1. **[12 points]**

$f(x_1,\dots,x_n)=(f_1(x_1,\dots,x_n),\dots,f_k(x_1,\dots,x_n))$

(a) Carefully complete the following definition:

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if the partial derivatives $\frac{\partial f_i}{\partial x_j}$, $i=1,\dots,k$, $j=1,\dots,n$ of f exist at a and if

$$\lim_{x\to a} \frac{\|f(x)-f(a)-Df(a)(x-a)\|}{\|x-a\|}=0,$$

where $Df(a)$ is the $k\times n$ matrix of partial derivatives of at $x=a$. If the above limit exists and is equal to zero, then the derivative of f is

$$Df(a)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(a) & \cdots & \frac{\partial f_k}{\partial x_n}(a) \end{bmatrix}$$

(b) Carefully state the following theorems.

i. The Chain Rule for functions of more than one variable.

Let $U\subset\mathbb{R}^n$, $V\subset\mathbb{R}^m$ be open sets. Let $g: U\to V$ and $f: V\to\mathbb{R}^p$. Suppose g is differentiable at $x_0\in U$ and f is differentiable at $y_0=g(x_0)$. Then $(f\circ g)$ is differentiable at x_0 and

$$D(f\circ g)(x_0)=Df(g(x_0))\cdot Dg(x_0)$$

ii. The Extreme Value Theorem.

Let $f: D\subset\mathbb{R}^n\to\mathbb{R}$ be continuous on D, where D is closed and bounded in $\mathbb{R}^n$. Then f attains its absolute maximum and absolute minimum values on D.

iii. The Change of Variables Theorem for double integrals.

Let D and D^* be elementary regions in $\mathbb{R}^2$ and $T: D^*\to D$ be of class C^1 and one-to-one on D^*, except possibly on the boundary of D^*. Suppose $D=T(D^*)$. Then for any integrable function $f: D\to\mathbb{R}$, we have

$$\iint_D f(x,y)\,dx\,dy=\iint_{D^*} f\left(x(u,v),y(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

where $T(u,v)=(x(u,v),y(u,v))$.

2. **[5 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{2x^2 - 2xy + 4y^2}{x^2 + 2y^2} & , \text{ if } (x,y) \neq (0,0) \\ 2 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$$

Determine all values of (x,y) for which $f(x,y)$ is continuous.

If $(x,y) \neq (0,0)$, $f(x,y)$ is continuous as a ratio of two polynomial functions (at (x,y)) with non-zero denominator.

$(x,y) = (0,0)$: $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2 - 2xy + 4y^2}{x^2 + 2y^2}$

If $x = 0$, $y \to 0$, then $f(x,y) = \dfrac{4y^2}{2y^2} \xrightarrow[(y\to 0)]{} 2$

If $x = y \to 0$, then $f(x,x) = \dfrac{4x^2}{3x^2} \xrightarrow[x\to 0]{} \dfrac{4}{3}$

$\Rightarrow 2 \neq \frac{4}{3}$, the limit dne

$\Rightarrow f$ is not continuous at $(0,0)$.

Answer: f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ and discontinuous at $(0,0)$.

3. **[5 points]** Let S be the surface defined by the equation $z = \sqrt{\dfrac{5 + x^2 + 2y^4}{y^2 + x^4}}$. Find the tangent plane to S at the point $(1,-1,2)$.

Tangent Plane: $z = f(x_0,y_0) + \dfrac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \dfrac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0)$ ✓

or Define $F(x,y,z) = \sqrt{\dfrac{5+x^2+2y^4}{y^2+x^4}} - z$; $F(x,y,z) = 0$ - level curve

$\nabla F(x_0,y_0,z_0) = \left(\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}\right)$

Tangent Plane: $\dfrac{\partial F}{\partial x}(x_0,y_0,z_0)(x - x_0) + \dfrac{\partial F}{\partial y}(x_0,y_0,z_0)(y - y_0) + \dfrac{\partial F}{\partial z}(x_0,y_0,z_0)(z - z_0) = 0$.

