$\triangle \le \triangle \le \triangle$

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ and $f(x) \le g(x) \le h(x)$ for all x, then $\lim_{x\to a} g(x) = L$

$\frac{1}{2}\sin\theta\cos\theta \le \frac{\theta}{2} \le \frac{\tan\theta}{2} \Rightarrow \sin\theta\cos\theta \le \theta \le \tan\theta$

$\Rightarrow \cos\theta \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

$\Rightarrow \frac{1}{\cos\theta} \ge \frac{\sin\theta}{\theta} \ge \cos\theta$

$\lim_{x\to 0} \frac{1}{\cos\theta} = \frac{1}{1} = 1$

$\lim_{x\to 0} \cos\theta = 1$

Therefore, by squeeze thm,

$$\boxed{\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1}$$

$\lim_{x\to 0} (3x) = 0$

$f(\theta) = \frac{\sin\theta}{\theta}$ Compute $\lim_{x\to 0} f(3x) = \lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$