AMACSS MATB41 Final Review Seminar

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Question 1

(a) Carefully complete the following definition:
Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is differentiable at $a \in U$ if

(b) Carefully state the following theorems.

(i) The Chain Rule for functions of more than one variable.

(ii) The Extreme Value Theorem.

(iii) The Change of Variables Theorem for double integrals.

Solution to Question 1

(a) Differentiability in $\mathbb{R}^n$.

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ and $a \in U$. We say that f is *differentiable at* a if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0.$$

This linear map is unique and is called the derivative at a, denoted $Df(a)$.

(b) Theorems.

Chain Rule. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ with $f(U) \subset V$. Suppose f is differentiable at $a \in U$ and g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a),$$

or, in matrix form, $D(g \circ f)(a) = Dg(f(a))\, Df(a)$.

Extreme Value Theorem. Let $D \subset \mathbb{R}^n$ be nonempty, closed and bounded, and let $f : D \to \mathbb{R}$ be continuous. Then there exist points $x_{\min}, x_{\max} \in D$ such that

$$f(x_{\min}) \leq f(x) \leq f(x_{\max}) \quad \text{for all } x \in D.$$

So f attains its global minimum and maximum on D.

Change of Variables Theorem (double integrals).
Let $D \subset \mathbb{R}^2$ be open and let $T : D \to \mathbb{R}^2$ be a C^1 map, one-to-one on D, with continuous partial derivatives and Jacobian determinant $J_T(u, v) = \det DT(u, v)$ that is never zero on D. Let $E = T(D)$ and let $f : E \to \mathbb{R}$ be integrable. Then

$$\iint_E f(x, y)\, dx\, dy = \iint_D f(T(u, v)) \left| J_T(u, v) \right| du\, dv.$$

Question 2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{2x^2 - 2xy + 4y^2}{x^2 + 2y^2}, & (x,y) \neq (0,0) \\ 2, & (x,y) = (0,0). \end{cases}$$

Determine all values of (x,y) for which $f(x,y)$ is continuous.

Solution to Question 2

- For $(x, y) \neq (0, 0)$ we have a quotient of polynomials with denominator $x^2 + 2y^2$. The only point where the denominator is 0 is $(0, 0)$, so f is continuous at every $(x, y) \neq (0, 0)$.

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- To check continuity at $(0, 0)$, rewrite

$$f(x, y) = 2 - \frac{2xy}{x^2 + 2y^2}.$$

Along the x–axis, $y = 0$ gives

$$f(x, 0) = 2.$$

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Along the x–axis, $y = 0$ gives

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- Along the line $y = x$:

$$f(x, x) = \frac{2x^2 - 2x^2 + 4x^2}{x^2 + 2x^2} = \frac{4x^2}{3x^2} = \frac{4}{3}.$$

Since

$$\lim_{x \to 0} f(x, 0) = 2, \quad \lim_{x \to 0} f(x, x) = \frac{4}{3},$$

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Hence, f is continuous everywhere on $\mathbb{R}^2$ except at $(0, 0)$.

Question 3

Let S be the surface defined by the equation

$$z = \sqrt{\frac{5 + x^2 + 2y^4}{y^2 + x^4}}.$$

Find the tangent plane to S at the point $(1, -1, 2)$.

Solution to Question 3 (1/3)

We are given the surface

$$z = f(x, y) = \sqrt{\frac{5 + x^2 + 2y^4}{y^2 + x^4}},$$

and we want the tangent plane at the point $(1, -1, 2)$.

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The general formula for the tangent plane to $z = f(x, y)$ at (a, b) is:

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

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So we need to compute f_x and f_y at $(1, -1)$.

