MATB41 - Techniques of the Calculus of Several Variables I

Final Exam Review

Lecture Worksheet

1. Limits, Continuity, Differentiability.

Example: Investigate Continuity

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{2x^2 - 2xy + 4y^2}{x^2 + 2y^2}, & \text{if } (x,y) \neq (0,0) \\ 2, & \text{if } (x,y) = (0,0). \end{cases}$$

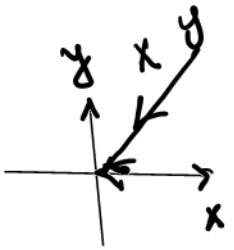
Determine all values of (x,y) for which $f(x,y)$ is continuous.

If $(x,y) \neq (0,0)$, f is continuous as a ratio of two polynomials with non-zero denominator.

If $(x,y) = (0,0)$,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^2 - 2xy + 4y^2}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \left(2 - \frac{2xy}{x^2+2y^2}\right) =$$

$$= 2 - \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$$



Along the x-axis $\Rightarrow$ $y=0$ $\Rightarrow$ $\dfrac{2xy}{x^2+2y^2} = \dfrac{2x\cdot 0}{x^2+0^2} \to 0$ as $(x,0)\to(0,0)$

Along the line $x=y$ $\Rightarrow$ $\dfrac{2x\cdot x}{x^2+2x^2} = \dfrac{2x^2}{3x^2} \to \dfrac{2}{3}$ as $(x,y)\to(0,0)$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+2y^2}$ does not exist

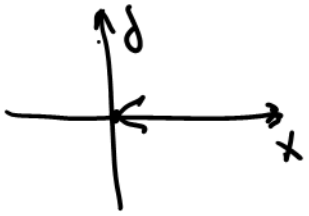
$\Rightarrow$ f is contin. on $\mathbb{R}^2 \setminus \{(0,0)\}$ and

discont at $(0,0)$.

Example: Limits

Evaluate the following limit or show that it does not exist. Justify your answer.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(2x) - 2x + y}{x^3 + y}.$$



Along the x-axis:

$$f(x,0) = \frac{\sin(2x) - 2x + 0}{x^3 + 0} = \frac{\sin(2x) - 2x}{x^3}$$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{\sin(2x) - 2x}{x^3}$$

$$= \lim_{x\to 0} \frac{(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots - 2x}{x^3}$$

Taylor's series

$$= \lim_{x\to 0} \frac{-\frac{8x^3}{6} + x^5(\dots)}{x^3}$$

(or Apply L'Hopital's rule 3 times)

$$= -\frac{8}{6} = -\frac{4}{3}$$

Along the y-axis:

$$\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0+y}{0+y} = \lim_{y\to 0} \frac{y}{y} = 1 \neq -\frac{4}{3}$$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

Example: Derivatives/Partial Derivatives

(a) Calculate the derivative of $g(x,y) = (e^{x+y} + y, y^2 x)$.

$g: \mathbb{R}^2 \to \mathbb{R}^2$

(b) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{4x^3 + 3xy}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Evaluate $\dfrac{\partial f}{\partial x}(0,0)$.

(a) $g(x,y) = (g_1(x,y), g_2(x,y))$, $g_1(x,y) = e^{x+y} + y$

$g_2(x,y) = y^2 x$

$\Rightarrow \dfrac{\partial g_1}{\partial x}, \dfrac{\partial g_1}{\partial y}, \dfrac{\partial g_2}{\partial x}, \dfrac{\partial g_2}{\partial y_2}$ are continuous

(as polynom. and exponential f. and their combinations)

$\Rightarrow g \in C^1(\mathbb{R}^2) \Rightarrow g$ is differentiable on $\mathbb{R}^2$

$$\Rightarrow Dg = \begin{bmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} & e^{x+y} + 1 \\ y^2 & 2xy \end{bmatrix}.$$

(b) $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x}$

$= \lim\limits_{x \to 0} \dfrac{\frac{4x^3}{x^2} - 0}{x} = \lim\limits_{x \to 0} \dfrac{4x^3}{x^3} = \boxed{4}$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

