STAB52 Final Review Seminar

Expectation, Variance, Conditioning, Transformations

Continuous Random Variables

- A continuous random variable is described by a PDF $f_X(x)$

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- A continuous random variable is described by a PDF $f_X(x)$
- Properties:

$$f_X(x) \geq 0, \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

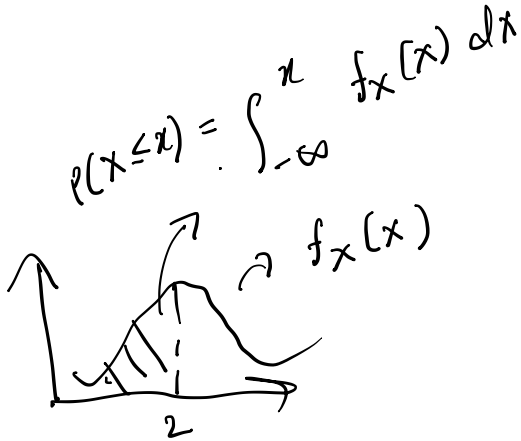
↳ $\sum_i p(x_i) = 1$

Continuous Random Variables

- A continuous random variable is described by a PDF $f_X(x)$
- Properties:

$$f_X(x) \geq 0, \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

- Probabilities are areas under the curve



Question: Continuous RV

Let

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

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1. Find $P(X > 0.5)$

Question: Continuous RV

Let

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find $P(X > 0.5)$
2. Find the CDF $F_X(x)$

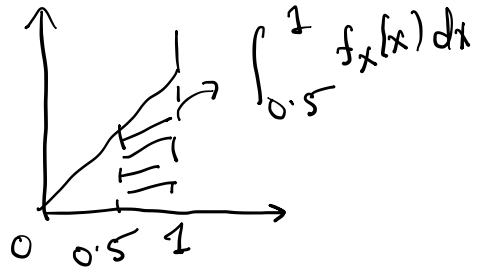
Solution: Continuous RV

(a) Probability

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(a) Probability

$$P(X > 0.5) = \int_{0.5}^{1} 2x\, dx$$



Solution: Continuous RV

(a) Probability

$$P(X > 0.5) = \int_{0.5}^{1} 2x\, dx$$

$$= \left[x^2\right]_{0.5}^{1} = 1 - 0.25 = 0.75$$

Solution: Continuous RV

(a) Probability

$$P(X > 0.5) = \int_{0.5}^{1} 2x\, dx$$

$$= \left[x^2\right]_{0.5}^{1} = 1 - 0.25 = 0.75$$

(b) CDF

Solution: Continuous RV

(a) Probability

$$P(X > 0.5) = \int_{0.5}^{1} 2x\,dx$$

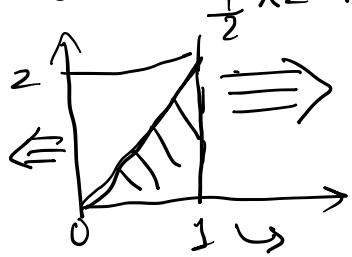
$$= \left[x^2\right]_{0.5}^{1} = 1 - 0.25 = 0.75$$

(b) CDF

For $0 < x < 1$:

$$F_X(x) = \int_0^x 2t\,dt = x^2 = \int_{-\infty}^{x} f_X(t)\,dt$$

$$P(X \le x) = \begin{cases} 0, & 0 \le x \\ x^2, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$



$\frac{1}{2} \times 2 \times 1 = 1$

$$= \int_{-\infty}^{0} f_X(t)\,dt + \int_0^x f_X(t)\,dt$$

$\downarrow$ 0

Expectation

$X_1 \dots X_n \overset{i.i.d}{\sim} P(x)$

- Discrete:

$$E[X] = \sum_x x p_X(x)$$

Expectation

- Discrete:

$$E[X] = \sum_x x p_X(x) \rightarrow \text{PMF}$$

- Continuous:

$$E[X] = \int x f_X(x)\, dx$$

↳ PDF

Expectation

- Discrete:

$$E[X] = \sum_x x p_X(x)$$

- Continuous:

$$E[X] = \int x f_X(x)\, dx$$

- Linearity always holds

$$E[c] = c$$

$$E[cX] = c\,E[X]$$

$$E[X+Y] = E[X] + E[Y]$$

If $X \perp Y$ $\quad E[XY] = E[X]E[Y]$ ✓

Question: Expectation

Pareto Distribution

Let

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Question: Expectation

Let

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find $E[X]$

Question: Expectation

Let

$$f_X(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find $E[X]$
2. Find $E[X^2]$