$$z_x = \frac{1}{2}\left(\frac{5+x^2+2y^2}{y^2+x^4}\right)^{-1/2} \cdot \frac{2x(y^2+x^4) - (5+x^2+2y^4)\cdot(4x^3)}{(y^2+x^4)^2}$$

$$z_x(1,-1) = \frac{1}{2}\left(\frac{5+1+2}{2}\right)^{-1/2} \cdot \frac{2\cdot 2 - (5+1+2)\cdot 4}{2^2} = \boxed{-\frac{7}{4}}$$

$$z_y = \frac{1}{2}\left(\frac{5+x^2+2y^2}{y^2+x^4}\right)^{-1/2} \frac{8y^3(y^2+x^4) - (5+x^2+2y^4)\cdot 2y}{(y^2+x^4)^2}$$

$$z_y(1,-1) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{-8(2) - 8(-2)}{2^2} = \boxed{0}$$

Tangent Plane: $z = 2 - \frac{7}{4}(x-1) + 0(y+1) \Rightarrow 4z = 8 - 7(x-1)$

$4z = 8 - 7x + 7 \Rightarrow \boxed{7x + 4z = 15}$

4. **[7 points]** Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2^2$$
$$= (1-\lambda-2)(1-\lambda+2)$$
$$= (-1-\lambda)(3-\lambda)$$

$\lambda_1 = -1$, $\lambda_2 = 3$ $\Rightarrow$ 2 eigenvalues

   (b) If A is the Hessian matrix for $f : \mathbb{R}^2 \to \mathbb{R}$ at a critical point in the domain of f, use the eigenvalues from part (a) to classify the critical point.

$\lambda_1 = -1 < 0$, $\lambda_2 = 3 > 0$ $\rightarrow$ mixed definite $\Rightarrow$ a saddle point

5. **[10 points]**

(a) Compute the 4th degree Taylor polynomial about the origin of $f(x,y) = e^{y^2}\sin(x+y)$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad , |x| < \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad , |x| < \infty$$

$$e^{y^2} = 1 + y^2 + \frac{y^4}{2!} + O(y^6)$$

$$\sin(x+y) = (x+y) - \frac{(x+y)^3}{3!} + O((x+y)^5)$$

$$e^{y^2}\cdot \sin(x+y) = \left(1+y^2+\frac{y^4}{2!} + O(y^6)\right)\left((x+y) - \frac{(x+y)^3}{3!} + O((x+y)^5)\right)$$

$$= (x+y) - \frac{(x+y)^3}{3!} + y^2(x+y) + \underbrace{\dots}_{\text{higher-order terms of degree} \geq 5}$$

$$T_4(x,y) = (x+y) - \frac{(x+y)^3}{6} - y^2(x+y) \quad \checkmark$$

(b) Find the linear approximation to the function $f(x,y) = \dfrac{x+2}{4y-2}$ at the point $(2,3)$ and use it to estimate $f(2.1, 2.9)$.

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)$$

$$f(2,3) = \frac{4}{10} = \frac{2}{5}$$

$$\frac{\partial f}{\partial x} = \frac{1}{4y-2} \Rightarrow \frac{\partial f}{\partial x}(2,3) = \frac{1}{10}$$

$$\frac{\partial f}{\partial y} = \frac{(x+2)(-1)\cdot(4)}{(4y-2)^2} \Rightarrow \frac{\partial f}{\partial y}(2,3) = \frac{4\cdot(-4)}{10^2} = \frac{-4}{25}$$

$$f(x,y) \approx \frac{2}{5} + \frac{1}{10}(x-2) - \frac{4}{25}(y-3)$$

$$f(2.1, 2.9) \approx \frac{2}{5} + \frac{1}{10}(2.1-2) - \frac{4}{25}(2.9-3)$$

$$= 0.4 + 0.01 + 0.016 = \boxed{0.426}$$

6. **[8 points]** Let $f(x,y,z) = x^3 + x^2 + y^2 + z^2 - xy + xz$. Find all the critical points of f. Characterize each critical point as a local maximum, a local minimum, or a saddle point.

$\nabla f = (3x^2+2x-y+z,\ 2y-x,\ 2z+x)$ ✓

$$\begin{cases} 3x^2+2x-y+z=0 \\ 2y-x=0 \\ 2z+x=0 \end{cases} \qquad \begin{cases} 3x^2+2x-\frac{x}{2}-\frac{x}{2}=0 \\ y=\frac{x}{2} \\ z=-\frac{x}{2} \end{cases}$$

$3x^2+x=0$

$x(3x+1)=0$

$x_1=0,\ y_1=0,\ z_1=0$ $\qquad$ $x_2=-\frac{1}{3},\ y_2=-\frac{1}{6},\ z_2=\frac{1}{6}$

Critical points: $(0,0,0), (-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})$

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \vdots & & \end{bmatrix} = \begin{bmatrix} 6x+2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \leftarrow \text{the Hessian matrix}$$

At $(0,0,0)$: $Hf(0,0,0) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\det A_1 = 2 > 0$

$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$

$\det A_3 = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= -2 + 2 \cdot (4-1) = 4 > 0$

$+ + +$ positive definite

$\Rightarrow (0,0,0)$ is a local min.

