AMACSS MATB41 TT2 Review Seminar

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Directional Derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{a} \in \mathbb{R}^n$. For a *unit* vector $\mathbf{u} \in \mathbb{R}^n$, the **directional derivative** of f at $\mathbf{a}$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h},$$

when this limit exists.

If f is differentiable at $\mathbf{a}$, then

$$\boxed{D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}}$$

(where $\nabla f(\mathbf{a})$ is the gradient at $\mathbf{a}$).

Intuition

- Think of f as a height field (surface). $D_{\mathbf{u}}f(\mathbf{a})$ is the instantaneous slope you feel walking from $\mathbf{a}$ in direction $\mathbf{u}$.
- $\nabla f(\mathbf{a})$ points in the direction of *steepest ascent*; its length $\|\nabla f(\mathbf{a})\|$ is the *maximum* slope.
- Extremes over directions:

$$\max_{\|\mathbf{u}\|=1} D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \quad \text{(achieved at } \mathbf{u} = \frac{\nabla f}{\|\nabla f\|}\text{)},$$

$$\min_{\|\mathbf{u}\|=1} D_{\mathbf{u}}f(\mathbf{a}) = -\|\nabla f(\mathbf{a})\|.$$

How to compute in practice

Given $f(x_1, \ldots, x_n)$, point $\mathbf{a}$, and direction $\mathbf{v} \neq \mathbf{0}$:

1. Compute the gradient ∇f.
2. Evaluate it at the point: $\nabla f(\mathbf{a})$.
3. Normalize the direction: $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$.
4. Take the dot product: $D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$.

Example (you try first)

Question. Let

$$f(x, y) = x^2 y + e^{xy}, \qquad \mathbf{a} = (1, 0),$$

and $\mathbf{v} = \langle 3, 4 \rangle$. Compute the directional derivative of f at $\mathbf{a}$ in the direction of $\mathbf{v}$.

Hint: Remember to use the *unit* direction.

Solution

Step 1: Gradient.

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Step 2: Evaluate at $\mathbf{a} = (1, 0)$.

$$e^{xy}\big|_{(1,0)} = e^0 = 1, \quad y = 0 \Rightarrow \nabla f(1,0) = \langle 0,\ 1^2 + 1 \cdot 1 \rangle = \langle 0, 2 \rangle.$$

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Step 3: Unit direction.

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$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, 4 \rangle}{5} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle.$$

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Step 4: Dot product.

Solution

Step 1: Gradient.

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$$D_{\mathbf{u}} f(1,0) = \nabla f(1,0) \cdot \mathbf{u} = \langle 0, 2 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{8}{5}.$$

Solution

Step 1: Gradient.

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Answer

Answer: $D_{\mathbf{u}} f(1,0) = \frac{8}{5} = 1.6.$

Quick checks and tips

- If you forget to normalize, your answer is off by a factor of $\|\mathbf{v}\|$.
- If $\nabla f(\mathbf{a}) = \mathbf{0}$, every directional derivative is 0 (critical point).
- Orthogonal directions to $\nabla f(\mathbf{a})$ have zero directional derivative. (why?)

Chain Rule (M&T p. 126)

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ be open sets. Suppose

$$g : U \to \mathbb{R}^k, \quad f : V \to \mathbb{R}^m, \quad \text{and } g(U) \subset V.$$

If g is differentiable at $\mathbf{x}_0 \in U$ and f is differentiable at $\mathbf{y}_0 = g(\mathbf{x}_0)$, then $f \circ g$ is differentiable at $\mathbf{x}_0$ and

$$\boxed{D(f \circ g)(\mathbf{x}_0) = Df(\mathbf{y}_0)\, Dg(\mathbf{x}_0).}$$

Example — Setup and Steps

Example. Let

$$g(x,y) = \begin{bmatrix} x^2 y \\ \sin(xy) \\ e^{x+y} \end{bmatrix}, \qquad f(u,v,w) = uv + w^2.$$

Find $D(f \circ g)(1,0)$.