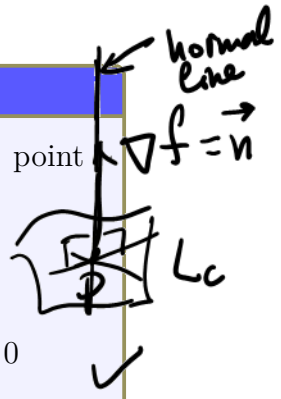
Tangent Plane to a Surface in $\mathbb{R}^3$

Let L_c be the level surface $f(x,y,z) = c$ in $\mathbb{R}^3$. The tangent plane of L_c at the point $P(x_0,y_0,z_0)$ of L_c is given by the equation

$$\nabla f(x_0,y_0,z_0)\cdot(x-x_0,y-y_0,z-z_0)=0$$

or

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0)\cdot(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0,z_0)\cdot(y-y_0)+\frac{\partial f}{\partial z}(x_0,y_0,z_0)\cdot(z-z_0)=0$$



Example: Tangent Plane to an Ellipsoid

Find the equations of the tangent plane and the normal line to the ellipsoid

$$\frac{x^2}{4}+y^2+\frac{z^2}{9}=3$$

at the point $P(-2,1,-3)$. $\in$ ellipsoid

$f(x,y,z) = \frac{x^2}{4}+y^2+\frac{z^2}{9}$; $f(x,y,z) = 3$ - Surface

$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{x}{2}, 2y, \frac{2z}{9}\right)$

$\nabla f(-2,1,-3) = \left(-1, 2, -\frac{2}{3}\right)$

$\left(-1,2,-\frac{2}{3}\right)\cdot(x+2, y-1, z+3) = 0$

$-(x+2)+2(y-1)-\frac{2}{3}(z+3)=0$

$-x-2+2y-2-\frac{2}{3}z-2=0$

$-x+2y-\frac{2}{3}z-6=0 \quad \times(-3)$

$\boxed{3x-6y+2z+18=0}$ ← Tangent Plane

Normal line:

$\nabla f(-2,1,-3) = \left(-1,2,-\frac{2}{3}\right)$

$P(-2,1,-3)$

$$\begin{cases} x = -2-t \\ y = 1+2t \\ z = -3-\frac{2}{3}t \end{cases}, t\in\mathbb{R}$$

$F(x,y,z) = z - f(x,y)$

$F(x,y,z) = 0$

Remarks $z = f(x,y)$ graph $\Rightarrow$ Tangent Plane at (x_0, y_0)

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

2. Taylor's Series, Multivariable Chain Rule

Formulas for Taylor Series Around Zero From One-Variable Calculus

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad |x| < \infty$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \qquad |x| < \infty$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \qquad |x| < \infty$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots, \qquad |x| < 1$$

$\frac{1}{1+x} = 1 - x + x^2 + \dots$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{(k+1)} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \qquad |x| < 1$$

$\int \frac{1}{1+x^2}\,dx = \arctan x$

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \qquad |x| < 1$$

Example: Taylor's Series/Polynomials

(a) Compute the fourth degree Taylor polynomial about the origin of

$$f(x,y) = e^{y^2} \sin(x+y).$$

(b) Give the fourth degree Taylor polynomial about the origin of

$$f(x,y) = \frac{e^{-xy}}{1-x^2}.$$

(a) $e^{y^2} = 1 + y^2 + \frac{(y^2)^2}{2!} + \underbrace{\frac{(y^2)^3}{3!} + \dots}_{\text{degree 6 and higher}}$

$\sin(x+y) = (x+y) - \frac{(x+y)^3}{3!} + \underbrace{\frac{(x+y)^5}{5!} - \dots}_{\text{degree 5 and higher}}$

$e^{y^2} \cdot \sin(x+y) = \left(1 + y^2 + \frac{y^4}{2} + \dots\right)\left((x+y) - \frac{(x+y)^3}{6} + \dots\right)$