Solution: Expectation

(a) $E[X]$

Solution: Expectation

(a) $E[X]$

$$E[X] = \int_0^1 x(3x^2)\,dx$$

Solution: Expectation

(a) $E[X]$

$$E[X] = \int_0^1 x(3x^2)\,dx$$

$$= 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$$

Solution: Expectation

(a) $E[X]$

$$E[X] = \int_0^1 x(3x^2)\,dx$$

$$= 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$$

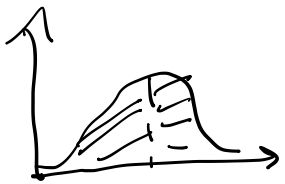
(b) $E[X^2]$

Solution: Expectation

(a) $E[X]$

$$E[X] = \int_0^1 x(3x^2)\,dx$$

$$= 3\int_0^1 x^3\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$$



(b) $E[X^2]$

$$E[X^2] = \int_0^1 x^2(3x^2)\,dx = 3\int_0^1 x^4\,dx = \frac{3}{5}$$

Variance

- Definition:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

$E[(X-\mu)^2]$

RV, const

$$= E[X^2] - (E[X])^2$$

Variance

- Definition:

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

- Shortcut:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Question: Variance

Let

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

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1. Find $E[X]$

Question: Variance

Let

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

1. Find $E[X]$
2. Find $\mathrm{Var}(X)$

Solution: Variance

(a) Expectation

Solution: Variance

(a) Expectation

$$E[X] = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

Solution: Variance

(a) Expectation

$$E[X] = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

(b) Variance

Solution: Variance

(a) Expectation

$$E[X] = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

(b) Variance

$$E[X^2] = \int_0^1 x^2(2x)\,dx = 2\int_0^1 x^3\,dx = \frac{1}{2}$$

Solution: Variance

(a) Expectation

$$E[X] = \int_0^1 x(2x)\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

(b) Variance

$$E[X^2] = \int_0^1 x^2(2x)\,dx = 2\int_0^1 x^3\,dx = \frac{1}{2}$$

$$\mathrm{Var}(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Conditional PDFs

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Not in all cases $= f_X(x) * f_Y(y)$

Conditional PDFs

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Defined only where $f_X(x) > 0$ ✓ ~~$f_X(x) < 0$~~ ✗ By prop'y PDF non-negative