Steps:

1. Compute $Dg(x,y)$ — a 3×2 matrix.
2. Compute $Df(u,v,w)$ — a 1×3 row vector.
3. Evaluate each at
$$\mathbf{x}_0 = (1,0), \quad \mathbf{y}_0 = g(\mathbf{x}_0).$$
4. Apply the chain rule:
$$D(f \circ g)(\mathbf{x}_0) = Df(\mathbf{y}_0)\, Dg(\mathbf{x}_0).$$

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2. **Compute** $Df(u, v, w)$.

$$Df(u, v, w) = \begin{bmatrix} v & u & 2w \end{bmatrix}.$$

3. **Evaluate at points.**

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$$Df(u,v,w) = \begin{bmatrix} v & u & 2w \end{bmatrix}.$$

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$$\mathbf{y}_0 = g(1,0) = (0,0,e), \quad Dg(1,0) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ e & e \end{bmatrix}, \quad Df(\mathbf{y}_0) = \begin{bmatrix} 0 & 0 & 2e \end{bmatrix}.$$

4. **Multiply (Chain Rule).**

Example : Solution

1. **Compute** $Dg(x, y)$.

$$Dg(x,y) = \begin{bmatrix} 2xy & x^2 \\ y\cos(xy) & x\cos(xy) \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

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4. **Multiply (Chain Rule).**

$$D(f \circ g)(1,0) = Df(\mathbf{y}_0)\, Dg(1,0) = \begin{bmatrix} 0 & 0 & 2e \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ e & e \end{bmatrix} = \boxed{\begin{bmatrix} 2e^2 & 2e^2 \end{bmatrix}}.$$

Higher-Order Partial Derivatives

Definition

If $f(x, y, z)$ has continuous second partial derivatives, then higher-order partials are defined recursively as:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^3 f}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y\, \partial z}\right).$$

If f is C^2 (i.e., all second partials are continuous), then

$$f_{xy} = f_{yx}, \quad f_{xyz} = f_{xzy} = f_{yzx} = \cdots$$

by Clairaut's theorem (theorem of mixed partials).

Example — Compute Mixed Third Derivatives

Question.

$$(a)\ V = \ln(r + s^2 + t^3), \quad \frac{\partial^3 V}{\partial r\, \partial s\, \partial t}\ ?$$

$$(b)\ w = \frac{x}{y + 2z}, \quad \frac{\partial^3 w}{\partial z\, \partial y\, \partial x}, \quad \frac{\partial^3 w}{\partial^2 x\, \partial y}.$$

Solution

(a)

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$$V_r = \frac{1}{r + s^2 + t^3}, \quad V_{rs} = -\frac{2s}{(r + s^2 + t^3)^2}, \quad V_{rst} = \frac{6st^2}{(r + s^2 + t^3)^3}.$$

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(b)

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$$\boxed{\frac{\partial^3 V}{\partial r\, \partial s\, \partial t} = \frac{6st^2}{(r + s^2 + t^3)^3}}.$$

(b)

$$w_x = \frac{1}{y + 2z}, \quad w_{xy} = -\frac{1}{(y + 2z)^2}, \quad w_{xyz} = \frac{4}{(y + 2z)^3}.$$

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$$w_{xxy} = \frac{\partial}{\partial x}(w_{xy}) = 0 \quad \Rightarrow \quad \boxed{\frac{\partial^3 w}{\partial^2 x\, \partial y} = 0.}$$

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$$\boxed{\frac{\partial^3 w}{\partial z\, \partial y\, \partial x} = \frac{4}{(y + 2z)^3}}.$$

Unconstrained Optimization (Hessian Test)

Theory

To find local extrema of $f(x, y, z)$:

1. Compute $\nabla f = 0$ to find critical points.
2. Compute Hessian matrix:

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}.$$

3. Classify using leading principal minors:
   - Positive definite $\Rightarrow$ local min
   - Negative definite $\Rightarrow$ local max
   - Indefinite $\Rightarrow$ saddle

Example — Find and Classify Critical Points

$$f(x, y, z) = x^3 + x^2 + y^2 + z^2 - xy + xz.$$

Solution — Part 1: Finding Critical Points

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$$\Rightarrow \begin{cases} 3x^2 + 2x - y + z = 0, \\ 2y - x = 0, \\ 2z + x = 0. \end{cases}$$

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$$\Rightarrow \begin{cases} 3x^2 + 2x - y + z = 0, \\ 2y - x = 0, \\ 2z + x = 0. \end{cases}$$

Substitute $y = \frac{x}{2},\ z = -\frac{x}{2}$:

$$3x^2 + 2x - \tfrac{x}{2} - \tfrac{x}{2} = 3x^2 + x = 0 \Rightarrow x = 0, -\tfrac{1}{3}.$$

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Critical points: $(0, 0, 0)$ and $\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$.