$= \underbrace{x+y - \frac{(x+y)^3}{6} + y^2(x+y)}_{5} + \underbrace{\dots}_{\text{degree 5 and higher}}$

$$T_4(x,y) = x+y - \frac{(x+y)^3}{6} + y^2(x+y)$$

(b) Give the fourth degree Taylor polynomial about the origin of

$$f(x,y) = \frac{e^{-xy}}{1-x^2}.$$

$$e^{-xy} = 1 - xy + \frac{x^2y^2}{2!} - \underbrace{\frac{x^3y^3}{3!} + \cdots}_{\text{higher-order terms}}$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \underbrace{\cdots}_{\text{higher-order terms}}$$

$$\frac{e^{-xy}}{1-x^2} = \left(1 - xy + \frac{x^2y^2}{2!} + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right)$$

$$= \underbrace{1 + x^2 + x^4 - xy - x^3y + \frac{x^2y^2}{2}}_{T_4(x,y)} + \underbrace{\cdots}_{\text{higher-order terms}}$$

$$T_4(x,y) = 1 + x^2 + x^4 - xy - x^3y + \frac{x^2y^2}{2}$$

See solutions for hw 5 and lectures of week 5 for different approaches.

Example: Chain Rule

Let $f: \mathbb{R}^3 \to \mathbb{R}^4$ be given by $f(x,y,z) = (xy^2, yz^2, x^2z, xyz)$ and let $g: \mathbb{R}^4 \to \mathbb{R}$ be given by $g(x,y,z,w) = xy + zw$. Use the Chain Rule to compute the gradient of $g \circ f$.

f_1 f_2 f_3 f_4

$g \circ f : \mathbb{R}^3 \to \mathbb{R}$

$\nabla(g \circ f) = D(g \circ f) = Dg(f(x,y,z)) \cdot Df(x,y,z)$ (Chain Rule)

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \vdots & \vdots \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} \end{bmatrix} = \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \\ yz & xz & xy \end{bmatrix}$$

$$Dg(x,y,z,w) = \nabla g = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{bmatrix} = \begin{bmatrix} y & x & w & z \end{bmatrix}$$

$$Dg(f(x,y,z)) = Dg(xy^2, yz^2, x^2z, xyz)$$

$$= \begin{bmatrix} yz^2 & xy^2 & xyz & x^2z \end{bmatrix}$$

$$D(g \circ f) = \begin{bmatrix} yz^2 & xy^2 & xyz & x^2z \end{bmatrix} \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2xz & 0 & x^2 \\ yz & xz & xy \end{bmatrix}$$

(1×4) (4×3) (1×3)

$$= \begin{bmatrix} y^3z^2 + 2x^2z^2y + x^2yz^2 & 2xy^2z^2 + xy^2z^2 + x^3z^2 & 2xy^3z + x^3yz \end{bmatrix}$$

$$= \begin{bmatrix} y^3z^2 + 3x^2yz^2 & 3xy^2z^2 + x^3z^2 & 2xy^3z + 2x^3yz \end{bmatrix} \checkmark$$

3. Extrema of Multivariable Functions. Lagrange Multipliers

Example: Hessian Matrix and Local Extrema

Let $f(x,y,z) = x^3 + x^2 + y^2 + z^2 - xy + xz$. Find all the critical points of f. Characterize each critical point as a local maximum, local minimum, or a saddle point.

$$\nabla f(x,y,z) = (3x^2+2x-y+z,\ 2y-x,\ 2z+x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\begin{cases} 3x^2+2x-y+z=0 \\ 2y-x=0 \\ 2z+x=0 \end{cases} \quad \begin{cases} 3x^2+2x-\frac{x}{2}-\frac{x}{2}=0 \\ y=\frac{x}{2} \\ z=-\frac{x}{2} \end{cases} \rightarrow \quad 3x^2+x=0$$

$$x\cdot(3x+1)=0$$

$$\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \quad \text{or} \quad \begin{cases} x=-\frac{1}{3} \\ y=-\frac{1}{6} \\ z=\frac{1}{6} \end{cases}$$

$\Rightarrow$ Two critical points: $(0,0,0)$

$\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$.