Question: Conditional PDF

Given

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1,\ 0 < y < 1 - x$$

Question: Conditional PDF

Given

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1,\ 0 < y < 1 - x$$

1. Find $f_X(x)$ $\longrightarrow$ Marginal

$$f_{Y|X}(y|x) = \frac{f_{x,y}(x,y)}{f_X(x)}$$

Question: Conditional PDF

Given

$$f_{X,Y}(x,y) = x + y, \quad 0 < x < 1,\ 0 < y < 1 - x$$

1. Find $f_X(x)$ ✓
2. Find $f_{Y|X}(y|x)$

Solution: Conditional PDF

(a) Marginal

Solution: Conditional PDF

(a) Marginal

$$f_X(x) = \int_0^{1-x} (x + y)\, dy$$

Solution: Conditional PDF

(a) Marginal

$$f_X(x) = \int_0^{1-x} (x + y)\, dy$$

$$= x(1 - x) + \frac{(1 - x)^2}{2}$$

Solution: Conditional PDF

(a) Marginal

$$f_X(x) = \int_0^{1-x} (x + y)\, dy$$

$$= x(1 - x) + \frac{(1 - x)^2}{2}$$

(b) Conditional

Solution: Conditional PDF

(a) Marginal

$$f_X(x) = \int_0^{1-x} (x+y)\,dy$$

$$= x(1-x) + \frac{(1-x)^2}{2}$$

(b) Conditional

$$f_{Y|X}(y|x) = \frac{x+y}{x(1-x)+(1-x)^2/2}, \quad 0 < y < 1-x$$

Transformations

- CDF method

Transformations

- CDF method
- Jacobian method

Transformations

- CDF method ✓
- Jacobian method → plug and play
- Convolution ✓

Question: CDF Method

Let $X \sim \text{Uniform}(0, 1)$ and $Y = X^2$.

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Let $X \sim \text{Uniform}(0, 1)$ and $Y = X^2$.

1. Find $F_Y(y)$

Question: CDF Method

Let $X \sim \text{Uniform}(0, 1)$ and $Y = X^2$.

1. Find $F_Y(y)$ → Given CDF, PDF = $\frac{d}{dy}$(CDF)
2. Find $f_Y(y)$

Solution: CDF Method

Solution: CDF Method

$F_Y(y) = P(Y \leq y)$, but $Y = X^2$

$$F_Y(y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$$

← in terms of X, ∴ we can use CDF of X!

Solution: CDF Method

$$F_Y(y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$$

$$= \sqrt{y}, \quad 0 < y \leq 1$$

$X \sim \text{Unif}[0,1]$

$P(X \leq x) = \frac{x}{1}$

Solution: CDF Method

$$F_Y(y) = P(X^2 \leq y) = P(X \leq \sqrt{y})$$

$y < 0 \rightarrow$ x defined for $\mathbb{R}$

$$= \begin{cases} 0, & y < 0 \\ \sqrt{y}, & 0 < y < 1 \\ 1, & y > 1 \end{cases}$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0, & o/w \end{cases}$$

Question: Convolution

Let X and Y be independent random variables with

$$X \sim \text{Exponential}(1), \qquad Y \sim \text{Uniform}(0, 1).$$

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Let X and Y be independent random variables with

$$X \sim \text{Exponential}(1), \qquad Y \sim \text{Uniform}(0, 1).$$

Find the PDF of

$$Z = X + Y.$$

Solution: Convolution (Setup)

Since $Z = X + Y$ and $X \perp Y$, we use convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx.$$

$z = x + y$

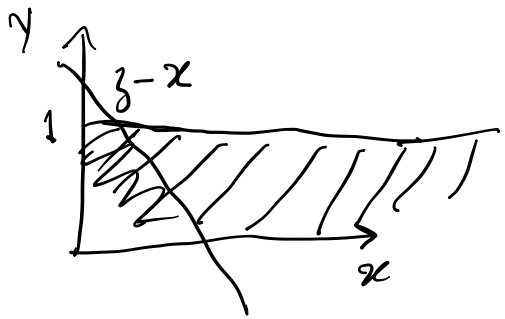
$y = z - x$ ✓

$f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$

Solution: Convolution (Setup)

Since $Z = X + Y$ and $X \perp Y$, we use convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx.$$



Recall the PDFs:

$$f_X(x) = e^{-x}, \quad x > 0$$

$$f_Y(y) = 1, \quad 0 < y < 1$$

$\hookrightarrow \{(x, y) : x > 0, 0 < y < 1\}$

For what values of y does $z - x$ lie in the strip

Lower bound for y

$\hookrightarrow 0 < y \Rightarrow 0 < z - x \Rightarrow \boxed{x < z}$

Upper bound for y

$z - x < 1 \Rightarrow \boxed{x > z - 1}$, $\boxed{x > 0}$

Combine these $\rightarrow \max(0, z-1) < x < z$

Solution: Convolution (Setup)

Since $Z = X + Y$ and $X \perp Y$, we use convolution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx.$$

Recall the PDFs:

$$f_X(x) = e^{-x}, \quad x > 0$$

$$f_Y(y) = 1, \quad 0 < y < 1$$

We must determine the integration limits from the support.