At $(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})$; $Hf(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\det A_1 = 0$

$\det A_2 = -1 < 0$

$\det A_3 = -4 < 0 \ (\neq 0)$

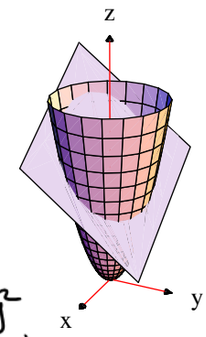
$0 - -$

mixed definite

$(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6})$ is a saddle point.

7. **[9 points]** Find the points on the intersection of the paraboloid $z = x^2 + y^2$ and the plane $x + y + z = 12$ that are closest to and farthest from the origin.

   Justify your answer including an explanation of why global extrema do exist.



Combining two equations $z = 12-x-y$ and $z = x^2+y^2$,

we get $\begin{cases} x^2+y^2 = 12-x-y \\ z = 12-x-y \end{cases}$ or $\begin{cases} (x+\frac{1}{2})^2 + (y+\frac{1}{2})^2 = 12.5 \\ z = 12-x-y \end{cases}$

Therefore, the intersection is a circle given by

$D = \{(x,y,z) \mid (x+\frac{1}{2})^2 + (y+\frac{1}{2})^2 = 12.5,\ z = 12-x-y\}$, which is closed and bounded in $\mathbb{R}^2$.

Let (x,y,z) be a point on this circle. The distance to the origin is $d = \sqrt{x^2+y^2+z^2}$. We will replace it by $f(x,y,z) = d^2 = x^2+y^2+z^2$.

We need to find the absolute extrema of f on D, which is closed and bounded. Since f is continuous everywhere in $\mathbb{R}^2$, by the EVT, absolute max and min exist on D.

We will apply the method of Lagrange multipliers with two constraints.

Let $h(x,y,z) = x+y+z = 12$, $k(x,y,z) = x^2+y^2-z = 0$.

Then $\nabla f = (2x, 2y, 2z)$, $\nabla g = (1,1,1)$, $\nabla h = (2x, 2y, -1)$.

The Lagrange multipliers equations are

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g(x,y,z) = 12 \\ h(x,y,z) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x = \lambda + 2\mu x & (1) \\ 2y = \lambda + 2\mu y & (2) \\ 2z = \lambda - \mu & (3) \\ x+y+z = 12 & (4) \\ x^2+y^2-z = 0 & (5) \end{cases}$$

(1)-(2) gives:

$2(x-y) = 2\mu(x-y)$

or $(x-y)(1-\mu) = 0$

Either $x=y$ or $\mu = 1$.

If $x=y$, then from (4), (5) we get

$\begin{cases} 2x+z = 12 \\ 2x^2-z = 0 \end{cases}$ ⊕

$2x^2+2x = 12$

$x^2+x-6 = 0$

$(x-2)(x+3) = 0$

$x_1 = 2$, $x_2 = -3$

$y_1 = 2$, $y_2 = -3$

$z_1 = 12-2x_1 = 8$, $z_2 = 12-2x_2 = 18$

We got 2 critical points $(2,2,8)$ and $(-3,-3,18)$ in this case with $f(2,2,8) = 4+4+64 = 72$, $f(-3,-3,18) = 9+9+324 = 342$.

If $\mu = 1$, then from (1) and (3) we get $\begin{cases} 2x+\lambda = 2x \\ 2z = \lambda - 1 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ z = -\frac{1}{2} \end{cases}$

Then equation (5) becomes $x^2+y^2+\frac{1}{2} = 0$, which is impossible

Therefore, $(2,2,8)$ is the closest point on the circle to the origin, and $(-3,-3,18)$ is the farthest to the origin.

8. **[10 points]** Find the maximum value of $x^2 - 4x + y^2 - 2y + z^2 - 4z - 1$ on the solid ball $x^2 + y^2 + z^2 \leq 9$.

Justify your answer including an explanation of why global extrema do exist.

