MATB44 Review Seminar

October 19, 2025 (5 – 7pm)

Separable

It comes from the fact that the variables are separable.

$$Q(x,y)\,\frac{dy}{dx} + P(x,y) = 0.$$

$$f(x)\,dx + g(y)\,dy = \int f(x)\,dx + \int g(y)\,dy = C,$$

Separable Equation Questions:

1. $e^{x+1} \tan y \, dx + \cos y \, dy = 0.$

2. $(x - 1) \cos y \, dy = 2x \sin y \, dx.$

Solution for Question 1:

We want to solve the differential equation:

$$e^{x+1}\tan y\,dx + \cos y\,dy = 0.$$

Divide the entire equation by $\tan y$:

$= \mathrm{e}^{x+1}\,dx + \frac{\cos^2 y}{\sin y}\,dy = 0$

$= e^{x+1}\,dx + \frac{1-\sin^2 y}{\sin y}\,dy = 0 \qquad$ (**By** $\sin^2 y + \cos^2 y = 1$)

$= e^{x+1}\,dx + (\csc y - \sin y)\,dy = 0$

Now integrate both sides:

$$\int e^{x+1}\,dx + \int (\csc y - \sin y)\,dy = 0$$

$$e^{x+1} + \int \csc y\,dy - \int \sin y\,dy = 0$$

$$e^{x+1} + \ln\left|\csc y - \cot y\right| + \cos y = C$$

Thus, the solution is:

$$e^{x+1} + \ln\left|\csc y - \cot y\right| + \cos y = C$$

Solution for Question 2:

We want to solve:

$$(x-1)\cos y\,dy - 2x\sin y\,dx = 0.$$

Divide through by $(siny)(x-1)$:

$$\frac{\cos y}{\sin y}\,dy = \frac{2x}{x-1}\,dx.$$

This is of the form:

$$\int \cot y\,dy = \int \frac{2x}{x-1}\,dx.$$

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Step 1. Integrate left side:

$$\int \cot y\,dy = \ln|\sin y| + C.$$

So we have:

$$\ln|\sin y| = \int \frac{2x}{x-1}\,dx.$$

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Step 2. Solve the integral on the right:
Let $u = x-1$, so $du = dx$ and $x = u+1$.

$$\int \frac{2x}{x-1}\,dx = \int \frac{2(u+1)}{u}\,du = \int \left(2+\frac{2}{u}\right)du.$$

$$= 2u + 2\ln|u| + C.$$

Substituting back $u = x-1$:

$$\int \frac{2x}{x-1}\,dx = 2(x-1) + 2\ln|x-1| + C.$$

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Step 3. Combine results:

$$\ln|\sin y| = 2(x-1) + 2\ln|x-1| + C.$$

Rearrange:

$$\ln|\sin y| - 2\ln|x-1| = 2(x-1) + C.$$

$$\ln\left(\frac{\sin y}{(x-1)^2}\right) = 2(x-1) + C.$$

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Step 4. Exponentiate both sides:

$$\frac{\sin y}{(x-1)^2} = e^{2(x-1)+C}.$$

$$\sin y = (x-1)^2 e^{2(x-1)+C}.$$

Absorb $-2+C$ into the constant C_1:

$$\sin y = (x-1)^2 e^{2x+C_1}.$$

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Final Answer:

$$\sin y = (x-1)^2 e^{2(x)+C_1}$$

Answer Key:

$$e^{x+1} + \log(\csc y - \cot y) + \cos y = c,$$

$$\sin y = (x - 1)^2 e^{2x+c},$$

Homogenous

Definition 7.1. Let $z = f(x,y)$ define z as a function of x and y in a region R. The function $f(x,y)$ is said to be **homogeneous of order n** if it can be written as

$$f(x,y) = x^n g(u), \tag{7.11}$$

where $u = y/x$ and $g(u)$ is a function of u; or alternately if it can be written as

$$f(x,y) = y^n h(u), \tag{7.12}$$

where $u = x/y$ and $h(u)$ is a function of u.