$\text{Max}(0, z-1) = \begin{cases} 0, & 0 < z < 1 \\ z-1, & z \geq 1 \end{cases}$

$z = x + y$, $x > 0$, $y > 0$

$\hookrightarrow z > 0$

Solution: Convolution (Case 1)

Case 1: $0 < z < 1$

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Case 1: $0 < z < 1$

Here:

$$\max(0, z-1) < x < z \iff 0 < x < z$$

$$0 < x < z$$

Solution: Convolution (Case 1)

Case 1: $0 < z < 1$

Here:

$$0 < x < z$$

$$f_Z(z) = \int_0^z e^{-x} \cdot 1 \, dx$$

Solution: Convolution (Case 1)

Case 1: $0 < z < 1$
Here:

$$0 < x < z$$

$$f_Z(z) = \int_0^z e^{-x} \cdot 1 \, dx$$

$$= \left[-e^{-x}\right]_0^z = 1 - e^{-z}$$

Solution: Convolution (Case 2)

Case 2: $z \geq 1$

Solution: Convolution (Case 2)

Case 2: $z \geq 1$

Here:

$$\max(0, z-1) < x < z$$

$$z - 1 < x < z$$

Solution: Convolution (Case 2)

Case 2: $z \geq 1$

Here:

$$z - 1 < x < z$$

$$f_Z(z) = \int_{z-1}^{z} e^{-x} \cdot 1 \, dx$$

Solution: Convolution (Case 2)

Case 2: $z \geq 1$

Here:

$$z - 1 < x < z$$

$$f_Z(z) = \int_{z-1}^{z} e^{-x} \cdot 1 \, dx$$

$$= \left[-e^{-x}\right]_{z-1}^{z} = e^{-(z-1)} - e^{-z}$$

Solution: Convolution (Final Answer)

$$f_Z(z) = \begin{cases} 1 - e^{-z}, & 0 < z < 1 \\ e^{-(z-1)} - e^{-z}, & z \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Solution: Convolution (Final Answer)

$$f_Z(z) = \begin{cases} 1 - e^{-z}, & 0 < z < 1 \\ e^{-(z-1)} - e^{-z}, & z \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Always split convolution by support

Order Statistics

Let $X_1 \dots X_n \sim i.i.d \; f_X(x)$

$$F_{X_{(n)}}(x) = [F(x)]^n$$

$$P(\max(X_1 \dots X_n) \le x) = P(X_1 \le x \text{ and } X_2 \le x \dots X_n \le x)$$

$$= \prod_{i=1}^{n} P(X_i \le x) \longrightarrow i.i.d$$

$$= [F(x)]^n$$

Order Statistics

$$F_{X_{(n)}}(x) = [F(x)]^n$$

Used for max and min problems

Question: Order Statistic

Let $X_1, X_2, X_3 \sim \text{Uniform}(0, 1)$ i.i.d.

Question: Order Statistic

Let $X_1, X_2, X_3 \sim \text{Uniform}(0, 1)$ i.i.d.
Find

$$P(\max\{X_1, X_2, X_3\} \leq 0.5)$$

Solution: Order Statistic

Solution: Order Statistic

$$P(\max \leq 0.5) = [F(0.5)]^3$$

Solution: Order Statistic

Because $F(x) = x$

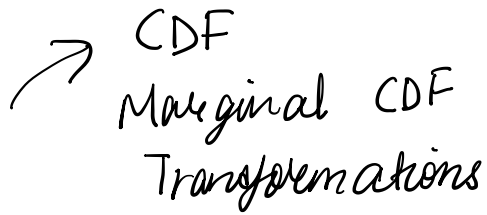
$$P(\max \leq 0.5) = [F(0.5)]^3 = \left(\frac{1}{2}\right)^3 =$$

$$= (0.5)^3 = \frac{1}{8} = \frac{1}{8}$$

Exercise: Derive order stat for min

Final Exam Advice

- Show integration limits



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- Show integration limits
- State distributions explicitly

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- Structure earns partial credit

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Clear logic beats perfect arithmetic