Let $f(x,y,z) = x^2 - 4x + y^2 - 2y + z^2 - 4z - 1$.
Absolute maximum and minimum of f exist on the solid ball $x^2+y^2+z^2 \leq 9$ by the EVT, since f is continuous in $\mathbb{R}^2$ and the ball is a closed and bounded set in $\mathbb{R}^2$.

Interior: $x^2+y^2+z^2 < 9$

Critical points of f are zeros of the derivative.

$\nabla f = (2x-4, 2y-2, 2z-4)$

$\nabla f = 0 \iff \begin{cases} x=2 \\ y=1 \\ z=2 \end{cases} \Rightarrow$ Critical point $(2,1,2)$, but $2^2+1^2+2^2 = 9 \Rightarrow (2,1,2)$ is not in the interior.

$\Rightarrow$ There are no critical points in the interior.

Boundary: We apply Lagrange multipliers with one constraint: $g(x,y,z) = x^2+y^2+z^2 = 9$

$\nabla g = (2x, 2y, 2z)$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 9 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} 2x-4 = 2\lambda x \\ 2y-2 = 2\lambda y \\ 2z-4 = 2\lambda z \\ x^2+y^2+z^2 = 9 \end{cases} \iff \begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 1 \\ z(1-\lambda) = 2 \\ x^2+y^2+z^2 = 9 \end{cases} \quad \lambda \neq 1 \Rightarrow \quad x = z = \frac{2}{1-\lambda}, \quad y = \frac{1}{1-\lambda}$$

$$\frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 9$$

$$\frac{9}{(1-\lambda)^2} = 9$$

$$(1-\lambda)^2 = 1$$

$$1-\lambda = \pm 1$$

$$\lambda_1 = 0, \quad \lambda_2 = 2.$$

If $\lambda_1 = 0 \Rightarrow \begin{cases} x=2 \\ y=1 \\ z=2 \end{cases}$

If $\lambda_2 = 2 \Rightarrow \begin{cases} x=-2 \\ y=-1 \\ z=-2 \end{cases}$

We got 2 critical points $(2,1,2)$ and $(-2,-1,-2)$ with

$f(2,1,2) = 4-8+1-2+4-8-1 = -10$

$f(-2,-1,-2) = 4+8+1+2+4+8-1 = 26$.

Therefore, $\max_{x^2+y^2+z^2 \leq 1} f = 26$ at $(-2,-1,-2)$.

9. **[5 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $f(x, y, z) = (xy^2, yz^2, x^2z, xyz)$ and let $g : \mathbb{R}^4 \to \mathbb{R}$ be given by $g(x, y, z, w) = xy + zw$. Use the Chain Rule to compute the gradient of $g \circ f$.

Solution 1. $g(f(x,y,z)) = g(xy^2, yz^2, x^2z, xyz) = xy^3z^2 + x^3yz^2$

$D(g\circ f) = (y^3z^2 + 3x^2yz^2,\ 3xy^2z^2 + x^3z^2,\ 2xy^3z + 2x^3yz)$

Solution 2. $D(g\circ f) = Dg(f(x,y,z)) \cdot Df(x,y,z)$

$$Df = \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \\ yz & xz & xy \end{bmatrix} \quad ; \quad Dg = [\, y \ \ x \ \ w \ \ z \,]$$

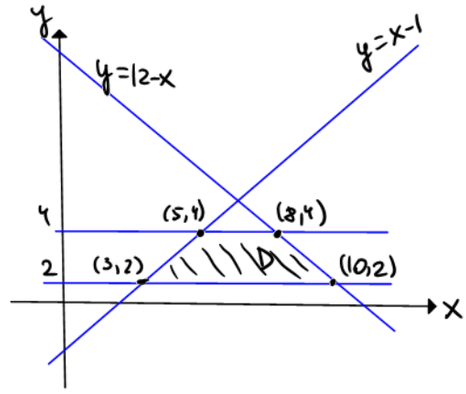
$$Dg(f(x,y,z)) = [\, yz^2 \ \ xy^2 \ \ xyz \ \ x^2z \,]$$

$$D(g\circ f) = [\, yz^2 \ \ xy^2 \ \ xyz \ \ x^2z \,] \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \\ yz & xz & xy \end{bmatrix} =$$

$$= [\, y^3z^2 + 3x^2yz^2 \ , \ 3xy^2z^2 + x^3z^2 \ , \ 2xy^3z + 2x^3yz \,] .$$

10. **[15 points]**

(a) Evaluate $\int_D e^{x+y}\, dA$, where D is the region bounded by $y = x - 1$ and $y = 12 - x$ for $2 \le y \le 4$.