Homogenous Equation Questions:

8. $\frac{y}{x}\cos\frac{y}{x}\,dx - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)dy = 0.$

9. $y\,dx + x\log\frac{y}{x}\,dy - 2x\,dy = 0.$

Solutions for Question 1:

$$\frac{y}{x}\cos\left(\frac{y}{x}\right)\,dx-\left(\frac{x}{y}\sin\left(\frac{y}{x}\right)+\cos\left(\frac{y}{x}\right)\right)\,dy=0$$

Let

$$u=\frac{y}{x},\quad\Rightarrow\quad y=ux,\quad dy=u\,dx+x\,du.$$

Substitute into the equation:

$$u\cos(u)\,dx-\left(\frac{1}{u}\sin(u)+\cos(u)\right)(u\,dx+x\,du)=0$$

$$u\cos(u)\,dx-\left(\sin(u)dx+\frac{x}{u}\sin(u)du+u\cos(u)dx+x\cos(u)du\right)=0$$

$$u\cos(u)\,dx-\sin(u)dx-\frac{x}{u}\sin(u)du-u\cos(u)dx-x\cos(u)du=0$$

$$-\sin(u)dx=\frac{x}{u}\sin(u)du+x\cos(u)du=0$$

Divide through by $x\sin(u)$:

$$-\frac{1}{x}\,dx=\left(\frac{\sin(u)+u\cos(u)}{u\sin(u)}\right)\,du$$

$$\frac{1}{x}\,dx=-\left(\frac{1}{u}+\cot(u)\right)\,du$$

Integrating both sides:

$$\int\frac{1}{x}\,dx=-\int\left(\frac{1}{u}+\cot(u)\right)\,du$$

$$\ln|x|=-\ln|u|-\ln|\sin(u)|+C$$

$$C-\ln|x|=\ln|u\sin(u)|$$

$$\frac{y}{x}\sin(\frac{y}{x})=\frac{C}{x}$$

Multiply by x:

$$C=ysin(\frac{y}{x})$$

Solutions for Question 2

$$(y\,dx + x\log(yx)\,dy - 2x\,dy = 0)$$

Divide through by x:

$$y\,dx + \log(yx)\,dy - 2\,dy = 0$$

Let

$$u = \frac{y}{x}, \quad \Rightarrow \quad y = ux, \quad dy = u\,dx + x\,du.$$

Substitute:

$$u\,dx + \log(u)\,(u\,dx + x\,du) - 2(u\,dx + x\,du) = 0$$

$$u\,dx + u\log(u)\,dx + x\log(u)\,du - 2u\,dx - 2x\,du = 0$$

$$(u\log(u) - u)\,dx + x(\log(u) - 2)\,du = 0$$

$$\frac{dx}{x} = \frac{(2 - \log(u))}{u(\log(u) - 1)}\,du$$

—

Integration

$$\int \frac{dx}{x} = \int \frac{2 - \log(u)}{u(\log(u) - 1)}\,du$$

$$(u\log(u) - u)\,dx + x(\log(u) - 2)\,du = 0$$

$$\frac{dx}{x} = \frac{(2 - \log(u))}{u(\log(u) - 1)}\,du$$

—

Integration

$$\int \frac{dx}{x} = \int \frac{2 - \log(u)}{u(\log(u) - 1)}\,du$$

Let $w = \log(u) - 1 \quad \Rightarrow \quad dw = \frac{1}{u}\,du.$

$$\int \frac{dx}{x} = \int \frac{1 - w}{w}\,dw$$

$$\ln|x| = \int \left(\frac{1}{w} - 1\right)\,dw$$

$$\ln|x| = \ln|w| - w + C$$

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Back-substitution

$$\ln|x| = \ln|\log(u) - 1| - (\log(u) - 1) + C$$

Since $u = \frac{y}{x}$:

$$\ln|x| = \ln\left|\log\left(\frac{y}{x}\right) - 1\right| - \log\left(\frac{y}{x}\right) + 1 + C$$