Solution to Question 3 (2/3)

Write

$$f(x,y) = (g(x,y))^{1/2}, \qquad g(x,y) = \frac{5 + x^2 + 2y^4}{y^2 + x^4}.$$

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$$f(x,y) = (g(x,y))^{1/2}, \qquad g(x,y) = \frac{5 + x^2 + 2y^4}{y^2 + x^4}.$$

Then

$$f_x = \frac{1}{2\sqrt{g}}\, g_x, \qquad f_y = \frac{1}{2\sqrt{g}}\, g_y.$$

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Evaluating:

$$f(1,-1) = 2, \qquad f_x(1,-1) = -\frac{7}{4}, \qquad f_y(1,-1) = 0.$$

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Now plug these into the tangent plane formula.

Solution to Question 3 (3/3)

We use

$$z - 2 = f_x(1, -1)(x - 1) + f_y(1, -1)(y + 1).$$

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$$\boxed{7x + 4z = 15}.$$

This is the tangent plane to the surface at $(1, -1, 2)$.

Question 4

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) If A is the Hessian matrix for $f : \mathbb{R}^2 \to \mathbb{R}$ at a critical point in the domain of f, use the eigenvalues from part (a) to classify the critical point.

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$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}.$$

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Expand:

$$(1 - 2\lambda + \lambda^2) - 4 = \lambda^2 - 2\lambda - 3.$$

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Expand:

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So the characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0.$$

Solution to Question 4 (2/3)

We solve the quadratic:

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Since the eigenvalues have opposite signs, the matrix is **indefinite**. Next we classify the corresponding critical point of a function.

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If A is the Hessian matrix of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at a critical point, then the eigenvalues determine the type of critical point.

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- One eigenvalue is negative $\to$ the function curves downward in another direction.
- The Hessian is **indefinite**.

Therefore, the critical point is a

$$\boxed{\text{saddle point}}.$$

Question 5

(a) Compute the 4th degree Taylor polynomial about the origin of

$$f(x, y) = e^{y^2} \sin(x + y).$$

(b) Find the linear approximation to the function $f(x, y) = \dfrac{x + 2}{4y - 2}$ at the point $(2, 3)$ and use it to estimate $f(2.1, 2.9)$.

Solution to Question 5(a) (1/4)

We want the 4th degree Taylor polynomial of

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about the origin.

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We only keep terms up to total degree 4 when multiplying.

Solution to Question 5(a) (2/4)

Multiply the series:

$$f(x,y) = \left(1 + y^2 + \frac{y^4}{2}\right)\left((x+y) - \frac{(x+y)^3}{6}\right) + O(5).$$

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Step 3: Multiply by $\frac{y^4}{2}$:

$$\frac{y^4}{2}(x+y),$$

but these terms have degree 5, so they are discarded.

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Step 3: Multiply by $\frac{y^4}{2}$:

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but these terms have degree 5, so they are discarded.
Now expand the cubic:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Solution to Question 5(a) (3/4)

Combine all degree 4 terms:

$$f(x,y) \approx (x+y) - \frac{1}{6}(x^3 + 3x^2y + 3xy^2 + y^3) + y^2(x+y).$$

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Expand:

$$x + y - \frac{x^3}{6} - \frac{x^2y}{2} - \frac{xy^2}{2} - \frac{y^3}{6} + xy^2 + y^3.$$

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Simplify coefficients:

$$xy^2 - \frac{xy^2}{2} = \frac{xy^2}{2}, \qquad y^3 - \frac{y^3}{6} = \frac{5y^3}{6}.$$

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Simplify coefficients:

$$xy^2 - \frac{xy^2}{2} = \frac{xy^2}{2}, \qquad y^3 - \frac{y^3}{6} = \frac{5y^3}{6}.$$

$$\boxed{T_4(x,y) = x + y - \frac{x^3}{6} - \frac{x^2y}{2} + \frac{xy^2}{2} + \frac{5y^3}{6}.}$$

Solution to Question 5(b) (4/4)

Given

$$f(x,y) = \frac{x+2}{4y-2}, \qquad (a,b) = (2,3).$$

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Compute $f(a, b)$:

$$f(2, 3) = \frac{4}{10} = \frac{2}{5}.$$

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Compute $f(a,b)$:

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Partial derivatives:

$$f_x(x,y) = \frac{1}{4y-2} \quad \Rightarrow \quad f_x(2,3) = \frac{1}{10},$$

$$f_y(x,y) = -\frac{4(x+2)}{(4y-2)^2} \Rightarrow f_y(2,3) = -\frac{4}{25}.$$

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Linear approximation:

$$L(x,y) = \frac{2}{5} + \frac{1}{10}(x-2) - \frac{4}{25}(y-3).$$

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Estimate $f(2.1, 2.9)$:

$$L(2.1, 2.9) = 0.4 + 0.01 + 0.016 = 0.426.$$

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Question 6

Let $f(x, y, z) = x^3 + x^2 + y^2 + z^2 - xy + xz$.
Find all the critical points of f. Characterize each critical point as a local maximum, a local minimum, or a saddle point.

Solution to Question 6 (1/5)

We are given

$$f(x, y, z) = x^3 + x^2 + y^2 + z^2 - xy + xz.$$

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$$\nabla f = (f_x, f_y, f_z).$$

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Set $\nabla f = 0$:

$$\begin{cases} 3x^2 + 2x - y + z = 0, \\ 2y - x = 0, \\ 2z + x = 0. \end{cases}$$

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Set $\nabla f = 0$:

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We next solve this system for (x, y, z).

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From the last two equations:

$$2y - x = 0 \quad \Rightarrow \quad y = \frac{x}{2},$$

$$2z + x = 0 \quad \Rightarrow \quad z = -\frac{x}{2}.$$

Substitute into

$$3x^2 + 2x - y + z = 0.$$

Solution to Question 6 (2/5)

From the last two equations:

$$2y - x = 0 \quad \Rightarrow \quad y = \frac{x}{2},$$

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Substitute into

$$3x^2 + 2x - y + z = 0.$$

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Thus the critical points are:

$$\left(\quad , \quad , \quad \right)$$

Solution to Question 6 (3/5)

Compute the Hessian:

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}.$$

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Second partials:

$$f_{xx} = 6x + 2, \quad f_{yy} = 2, \quad f_{zz} = 2,$$

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Thus

$$H_f(x, y, z) = \begin{pmatrix} 6x + 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

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All principal minors are positive $\rightarrow$ Hessian is **positive definite**. Therefore:

$$\boxed{(0,0,0) \text{ is a local minimum.}}$$

Solution to Q6(5/5)

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At $\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$:

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The Hessian is **indefinite**. Therefore:

$$\boxed{\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right) \text{ is a saddle point.}}$$

Question 7

Find the points on the intersection of the paraboloid $z = x^2 + y^2$ and the plane $x + y + z = 12$ that are closest to and farthest from the origin. Justify your answer including an explanation of why global extrema do exist.

Solution to Question 7 (1/4)

We want extrema of the distance from the origin on the curve

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Let

$$g(x, y) = x + y + x^2 + y^2 - 12 = 0.$$

Use Lagrange multipliers on $D(x, y)$ with constraint $g(x, y) = 0$.

Solution to Question 7 (2/4)

Compute gradients:

$$\nabla D = (2x + 4x(x^2 + y^2),\ 2y + 4y(x^2 + y^2)),$$

$$\nabla g = (2x + 1,\ 2y + 1).$$

Solution to Question 7 (2/4)

Compute gradients:

$$\nabla D = (2x + 4x(x^2 + y^2),\ 2y + 4y(x^2 + y^2)),$$

$$\nabla g = (2x + 1,\ 2y + 1).$$

Set up Lagrange system:

$$\nabla D = \lambda \nabla g.$$

This gives:

$$\begin{cases} 2x(1 + 2(x^2 + y^2)) = \lambda(2x + 1), \\ 2y(1 + 2(x^2 + y^2)) = \lambda(2y + 1), \\ x + y + x^2 + y^2 = 12. \end{cases}$$

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We next solve for symmetry.