Considering D as an x-region, we have

$D = \{(x,y) \in \mathbb{R}^2 \mid 2 \le y \le 4,\ y+1 \le x \le 12-y\}$.

Therefore,

$$\iint_D e^{x+y}\, dA = \int_2^4 \int_{y+1}^{12-y} e^x \cdot e^y \, dx\, dy$$

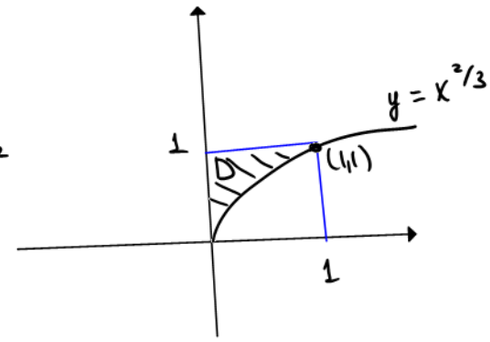
$$= \int_2^4 e^y \cdot e^x \Big|_{x=y+1}^{x=12-y} dy$$

$$= \int_2^4 e^y \left(e^{12-y} - e^{y+1}\right) dy = \int_2^4 \left(e^{12} - e^{2y+1}\right) dy = \left(e^{12}y - \frac{e^{2y+1}}{2}\right)\Big|_2^4$$

$$= 2e^{12} - \frac{1}{2}\left(e^9 - e^5\right) = 2e^{12} - \frac{e^9}{2} + \frac{e^5}{2} .$$

Question 10. (cont'd)

(b) Evaluate $\int_0^1 \int_{x^{2/3}}^1 x\, e^{y^4}\, dy\, dx$.



Domain D: $0 \le x \le 1$ $\Leftrightarrow$ $0 \le y \le 1$, $x^{2/3} \le y \le 1$ $\Leftrightarrow$ $0 \le x \le y^{3/2}$

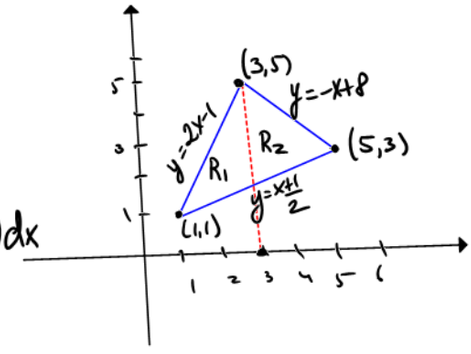
Changing the order of integration, we have

$$\int_0^1 \int_{x^{2/3}}^1 x e^{y^4}\, dy\, dx = \int_0^1 \int_0^{y^{3/2}} x e^{y^4}\, dx\, dy$$

$$= \int_0^1 e^{y^4} \cdot \frac{1}{2} x^2 \Big|_{x=0}^{x=y^{3/2}} dy = \frac{1}{2} \int_0^1 e^{y^4} \cdot y^3\, dy$$

$$= \frac{1}{2} \frac{e^{y^4}}{4} \Big|_0^1 = \frac{1}{8}(e-1)$$

(c) Integrate $f(x,y) = x+1$ over the interior of the triangle with vertices $(1,1)$, $(3,5)$ and $(5,3)$.



$$\iint_D (x+1)\, dx\, dy = \int_1^3 \int_{\frac{x+1}{2}}^{2x-1} (x+1)\, dy\, dx + \int_3^5 \int_{\frac{x+1}{2}}^{8-x} (x+1)\, dy\, dx$$

$$= \int_1^3 (x+1)\left(2x-1-\frac{x}{2}-\frac{1}{2}\right) dx + \int_3^5 (x+1)\cdot\left(8-x-\frac{x}{2}-\frac{1}{2}\right) dx$$

$$= \int_1^3 (x+1)\left(\frac{3}{2}x-\frac{3}{2}\right) dx + \int_3^5 (x+1)\left(\frac{15}{2}-\frac{3x}{2}\right) dx$$

$$= \frac{3}{2}\int_1^3 (x^2-1)\, dx + \frac{3}{2}\int_3^5 (x+1)(5-x)\, dx$$

$$= \frac{3}{2}\left(\frac{x^3}{3}-x\right)\Big|_1^3 + \frac{3}{2}\int_3^5 (-x^2+4x+5)\, dx$$