$$Hf(x,y,z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} 6x+2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

↑ Hessian matrix

Critical Point $(0,0,0)$: $\rightarrow$ $Hf(0,0,0) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$\det A_1 = 2 > 0$

$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3 > 0$

$\det A_3 = 1\cdot\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = -2 + 2(4-1)$

$= 4 > 0$

$+\ +\ +$ $\Rightarrow$ positive definite $\rightarrow$ $(0,0,0)$ is a local min

Critical point: $\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$

$$Hf\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$\det A_1 = 0$

$\det A_2 = -1 < 0$

$$\det A_3 = 1 \cdot \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} \quad \end{vmatrix} + 2 \begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -2 - 2 = -4 < 0$$

$0 \; - \; - \Rightarrow$ neither postive nor negative definite $\Rightarrow$ $\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$ is a saddle point

Remark: (a) $- + - \Rightarrow$ negative definite $\Rightarrow$ local max.

(b) if $\det Hf(x_0, y_0, z_0) = 0 \Rightarrow$ the test is inconclusive, apply other tools.

Example: Absolute Extrema and Lagrange Multipliers

Find the maximum value of $f(x,y,z) = x^2 - 4x + y^2 - 2y + z^2 - 4z - 1$ on the solid ball $x^2 + y^2 + z^2 \leq 9$. Justify your answer including an explanation of why global extrema do exist.

f is continuous in $\mathbb{R}^2$ and the solid $x^2+y^2+z^2 \leq 9$ is closed and bounded $\Rightarrow$ by the EVT, f has its absolute max and min on $x^2+y^2+z^2 \leq 9$.

Interior: $x^2+y^2+z^2 < 9$

$$\nabla f = (2x-4,\ 2y-2,\ 2z-4)$$

$$\nabla f = 0 \iff \begin{cases} x=2 \\ y=1 \\ z=2 \end{cases} \Rightarrow \text{Critical point } (2,1,2)$$

$$2^2+1^2+2^2 = 9$$

$(2,1,2)$ is not in the interior

$\Rightarrow$ there are no critical points in the interior.

Boundary: $x^2+y^2+z^2=9$ ← Constraint, Lagrange multipliers

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda(x^2+y^2+z^2-9)$$

$$\nabla \mathcal{L} = \nabla f - \lambda(2x, 2y, 2z)$$

$$\nabla \mathcal{L} = 0 \Rightarrow \begin{cases} 2x-4 = 2\lambda x \\ 2y-2 = 2\lambda y \\ 2z-4 = 2\lambda z \\ x^2+y^2+z^2=9 \end{cases} \Rightarrow \begin{cases} x(1-\lambda)=2 \\ y(1-\lambda)=1 \\ z(1-\lambda)=2 \\ x^2+y^2+z^2=9 \end{cases} \Rightarrow \lambda \neq 1$$

$$x = \frac{2}{1-\lambda}$$

$$y = \frac{1}{1-\lambda}$$

$$z = \frac{2}{1-\lambda}$$

$$\Rightarrow \begin{cases} x=2y \\ z=2y \\ x^2+y^2+z^2=9 \end{cases} \qquad 4y^2+y^2+4y^2=9$$

$$y^2=1 \Rightarrow y=\pm 1$$

$$\begin{cases} y=1 \\ x=2 \\ z=2 \end{cases} \text{ or } \begin{cases} y=-1 \\ x=-2 \\ z=-2 \end{cases}$$

2 Critical Points: $(2,1,2)$, $(-2,-1,-2)$

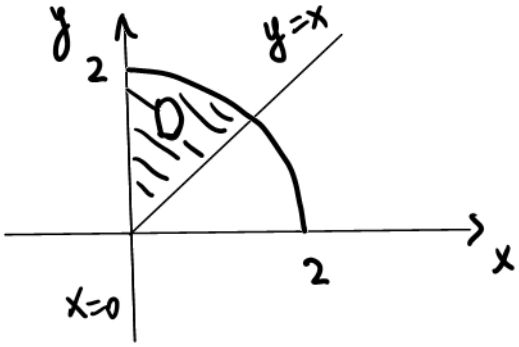
with $f(2,1,2) = 4-8+1-2+4-8-1 = -10$

$f(-2,-1,-2) = 4+8+1+2+4+8-1 = 26$

$\max_{\text{ball}} f = f(-2,-1,-2) = 26$

Example: Double Integral # 1

Evaluate the integral $\iint_D (2x-y)\,dA$, where D is the region in the first quadrant enclosed by the curve $x^2+y^2=4$ and the lines $x=0$ and $y=x$.