Solution — Part 2: Hessian Classification

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At $(0,0,0)$:

$$H = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

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At $\left(-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$:

$$H = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix},$$

which has both positive and negative eigenvalues $\Rightarrow$ indefinite $\Rightarrow$ saddle point.

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which has both positive and negative eigenvalues $\Rightarrow$ indefinite $\Rightarrow$ saddle point.

Answer:

$$\boxed{(0,0,0) : \text{local minimum}, \quad (-\tfrac{1}{3}, -\tfrac{1}{6}, \tfrac{1}{6}) : \text{saddle point}.}$$

Constrained Optimization — Lagrange Multipliers

Theory

To optimize $f(x, y)$ subject to $g(x, y) = 0$:

$$\nabla f = \lambda \nabla g, \quad g(x, y) = 0.$$

If f and g are continuous on a closed and bounded region, global extrema exist by the Extreme Value Theorem.

Example — Elliptic Constraint

$$f(x,y) = 2x - 3y + 5, \quad g(x,y) = \frac{x^2}{9} + \frac{y^2}{16} - 1 = 0.$$

Find absolute max and min.

Solution

1. Lagrange system:

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$$\nabla f = (2, -3), \quad \nabla g = \left(\frac{2x}{9}, \frac{2y}{16} \right).$$

$$(2, -3) = \lambda \left(\frac{2x}{9}, \frac{2y}{16} \right).$$

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From first: $\lambda = \frac{9}{x}$, second: $\lambda = \frac{-24}{y}$, so $9/x = -24/y \Rightarrow y = -\frac{8}{3}x$.

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From first: $\lambda = \frac{9}{x}$, second: $\lambda = \frac{-24}{y}$, so $9/x = -24/y \Rightarrow y = -\frac{8}{3}x$. Plug into ellipse:

$$\frac{x^2}{9} + \frac{64x^2/9}{16} = 1 \Rightarrow x^2 \left(\frac{1}{9} + \frac{4}{9} \right) = 1 \Rightarrow x = \pm \frac{3}{\sqrt{5}}.$$

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Then $y = \mp \frac{8}{\sqrt{5}}$.

$$f(x, y) = 2x - 3y + 5.$$

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$$f(x, y) = 2x - 3y + 5.$$

At $(\frac{3}{\sqrt{5}}, -\frac{8}{\sqrt{5}})$: $f = 5 + \frac{30}{\sqrt{5}} = 5 + 13.42 = 18.42$.

At $(-\frac{3}{\sqrt{5}}, \frac{8}{\sqrt{5}})$: $f = 5 - \frac{30}{\sqrt{5}} = -8.42$.

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$$f(x, y) = 2x - 3y + 5.$$

At $(\frac{3}{\sqrt{5}}, -\frac{8}{\sqrt{5}})$: $f = 5 + \frac{30}{\sqrt{5}} = 5 + 13.42 = 18.42$.

At $(-\frac{3}{\sqrt{5}}, \frac{8}{\sqrt{5}})$: $f = 5 - \frac{30}{\sqrt{5}} = -8.42$.

$$\boxed{\text{Max } = 18.42, \quad \text{Min } = -8.42.}$$

Multivariable Taylor Series

Theory

The nth-degree Taylor polynomial of $f(x, y)$ about (a, b) is:

$$T_n(x, y) = \sum_{i+j \leq n} \frac{f^{(i,j)}(a, b)}{i!\, j!}(x - a)^i (y - b)^j.$$