$$= \frac{3}{2}\left((9-3)-\left(\frac{1}{3}-1\right)\right) + \frac{3}{2}\left(-\frac{x^3}{3}+2x^2+5x\right)\Big|_3^5$$

$$= \frac{3}{2}\left(6+\frac{2}{3}\right) + \frac{3}{2}\left(-\frac{125}{3}+50+25-(-9+18+15)\right)$$

$$= \frac{3}{2}\cdot\frac{20}{3} + \frac{3}{2}\cdot\left(-\frac{125}{3}+75-24\right) = 10 + \frac{3}{2}\left(-\frac{125}{3}+51\right)$$

$$= 10 - \frac{125}{2} + \frac{153}{2} = 10+14 = \boxed{24}$$

11. [**8 points**] Find the volume of the solid B bounded by the parabolic cylinder $x = (y-4)^2 + 3$ and the planes $z = x + 2y - 4$, $z = x + 4y - 7$ and $x + 2y = 11$.

We make change of variables:

$$\begin{cases} x = X+3 \\ y = Y+4 \\ z = Z+9 \end{cases}$$

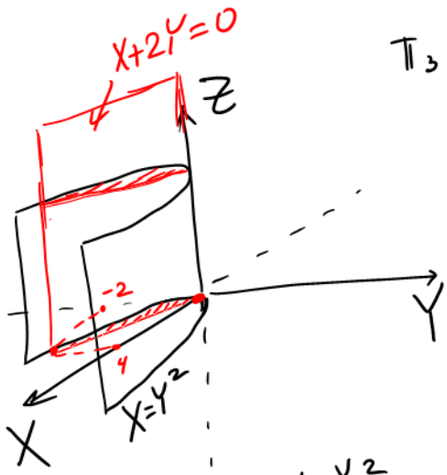
Then since we make only shifts, the jacobian matrix is the unit matrix and $\frac{\partial(x,y,z)}{\partial(X,Y,Z)} = 1$.

Then $V(B) = \iiint_B dx\,dy\,dz = \iiint_{B_1} dX\,dY\,dZ = V(B_1)$,

where B_1 is the solid bounded by the paraboloid $X = Y^2$ and the planes Π_1: $Z = X + 2Y - 2$

Π_2: $Z = X + 4Y + 3$

Π_3: $X + 2Y = 0$

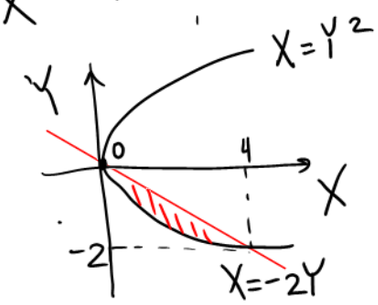


Interseection of the paraboloid $X = Y^2$ and the plane Π_3:

$Y^2 = -2Y$

$Y(Y+2) = 0 \Rightarrow Y = 0, X = 0$

$Y = -2, X = 4$



Then the solid B_1 is the set of points (X,Y,Z) such that

$-2 \le Y \le 0$

$Y^2 \le X \le -2Y$

$X + 2Y - 2 \le Z \le X + 4Y + 3$.

$$V(B_1) = \int_{-2}^{0}\int_{Y^2}^{-2Y}\int_{X+2Y-2}^{X+4Y+3} dZ\,dX\,dY = \int_{-2}^{0}\int_{Y^2}^{-2Y} (X+4Y+3-X-2Y+2)\,dX\,dY$$

$$= \int_{-2}^{0}\int_{Y^2}^{-2Y} (2Y+5)\,dX\,dY = \int_{-2}^{0} (2Y+5)(-2Y-Y^2)\,dY$$

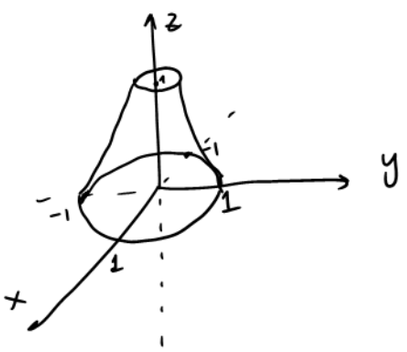
$$= \int_{-2}^{0} (-4Y^2 - 2Y^3 - 10Y - 5Y^2)\,dY = -\int_{-2}^{0} (2Y^3 + 9Y^2 + 10Y)\,dY$$

$$= -\left(\frac{Y^4}{2} + 3Y^3 + 5Y^2\right)\Big|_{-2}^{0} = -\left(0 - (8 - 24 + 20)\right)$$

$$= \boxed{4}$$

12. **[8 points]** Evaluate $\int_B z\, dV$, where B is the region bounded by the planes $z = 0$ and $z = 1$ and the surface $(z+1)\sqrt{x^2+y^2} = 1$.