Solutions for Question 2:

$$\ln|x| = \ln\left|\log\left(\frac{y}{x}\right) - 1\right| - \log\left(\frac{y}{x}\right) + C$$

$$\ln|x| - \ln\left|\log\left(\frac{y}{x}\right) - 1\right| = -\log\left(\frac{y}{x}\right) + C$$

$$\ln\left(\frac{|x|}{\left|\log\left(\frac{y}{x}\right) - 1\right|}\right) = -\log\left(\frac{y}{x}\right) + C$$

$$\frac{|x|}{\left|\log\left(\frac{y}{x}\right) - 1\right|} = e^{C}\, e^{-\log\left(\frac{y}{x}\right)} = C\frac{x}{y}$$

$$\frac{|y|}{\left|\log\left(\frac{y}{x}\right) - 1\right|} = C \quad \Longrightarrow \quad \frac{y}{\log\left(\frac{y}{x}\right) - 1} = C$$

$$\frac{y}{\log\left(\frac{y}{x}\right) - 1} = C \quad \Rightarrow \quad \frac{y}{-\log\left(\frac{x}{y}\right) - 1} = C \quad (since\ \log(y/x) = -\log(x/y))$$

$$\frac{y}{-(1+\log(x/y))} = C \quad \Rightarrow \quad y = -C\big(1+\log(x/y)\big).$$

Let $c := -C$. Then

$$y = c\big(1+\log(x/y)\big).$$

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Simplified implicit solution

$$\frac{y}{\ln\left(\frac{y}{x}\right) - 1} = Cx$$

Answer Key:

8. $y \sin \frac{y}{x} = c.$

9. $y = c(1 + \log x/y).$

Linear Coefficients Questions

3. $(x + y + 1)\,dx + (2x + 2y + 2)\,dy = 0.$

4. $(x + y - 1)\,dx + (2x + 2y - 3)\,dy = 0.$

Solutions

1. $(2x + y + 1)\,dx + (x + 2y)\,dy = 0 \quad (a)$

   We observe that it defines the same line.
   Divide: $(2x + y + 1)\,dx + (x + 2y)\,dy$. Then

$$(2x + y + 1)\,dx + (x + 2y)\,dy = 0$$

$$\Rightarrow \; dx + 2dy = 0$$

$$\Rightarrow x + 2y = C$$

Solutions

1. $(x+y-1)\,dx + (2x+2y-3)\,dy = 0$

   Notice that the lines are parallel but not coincident, so the substitution of a new variable for dx and dy will make it separable.

   Let

$$u = x + y - 1.$$

   Then

$$du = dx + dy. => dx = du - dy$$

So,

$$(x + y - 1)\,dx + (x + 2y - 3)\,dy = u\,(du - dy) + (2u - 1)\,dy.$$

$$= u\,(du - dy) + (2u - 1)\,dy$$

$$= u\,du - u\,dy + 2u\,dy - 1\,dy$$

$$= u\,du + (u - 1)\,dy = 0$$

Now the equation is separable:

$$\frac{u}{u-1}du + dy = 0$$

So, if we let v = u - 1, then dv = du. Moreover, u = v + 1

$$\frac{v+1}{v}dv + dy = 0.$$

$$(1 + \frac{1}{v})dv + dy = 0$$

$$v + ln|v| + y = C$$

Substitute back $v = u - 1$:

$$u - 1 + ln|u - 1| + y = C$$

Substitute back $u = x + y - 1$:

$$(x + y - 1) - 1 + ln|(x + y - 1) - 1| + y = C$$

$$x + y - 2 + ln|x + y - 2| + y = C$$

$$x + y + ln|x + y - 2| + y = C$$

Answer Key:

3. $x + 2y = c.$
4. $x + 2y + \log|x + y - 2| = c.$

Exact

The differential form $P(x,y)\,dx + Q(x,y)\,dy = 0$ is exact if we can find a function $f(x,y)$ so that:

$$\frac{\partial f}{\partial x} = P \text{ and } \frac{\partial f}{\partial y} = Q$$

The given differential form is closed if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Exact forms are closed. And if P, Q are C^1 on "nice" domains (such as rectangles) in $\mathbb{R}^2$, then closed forms are exact.

Questions

Determine if the ODE is exact. Solve the ODE if it is exact. No integrating factors for now.

$$(tx + 2)\,dt + (tx - 3)\,dx = 0$$

$$\frac{dy}{dt} = \frac{4 + ye^{ty}}{y - te^{ty}}$$

Solutions

Take $P(x,t) = tx + 2$ and $Q(x,t) = tx - 3$. Then $P_x = t$ and $Q_t = x$. Since $P_x \neq Q_t$, hence the ODE is not exact.