Use polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$D: \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 2$

$$\iint_D (2x-y)\,dA = \int_{\frac{\pi}{4}}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\cdot r\,dr\,d\theta$$

↑ jacobian

$$= \int_{\pi/4}^{\pi/2} \int_0^2 r^2\cdot(2\cos\theta - \sin\theta)\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/2} \frac{1}{3}r^3\Big|_{r=0}^{r=2} (2\cos\theta - \sin\theta)\,d\theta$$

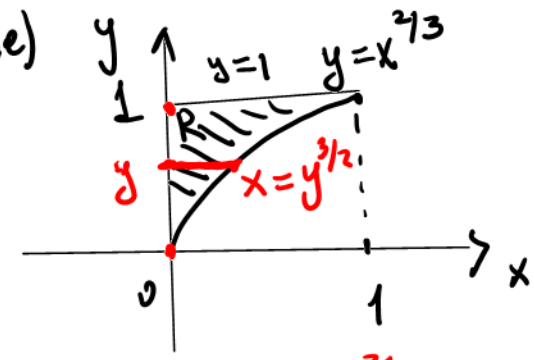
$$= \frac{8}{3}\int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\,d\theta$$

$$= \frac{8}{3}(2\sin\theta + \cos\theta)\Big|_{\pi/4}^{\pi/2} = \frac{16}{3} - 4\sqrt{2}.$$

Example: Double Integral # 2

Evaluate the integral $\int_0^1 \int_{x^{2/3}}^1 xe^{y^4}\, dydx$. → We can not integrate straightforward. Change the order of integration

Region R: $\begin{cases} x^{2/3} \le y \le 1 \\ 0 \le x \le 1 \end{cases}$ (y-simple)



R (x-simple): $\begin{cases} 0 \le y \le 1 \\ 0 \le x \le y^{3/2} \end{cases}$

$y = x^{2/3} \Leftrightarrow x = y^{3/2}$

$$\int_0^1 \int_{x^{2/3}}^1 xe^{y^4}\, dydx$$

$$= \int_0^1 \int_0^{y^{3/2}} xe^{y^4}\, dxdy$$

$$= \int_0^1 e^{y^4} \cdot \frac{1}{2}x^2 \Big|_{x=0}^{x=y^{3/2}} dy$$

$$= \int_0^1 e^{y^4} \cdot \frac{1}{2} \cdot y^3\, dy$$

$$= \frac{1}{2} \int_0^1 y^3 e^{y^4} dy = \frac{1}{2} \cdot \frac{1}{4} e^{y^4} \Big|_0^1$$

$$= \frac{1}{8}(e-1).$$

5. Change of Variables Formula

Theorem: Change of Variables (M&T p. 315)

If D and D^* are elementary regions in the plane, $T\colon D^* \to D$ is C^1 and one-to-one on D^*, and $D = T(D^*)$, then for any integrable function $f\colon D \to \mathbb{R}$ we have

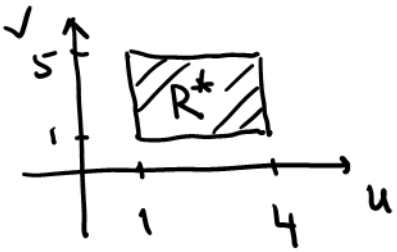
$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