Example — Compute 4th-degree Taylor Polynomial

$$f(x, y) = e^{y^2} \sin(x + y) \quad \text{about } (0, 0).$$

Solution

Solution

Expand both:

$$e^{y^2} = 1 + y^2 + \frac{y^4}{2} + \cdots, \quad \sin(x+y) = x + y - \frac{(x+y)^3}{6} + \cdots$$

Solution

Expand both:

$$e^{y^2} = 1 + y^2 + \frac{y^4}{2} + \cdots, \quad \sin(x+y) = x + y - \frac{(x+y)^3}{6} + \cdots$$

Multiply and keep degree ≤ 4:

$$f(x,y) \approx (x+y) + y^2(x+y) - \frac{(x+y)^3}{6} + \frac{y^4(x+y)}{2}.$$

Solution

Expand both:

$$e^{y^2} = 1 + y^2 + \frac{y^4}{2} + \cdots, \quad \sin(x+y) = x + y - \frac{(x+y)^3}{6} + \cdots$$

Multiply and keep degree ≤ 4:

$$f(x,y) \approx (x+y) + y^2(x+y) - \frac{(x+y)^3}{6} + \frac{y^4(x+y)}{2}.$$

Simplify:

$$T_4(x,y) = x + y + y^2x + y^3 - \frac{x^3 + 3x^2y + 3xy^2 + y^3}{6}.$$

Solution

Expand both:

$$e^{y^2} = 1 + y^2 + \frac{y^4}{2} + \cdots, \quad \sin(x+y) = x + y - \frac{(x+y)^3}{6} + \cdots$$

Multiply and keep degree ≤ 4:

$$f(x,y) \approx (x+y) + y^2(x+y) - \frac{(x+y)^3}{6} + \frac{y^4(x+y)}{2}.$$

Simplify:

$$T_4(x,y) = x + y + y^2x + y^3 - \frac{x^3 + 3x^2y + 3xy^2 + y^3}{6}.$$

$$\boxed{T_4(x,y) = x + y + \tfrac{5}{6}y^3 + y^2x - \tfrac{1}{6}(x^3 + 3x^2y + 3xy^2)}.$$

Riemann Sums for Multivariable Functions

Definition

Let $f(x, y)$ be defined on a closed region R in the xy-plane. Partition R into small rectangles $\Delta A_{ij} = \Delta x\, \Delta y$, and choose a sample point (x_{ij}^*, y_{ij}^*) in each subrectangle.

The **Riemann sum** is:

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A_{ij}.$$

If $\lim_{\max(\Delta x, \Delta y) \to 0} S$ exists, then

$$\boxed{\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A_{ij}.}$$

Geometric Meaning

- Each term $f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}$ represents the volume of a thin column with base ΔA_{ij} and height $f(x_{ij}^*, y_{ij}^*)$.
- The double integral $\iint_R f(x,y)\,dA$ is the total volume under the surface $z = f(x,y)$ above the region R.
- Riemann sums generalize the 1D concept of "area under a curve" to "volume under a surface."

Example — Approximate a Double Integral

Question. Approximate

$$\iint_R (x^2 + y^2)\, dA, \quad R = [0, 2] \times [0, 2],$$

using a Riemann sum with $m = n = 2$ subrectangles and sample points taken at the *upper-right corners.*

Solution

Partition $[0, 2] \times [0, 2]$ into 4 rectangles.

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Compute function values:

$$f(1,1) = 2, \quad f(2,1) = 5, \quad f(1,2) = 5, \quad f(2,2) = 8.$$

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Each $\Delta A = 1 \cdot 1 = 1$. Sum all:

$$S = (2+5+5+8)(1) = 20.$$

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$$\boxed{\iint_R (x^2 + y^2)\, dA \approx 20.}$$

Summary — Riemann Sums and Double Integrals

- Riemann sums approximate $\iint_R f(x, y)\, dA$ as the sum of f-values times small area pieces.
- The finer the partition (smaller Δx, Δy), the closer the approximation.
- Sample point choice (lower-left, midpoints, upper-right) affects accuracy.
- In the limit, the Riemann sum becomes the exact double integral — the total signed volume under $f(x, y)$.

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