The differential form of the given ODE is $(y - te^{ty})\,dy + (-4 - ye^{ty})\,dt = 0$. Then take $P(y,t) = y - te^{ty}$ and $Q(y,t) = -4 - ye^{ty}$. Testing for exactness, note that

$$P_t = -e^{ty} - tye^{ty} = Q_y$$

Hence, the ODE is exact.

Solution to

$$\frac{dy}{dt} = \frac{4 + ye^{ty}}{y - te^{ty}}.$$

So there exists a function f so that $f_y = P, f_t = Q$. Then:

$$f(y,t) = \int P(y,t)\,dy = y^2/2 - e^{ty} + \phi(t)$$

and ϕ is a function of t only. Then $f_t = Q$ implies

$$-ye^{ty} + \phi'(t) = -4 - ye^{ty}$$

$$\Rightarrow \phi'(t) = -4$$

$$\Rightarrow \phi(t) = -4t + C$$

for an arbitrary constant C. Hence, the solution to the ODE is

$$f(y,t) = y^2/2 - e^{ty} - 4t + C$$

Integrating Factors

A function $I(x, y)$ such that the form $I(x, y)[P(x, y)\,dx + Q(x, y)\,dy]$ is exact. Some common ways to obtain:

- If $\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = f(x)$ then $I(x, y) = \exp\left(\int f(x)\,dx\right)$
- If $\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = g(y)$ then $I(x, y) = \exp\left(-\int g(y)\,dy\right)$
- If $P = y \cdot \phi(xy)$ and $Q = x \cdot \psi(xy)$ then $I(x, y) = 1/(xP - yQ)$.

Questions

$$v^2\,du + uv\,dv = 0$$

$$\begin{cases} \frac{dy}{dx} + y = 3\cos(2x) \\ y(0) = -1 \end{cases}$$

$$x' + \frac{x}{t} = 2x^{-1.25} \text{ (Hint: } u = x^{1-a})$$

Solution to $v^2\,du + uv\,dv = 0$

Testing for exactness, take $P(u,v) = v^2$ and $Q(u,v) = uv$. Then

$$P_v = 2v, Q_u = v$$

So the given differential form is not exact. But notice that $(P_v - Q_u)/P = v/v^2 = 1/v$ is a function of v only. So a good integrating factor would be $I(u,v) = \exp(-\ln(v)) = 1/v$. Multipling the given differential form by I, we get:

$$v\,du + u\,dv = 0$$

Solution to $v^2\,du + uv\,dv = 0$

Now let $M(u,v) = v$ and $N(u,v) = u$. Then

$$M_v = 1 = N_u$$

so this alternative differential form is exact.

There exists a solution f where $f_u = M$ and $f_v = N$. Then,

$$f(u,v) = \int v\,du = uv + \phi(v)$$

where ϕ is a function of v. Then $f_v = N$ implies:

$$u + \phi'(v) = u \;\Rightarrow\; \phi(v) = C$$

for any real constant C. Hence, $f(u,v) = uv + C$ solves our ODE.

Solution to $\begin{cases} \frac{dy}{dx} + y = 3\cos(2x) \\ y(0) = -1 \end{cases}$

Consider the integrating factor $I(x) = \exp(\int dx) = e^x$. Multiplying our ODE by I, we get:

$$\begin{aligned} 3\cos(2x)e^x &= \frac{dy}{dx}e^x + ye^x \\ &= \frac{d}{dx}(ye^x) \\ \Rightarrow ye^x &= \int 3\cos(2x)e^x \, dx \\ &= \frac{3e^x}{5}(\cos(2x) + 2\sin(2x)) + C \\ \Rightarrow y &= \frac{3}{5}(\cos(2x) + 2\sin(2x)) + Ce^{-x} \end{aligned}$$

Solution to $\begin{cases} \frac{dy}{dx} + y = 3\cos(2x) \\ y(0) = -1 \end{cases}$

Plugging in the initial value, we get: $\frac{3}{5} + C = -1$, i.e. $C = -\frac{8}{5}$. Thus, the particular solution is given by $y = \frac{3}{5}(\cos(2x) + 2\sin(2x)) - \frac{8}{5}e^{-x}$.