Solution to Question 7 (3/4)

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Compute corresponding $z = x^2 + y^2$:

$$(2, 2) : \ z = 4 + 4 = 8, (-3, -3) : \ z = 9 + 9 = 18.$$

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Compute distances:

$$D(2,2,8) = 2^2 + 2^2 + 8^2 = 4 + 4 + 64 = 72.$$

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- Distance is continuous.
- By the EVT, global max and min occur on this curve.

Question 8

Find the maximum value of

$$x^2 - 4x + y^2 - 2y + z^2 - 4z - 1$$

on the solid ball $x^2 + y^2 + z^2 \leq 9$.
Justify your answer including an explanation of why global extrema do exist.

Solution to Question 8

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$$\begin{cases} 2x - 4 = 0 \\ 2y - 2 = 0 \\ 2z - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \\ z = 2 \end{cases}$$

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  $\Rightarrow$ Critical point: $(2, 1, 2)$, but as $2^2 + 1^2 + 2^2 = 9$, it is not in the interior of D.
  $\Rightarrow$ The interior has no critical points.

Solution to Question 8 (cont'd)

- Boundary of D: $x^2 + y^2 + z^2 = 9$

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  $g(x, y, z) = x^2 + y^2 + z^2 = 9$
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$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x, y, z) = 9 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 2x - 4 = 2\lambda x \\ 2y - 2 = 2\lambda y \\ 2z - 4 = 2\lambda z \\ x^2 + y^2 + z^2 = 9 \end{cases}$$

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$\Leftrightarrow$

$$\begin{cases} x(1-\lambda) = 2 \\ y(1-\lambda) = 1 \\ z(1-\lambda) = 2 \\ x^2 + y^2 + z^2 = 9 \end{cases} \quad \xrightarrow{\lambda \neq 1} \quad \begin{cases} x = z = \frac{2}{1-\lambda} \\ y = \frac{1}{1-\lambda} \\ \frac{4}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 9 \end{cases}$$

Solution to Question 8 (cont'd)

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  If $\lambda = 0$, then $(x, y, z) = (2, 1, 2)$
  $\Rightarrow f(x, y, z) = 4 - 8 + 1 - 2 + 4 - 8 - 1 = -10$

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  If $\lambda = 2$, then $(x, y, z) = (-2, -1, -2)$
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  $\Rightarrow f(x, y, z) = 4 + 8 + 1 + 2 + 4 + 8 - 1 = 26$
- Therefore, $max(f) = 26$ in the solid ball $x^2 + y^2 + z^2 \leq 9$ at the point $(-2, -1, -2)$ on ∂D.

Question 9

Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$$f(x, y, z) = (xy^2, yz^2, x^2z, xyz)$$

and let $g : \mathbb{R}^4 \to \mathbb{R}$ be given by

$$g(x, y, z, w) = xy + zw.$$

Use the Chain Rule to compute the gradient of $g \circ f$.

Solution to Question 9

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$$= \begin{bmatrix} y^3z^2 + 3x^2yz^2, & 3xy^2z^2 + x^3z^2, & 2xy^3z + 2x^3yz \end{bmatrix}$$

Question 10

(a) Evaluate

$$\iint_D e^{x+y}\, dA,$$

where D is the region bounded by $y = x - 1$ and $y = 12 - x$ for $2 \leq y \leq 4$.

(b) Evaluate

$$\int_0^1 \int_{x^{2/3}}^1 x e^{y/4}\, dy\, dx.$$

(c) Integrate $f(x, y) = x + 1$ over the interior of the triangle with vertices $(1, 1), (3, 5), (5, 3)$.