The region is symmetrical about the z-axis, so we will use cylindrical coordinates.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Then equation of the surface is

$(z+1)\cdot r = 1 \Rightarrow r = \frac{1}{z+1}$ and

the region B is the set of points given by

$$B = \left\{ (r,\theta,z) \;\middle|\; 0 \le \theta \le 2\pi,\ 0 \le z \le 1,\ 0 \le r \le \frac{1}{z+1} \right\}.$$

Therefore, (jacobian)

$$\iiint_B z\, dV = \int_0^{2\pi}\int_0^1\int_0^{\frac{1}{z+1}} z\cdot r\, dr\, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 z\cdot\frac{1}{2}r^2\Big|_{r=0}^{r=\frac{1}{z+1}} dz\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{z}{(z+1)^2}\, dz\, d\theta = \pi\cdot\int_0^1 \frac{z}{(z+1)^2}\, dz$$

$$= \pi\int_0^1 \frac{z+1-1}{(z+1)^2}\, dz = \pi\int_0^1 \left(\frac{1}{z+1} - \frac{1}{(z+1)^2}\right) dz$$

$$= \pi\left(\ln(z+1) + \frac{1}{z+1}\right)\Big|_0^1 = \pi\left(\ln 2 + \frac{1}{2} - \ln 1 - 1\right)$$

$$= \pi\left(\ln 2 - \frac{1}{2}\right).$$

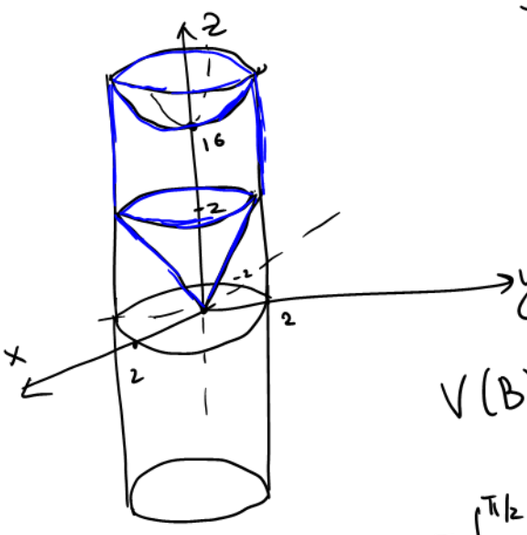
13. **[8 points]** Let B be the first octant region bounded by $z = x^2+y^2+16$, $z = \sqrt{x^2+y^2}$ and $x^2+y^2 = 4$. Use cylindrical or spherical polars to describe B and set up a triple integral to find the volume of B. What is the volume of B?

The shape is symmetrical with respect to the z-axis, we will use cylindrical coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Equation of the paraboloid $z = x^2+y^2+16$ in cylindrical coordinates is $z = r^2+16$.

The Cone $z = \sqrt{x^2+y^2}$ in cylindrical coordinates is $z = r$, and the cylinder $x^2+y^2 = 4$ is $r = 2$.



The region B in the first octant is given by

$$B = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le \tfrac{\pi}{2},\ r \le z \le r^2+16\}$$

The volume of B is

$$V(B) = \int_0^{\pi/2}\int_0^2\int_r^{r^2+16} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r\cdot(r^2+16-r)\,dr\,d\theta$$

$$= \frac{\pi}{2}\cdot\int_0^2 (r^3-r^2+16r)\,dr$$

$$= \frac{\pi}{2}\left(\frac{r^4}{4}-\frac{r^3}{3}+8r^2\right)\Big|_0^2$$

$$= \frac{\pi}{2}\left(4-\frac{8}{3}+32\right) = \frac{\pi}{2}\left(36-\frac{8}{3}\right) = \frac{\pi}{2}\cdot\frac{100}{3} = \boxed{\frac{50\pi}{3}}$$