Solution to $x' + \frac{x}{t} = 2x^{-1.25}$

Let $u = x^{2.25}$. Then, the ODE gives $x^{1.25}x' + x^{2.25}/t = 2$, so in terms of u, we get: $u' + u/t = 2$. Multiply this by $\exp(\int 1/t\,dt) = t$, we get: $tu' + u = 2t$. Since $(tu)' = u + tu'$, we have: $tu = t^2 + C$ or equivalently, $u = t + C/t, t \neq 0$

In terms of t, x instead of t, u, we have $x = (t + C/t)^{4/9}, t \neq 0$

nth-Order, Linear, Constant Coefficients

$$\sum_{k=0}^{n} a_k y^{(k)}(x) = f(x)$$

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$. If $f = 0$ then the ODE is homogeneous.

Characteristic (Find all λ): $\sum_{k=0}^{n} a_k \lambda^k = 0$

Example: General Solutions

Suppose:

$$\sum_{k=0}^{n} a_k \lambda^k = \underbrace{(\lambda + 1)}_{(1)} \underbrace{(\lambda^2 - 6\lambda + 13)}_{(2)} \underbrace{(\lambda - 2)^3}_{(3)}$$

The roots for (2) are $\lambda = 3 + 2i, 3 - 2i$. Then the general solution would be:

$$y_h = \underbrace{c_1 e^{-x}}_{(1)} + \underbrace{c_2 e^{3x} \cos(2x) + c_3 e^{3x} \sin(2x)}_{(2)} + \underbrace{c_4 e^{2x} + c_5 x e^{2x} + c_6 x^2 e^{2x}}_{(3)}$$

where $c_1, c_2, \ldots, c_6 \in \mathbb{R}$

Particular Solution

Recall that the final solution is given by $y = y_h + y_p$. If the ODE is homogeneous then $y_p = 0$. Otherwise, we must consider $f(x)$ from the RHS of the ODE. If for all $k \in \mathbb{Z}^{\geq 0}$,

$$f^{(k)}(x) = \sum_{m=1}^{n} b_m y_m(x)$$

where $y_1, \ldots, y_n$ are linearly independent functions and $b_1, \ldots, b_n \in \mathbb{R}$, then

$$y_p(x) = \sum_{m=1}^{n} A_m y_m(x)$$

for some $A_1, \ldots, A_n \in \mathbb{R}$. Then y_p can be found by plugging it back into the ODE and solving for $A_1, \ldots, A_n$.

Particular Solution

Furthermore, if any of those linearly independent functions y_j appears as terms in the general solution, then you can ignore y_j term in y_p.

For example, for the ODE $y'' - 3y' + 2y = e^x + x$, the general solution is $C_1e^x + C_2e^{2x}$ and the right-hand side of the ODE is $e^x + x$. Since e^x is already in the general solution, you only need to consider $y_p = Ax + B$. Then $y_p'' - 3y_p' + 2y_p = 0 - 3A + 2(Ax + B) = 2Ax + B - 3A$. Then $2A = 1, B - 3A = 0$. So $A = 1/2, B = 3A = 3/2$. Then the solution is $y = C_1e^x + C_2e^{2x} + x/2 + 3/2$.

Questions

$$y''' - 3y' - 2y = 0$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin(3x)$$

Solution to $y''' - 3y' - 2y = 0$

Consider the characteristic equation: $\lambda^3 - 3\lambda - 2 = 0$. Then $\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2(\lambda - 2)$. Hence the general solution is $y_h(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^{2t}$ where $C_1, C_2, C_3 \in \mathbb{R}$. Since the ODE is homogeneous, the solution is given by $y = y_h$.

Solution to $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin(3x)$

Consider the characteristic equation: $\lambda^2+2\lambda+2=0$. Then $\lambda^2+2\lambda+2 = (\lambda-(-1-i))(\lambda-(-1+i))$. So the general solution is given by the following equation: $y_h(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x)$. The particular solution is given by $y_p(x) = A\sin(3x) + B\cos(3x)$. Then

$$y_p'' + 2y_p' + 2y_p = (-7A-6B)\sin(3x) + (6A-7B)\cos(3x)$$

We have the following linear system: $-7A-6B = 1, 6A-7B = 0$. This implies $A = -7/85, B = -6/85$. Hence, the solution to the ODE is given by $y(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x) + -(7/85)\sin(3x) - (6/85)\cos(3x)$.