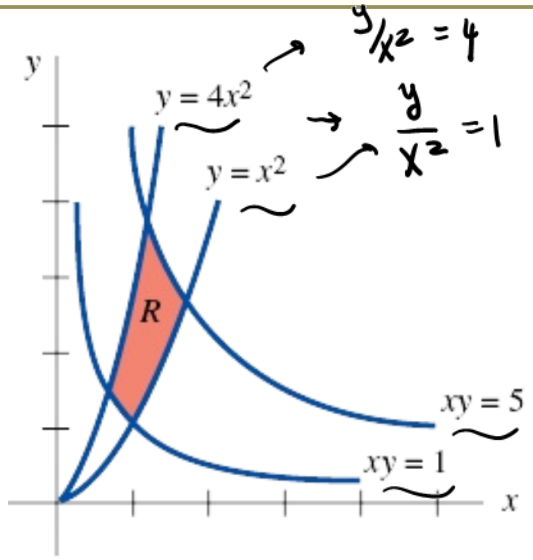
Example: Change of Variables

Evaluate $\iint_R xy\,dA$ over the region R shown in the diagram below.

Change of variables: R^*:

$$\begin{cases} u = \dfrac{y}{x^2} & 1 \le u \le 4 \\ v = xy & 1 \le v \le 5 \end{cases}$$





$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{3}u^{-4/3}v^{1/3} & \frac{1}{3}v^{-2/3}u^{-1/3} \\ \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}v^{-1/3}u^{1/3} \end{vmatrix}$$

$$= -\frac{2}{9}u^{-1} - \frac{1}{9}u^{-1} = -\frac{1}{3u}$$

$$\begin{cases} u = \dfrac{y}{x^2} \Rightarrow y = x^2u \\ v = xy \end{cases}$$

$$v = x \cdot x^2 u$$

$$v = x^3 u$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3u}$$

$$x = \left(\frac{v}{u}\right)^{1/3} = v^{1/3}u^{-1/3}$$

$$y = v^{\frac{2}{3}}u^{-\frac{2}{3}} \cdot u = v^{\frac{2}{3}}u^{1/3}$$

$$\iint_R xy\,dA = \int_1^4 \int_1^5 v^{1/3} u^{-1/3} \cdot v \cdot u^{1/3} \cdot \frac{1}{3u}\,dv\,du$$

$$= \int_1^4 \int_1^5 \frac{v}{3u}\,dv\,du = \frac{1}{3}\int_1^4 \frac{1}{u} \cdot \frac{1}{2}v^2\Big|_{v=1}^{5} du$$

$$= \frac{1}{6}(25-1)\int_1^4 \frac{1}{u}\,du$$

$$= 4 \ln u \Big|_1^4 = \boxed{4 \ln 4}$$

Example: Triple Integral

Evaluate $\iiint_B z\,dV$, where B is the region bounded by the planes $z = 0$ and $z = 1$ and the surface $(z+1)\sqrt{x^2+y^2} = 1$.

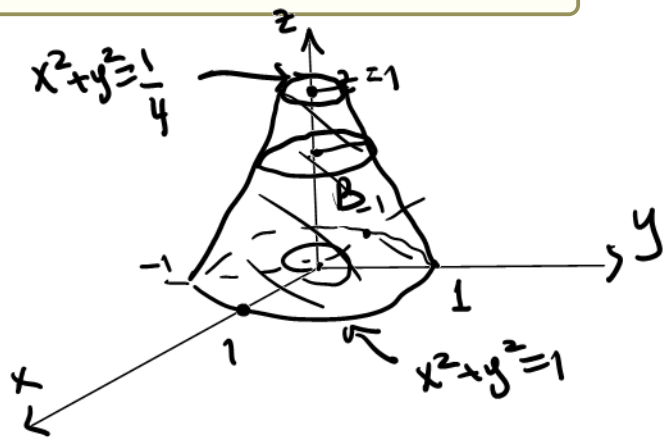
$(z+1)\sqrt{x^2+y^2} = 1$

$z=0 \Rightarrow \sqrt{x^2+y^2} = 1$

$z=1 \Rightarrow 2\sqrt{x^2+y^2} = 1$

$\sqrt{x^2+y^2} = \frac{1}{2}$

$x^2+y^2 = \frac{1}{4}$



$z = c \rightarrow (c+1)\sqrt{x^2+y^2} = 1$

$0 < c < 1$

$x^2+y^2 = \frac{1}{(c+1)^2} < 1$

We will use cylindrical coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$(z+1)\sqrt{x^2+y^2} = 1 \rightarrow (z+1)\cdot r = 1$

$\boxed{r = \frac{1}{z+1}}$ - Equation of the surface in cylindr. coord.