Solution to Question 10 (a)

- Consider D as an x-region, we have
  $D = \{(x, y) \in \mathbb{R}^2 | 2 \leq y \leq 4\,,\, y + 1 \leq x \leq 12 - y\}$

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-

$$\iint_D e^{x+y}\, dA = \int_2^4 \int_{y+1}^{12-y} e^x \cdot e^y \, dx\, dy$$

$$= \int_2^4 [e^y \cdot e^x]_{x=y+1}^{x=12-y} \, dy$$

$$= \int_2^4 e^y (e^{12-y} - e^{y+1}) dy = \int_2^4 (e^{12} - e^{2y+1}) dy$$

$$= \left[e^{12} y - \frac{1}{2} e^{2y+1} \right]_{y=2}^{y=4} = 2e^{12} - \frac{1}{2}(e^9 - e^5)$$

$$= \boxed{2e^{12} - \frac{e^9}{2} + \frac{e^5}{2}}$$

Solution to Question 10 (b)

- Domain $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1\,,\, x^{2/3} \leq y \leq 1\}$

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- Domain $D = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1\,,\, x^{2/3} \leq y \leq 1\}$
  $\Rightarrow D = \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq 1\,,\, 0 \leq x \leq y^{3/2}\}$
- Change the order of integration:

$$\int_0^1 \int_{x^{2/3}}^1 xe^{y^4}\, dy\, dx = \int_0^1 \int_0^{y^{3/2}} xe^{y^4}\, dx\, dy$$

$$= \int_0^1 \left[e^{y^4} \cdot \frac{1}{2}x^2 \right]_{x=0}^{x=y^{3/2}} dy = \frac{1}{2}\int_0^1 y^3 e^{y^4}\, dy$$

$$= \frac{1}{2}\left[\frac{e^{y^4}}{4} \right]_0^1 = \boxed{\frac{1}{8}(e-1)}$$

Solution to Question 10 (c)

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  $R_1 = \{(x, y) \in \mathbb{R}^2 | 1 \leq x \leq 3\,,\, 2x - 1 \leq y \leq \frac{x+1}{2}\}$
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Solution to Question 10 (c)

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- Integrate over each triangle and take the sum:

$$\begin{aligned}
\iint_D (x+1)\,dx\,dy &= \int_1^3 \int_{\frac{x+1}{2}}^{2x-1} (x+1)\,dy\,dx + \int_3^5 \int_{\frac{x+1}{2}}^{8-x} (x+1)\,dy\,dx \\
&= \int_1^3 (x+1)(\frac{3x}{2} - \frac{3}{2})\,dx + \int_3^5 (x+1)(\frac{15}{2} - \frac{3x}{2})\,dx \\
&= \frac{3}{2}\int_1^3 (x^2 - 1)\,dx + \frac{3}{2}\int_3^5 (-x^2 + 4x + 5)\,dx
\end{aligned}$$

Solution to Question 10(c) (cont'd)

$$= \frac{3}{2}\left[\frac{x^3}{3} - x\right]_1^3 + \frac{3}{2}\left[-\frac{x^3}{3} + 2x^2 + 5x\right]_3^5$$

$$= \frac{3}{2}\left(\left((9-3) - (\frac{1}{3} - 1)\right) + \left((-\frac{125}{3} + 50 + 25) - (-9 + 18 + 15)\right)\right)$$

$$= 10 + 14$$

$$= \boxed{24}$$

Question 11

Find the volume of the solid B bounded by the parabolic cylinder

$$x = (y-4)^2 + 3$$

and the planes

$$z = x + 2y - 4, \qquad z = x + 4y - 7$$

and

$$x + 2y = 11.$$

Solution to Question 11

- We make change of variables:

$$\begin{cases} x = X + 3 \\ y = Y + 4 \\ z = Z \end{cases}$$

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- Since we only make shifts, the Jacobian matrix is the unit matrix:

Question 12

Evaluate

$$\iiint_B z \, dV,$$

where B is the region bounded by the planes $z = 0$ and $z = 1$ and the surface

$$(z+1)\sqrt{x^2+y^2} = 1.$$

Solution to Question 12

Question 13

Let B be the first octant region bounded by

$$z = x^2 + y^2 + 16, \qquad z = \sqrt{x^2 + y^2}$$

and

$$x^2 + y^2 = 4.$$

Use cylindrical or spherical polars to describe B and set up a triple integral to find the volume of B. What is the volume of B?

Solution to Question 13