$B: \begin{cases} 0 \le r \le \frac{1}{z+1} \\ 0 \le \theta \le 2\pi \\ 0 \le z \le 1 \end{cases}$

$$\iiint_B z\,dV = \int_0^{2\pi} \int_0^1 \int_0^{\frac{1}{z+1}} z \cdot r\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^1 z\cdot\frac{1}{2}r^2\Big|_{r=0}^{\frac{1}{z+1}} dz\,d\theta$$

↖ jacobian

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{z}{(z+1)^2}\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \frac{z+1-1}{(z+1)^2}\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(\frac{1}{z+1} - \frac{1}{(z+1)^2}\right)dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\ln(z+1) + \frac{1}{z+1}\right)\Big|_{z=0}^{1} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\ln 2 - \ln 0 + \left(\frac{1}{2} - 1\right)\right) d\theta$$

$$= \frac{1}{2}\cdot 2\pi\cdot\left(\ln 2 - \frac{1}{2}\right) = \pi\left(\ln 2 - \frac{1}{2}\right).$$

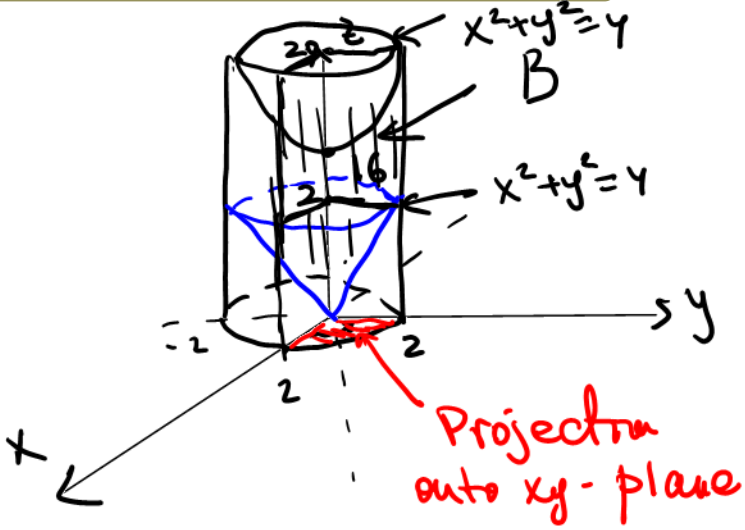
Example: Volume # 1

Let B be the first octant region bounded by $z = x^2+y^2+16$, $z = \sqrt{x^2+y^2}$ and $x^2+y^2 = 4$. Use cylindrical or spherical coordinates to describe B and set up a triple integral to find the volume of B. What is the volume of B?

$z = x^2+y^2+16$ - Paraboloid

$z = \sqrt{x^2+y^2}$ - Cone (upper)

$x^2+y^2 = 4$ - Cylinder

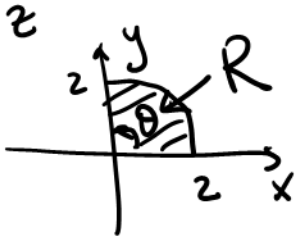


$$\begin{cases} x^2+y^2=4 \\ z=\sqrt{x^2+y^2} \end{cases} \Rightarrow z=2$$

$$\begin{cases} z=x^2+y^2+16 \\ x^2+y^2=4 \end{cases} \Rightarrow z=20$$

Use cylindrical coordinates:

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$$



Cylinder: $r=2$

Cone: $z=\sqrt{x^2+y^2} \Rightarrow z=r$

Paraboloid: $z=x^2+y^2+16 \Rightarrow z=r^2+16$

$$B: \begin{cases} r \le z \le r^2+16 \\ 0 \le r \le 2 \\ 0 \le \theta \le \pi/2 \end{cases}$$

$$V(B) = \iiint_B 1\,dV = \int_0^{\pi/2}\int_0^2\int_r^{r^2+16} 1 \cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r(r^2+16-r)\,dr\,d\theta = \boxed{\frac{50\pi}{3}}$$

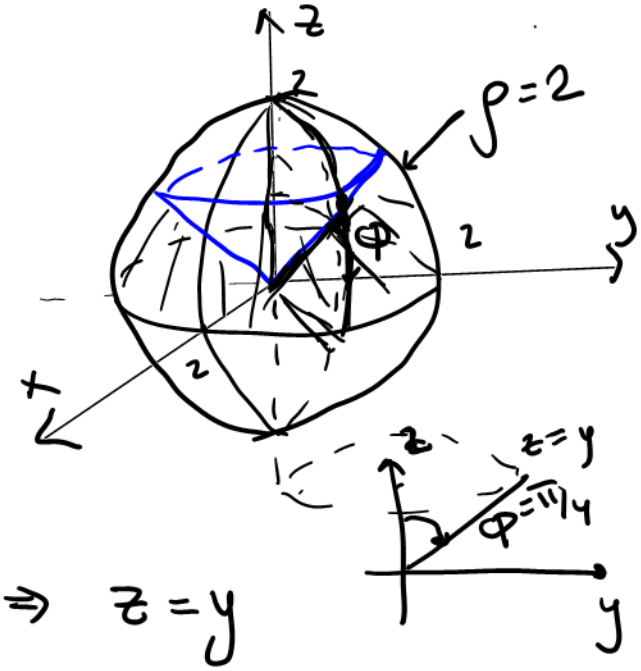
Remark.  $z = f(x,y)$ $V = \iint_R f(x,y)\,dx\,dy = \iint_R \left(x^2+y^2+16-\sqrt{x^2+y^2}\right)dx\,dy$

Example: Volume # 2

Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Use spherical coordinates:

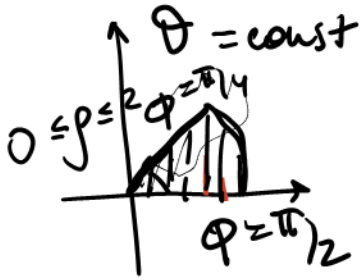
$$\begin{cases} x = \rho\cos\theta\cdot\sin\varphi \\ y = \rho\sin\theta\cdot\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$



Cone: $z = \sqrt{x^2+y^2}$

Projection onto yz-plane $\Rightarrow z = y$

$\boxed{\varphi = \frac{\pi}{4}}$ B: $\begin{cases} \frac{\pi}{4} \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \\ 0 \le \rho \le 2 \end{cases}$



$$V = \iiint_B 1\,dV = \int_0^{2\pi}\int_0^{2}\int_{\pi/4}^{\pi/2} 1\cdot \rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta$$

$\rho^2\sin\varphi$: jacobian

$$= \int_0^{2\pi}\int_0^{2} \rho^2(-\cos\varphi)\Big|_{\varphi=\pi/4}^{\varphi=\pi/2}\,d\rho\,d\theta$$

$$= \frac{\sqrt{2}}{2}\int_0^{2\pi}\int_0^{2}\rho^2\,d\rho\,d\theta = \frac{\sqrt{2}}{2}\cdot 2\pi\cdot\frac{1}{3}\rho^3\Big|_0^2$$

$$= \boxed{\frac{8\pi\sqrt{2}}{3}}$$

Summary of MAT B41

- Linear algebra and functions of several variables

- Differentiation of functions $f : \mathbb{R}^n \to \mathbb{R}^k$

- Multivariate chain rule

- Quadratic forms, Hessian matrices, and extrema of multivariate functions

- Constrained optimization and Lagrange multipliers

- Approximation by Taylor polynomials

- Integration of function $f : \mathbb{R}^n \to \mathbb{R}$

- Change of Coordinates: spherical, cylindrical and polar coordinates